Curved Casimir operators in conformal geometry

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- These operators are defined on all natural bundles, they lead to standard and non–standard operators and apply both in regular and singular infinitesimal character.
- Constructing invariant operators out of curved Casimirs usually boils down to finite dimensional representation theory.

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2. Curved Casimir operators
3. Applications

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- Representations of G_0 give rise to natural bundles and G_0–equivariant maps induce natural bundle maps.
- The bundles obtained in this way are conformally weighted tensor and spinor bundles.
- G_0 is reductive, so any representation and consequently any natural bundle splits into a direct sum of irreducibles.

connection to representation theory

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The homogeneous model

The group $G = SO(n+1,1)$ acts on S^n by conformal isometries. This action is transitive, thus inducing a diffeomorphism $S^n \cong G/P$, where $P \subset G$ is the stabilizer of an isotropic line in $\mathbb{R}^{n+1,1}$. For any natural bundle $E \to S^n$ for the conformal structure on S^n, the space $\Gamma(E)$ naturally is a representation of G.

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The subgroup $P \subset G$ is a *parabolic* subgroup in the sense of representation theory. The theory of curved Casimir operators extends to other geometries modeled on the quotients of semisimple Lie groups by parabolic subgroups ("parabolic geometries").

Let $E \to S^n$ be the natural vector bundle associated to an irreducible representation of G_0 ("irreducible bundle"). Then the natural representation of G on $\Gamma(E)$ is a principal series representation and hence well understood. Even better, homomorphisms between such representations which are differential operators are related by a duality to homomorphisms between generalized Verma modules, which are completely understood.

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Results

- Operators occur in patterns that can be nicely described by representation theory.
- For a fixed irreducible bundle E, there are at most 3 invariant operators with values in irreducible bundles defined on $\Gamma(E)$. E.g. on unweighted functions in even dimensions $n = 2m$, there is only the exterior derivative and the critical GJMS operator of order n.

Cartan connections

To work in an invariant setting, one has to leave the realm of irreducible bundles. For $S^n = G/P$, any representation of P gives rise to a bundle on which G naturally acts. For example, restrictions of representations of G lead to bundles which are G–equivariantly trivial, and hence admit G–invariant linear connections.

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It turns out that $P \cong G_0 \rtimes P_+$, where $G_0 = CO(n)$ and $P_+ \cong \mathbb{R}^{n*}$ is Abelian, so P is naturally an extension of G_0. This group also plays a crucial role in the curved case:

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Theorem (E. Cartan)

Let $(M, [g])$ be a conformal manifold of dimension $n \geq 3$. Then the conformal frame bundle of M can be naturally extended to a principal P–bundle, on which there is a canonical Cartan connection. This leads to an equivalence of categories.

Natural bundles 2

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- Restrictions to P of representations of G give rise to so–called *tractor bundles*.
- Under very mild conditions, a representation $\mathbb{V}$ of P admits a P–invariant filtration $\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset \cdots \supset \mathbb{V}^N$ such that for each i the representation $\mathbb{V}^i/\mathbb{V}^{i+1}$ is completely reducible.

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This carries over to natural bundles.

Structure

Let $E \to G/P$ be a homogeneous vector bundle corresponding to a representation $\mathbb{V}$ of P. Then $\Gamma(E) \cong C^\infty(G, \mathbb{V})^P$ and the quadratic Casimir element in $\mathcal{U}(\mathfrak{g})$ acts on this representation as the differential operator $\mathcal{C}(f) := -\sum_i R_{X_i} \cdot R_{X^i} \cdot f$, where $\{X_i\}$ and $\{X^i\}$ are dual bases for $\mathfrak{g}$ with respect to the Killing form, and R_X is the right invariant vector field generated by X.

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Some tractor calculus

Consider a conformal structure with Cartan bundle $\mathcal{G} \to M$, and put $\mathfrak{g} = \mathfrak{so}(n+1, 1)$.

- adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$

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- adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$
- sections of $\mathcal{A}M$ correspond to P–invariant vector fields on $\mathcal{G}$
- For any natural bundle $E \to M$, this induces the fundamental derivative $D : \Gamma(E) \to \Gamma(\mathcal{A}^*M \otimes E)$. For $\sigma \in \Gamma(E)$ and $s \in \Gamma(\mathcal{A}M)$ we write $D_s\sigma$ for $D\sigma(s)$.

Definition of the curved Casimir operator

The Killing form induces a non–degenerate invariant bilinear form on $\mathfrak{g}$ and hence also on $\mathfrak{g}^*$. This induces a natural bundle map $B : \mathcal{A}^*M \otimes \mathcal{A}^*M \to M \times \mathbb{R}$.

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Definition/Proposition

For any natural vector bundle $E \to M$, $\mathcal{C}(\sigma) := (B \otimes \mathrm{id}_E)(D^2\sigma)$ defines an invariant differential operator $\mathcal{C} : \Gamma(E) \to \Gamma(E)$. On the homogeneous model G/P, this operator specializes to the action of the quadratic Casimir, so it is called the *curved Casimir operator*.

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From this definition it follows immediately that $\mathcal{C}$ is a differential operator of order at most 2, which is natural with respect to *all* bundle maps coming from P–equivariant maps between the inducing representations.

For a local frame $\{s_i\}$ for $\mathcal{A}M$ with dual frame $\{s^i\}$, one has $\mathcal{C}(\sigma)=\sum_i D^2\sigma(s^i,s_i)$. Using special local frames, which are nicely adapted to the natural filtration of $\mathcal{A}M$, one proves:

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Theorem

(1) On any natural bundle E, $\mathcal{C}$ has order at most one, and its symbol is induced by the action of $\mathfrak{p}_+$ on the representation inducing E.

(2) On an irreducible bundle, $\mathcal{C}$ acts by a scalar multiple of the identity, and the scalar can be computed from the highest weight of the inducing representation.

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There are also explicit formulae for the action of $\mathcal{C}$ on tractor bundles and tractor bundle valued forms, which nicely relate it to tractor connections and BGG sequences.

Structure

The basic construction principle

Let $\mathbb{V}$ be a representation of P with P–invariant filtration $\{\mathbb{V}^i\}$ such that all subquotients $\mathbb{V}^i/\mathbb{V}^{i+1}$ are completely reducible.

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- For each index i, let $\alpha_i^1, \dots, \alpha_i^{n_i}$ be the different scalars by which $\mathcal{C}$ acts on sections of the bundles induced by the irreducible components of $\mathbb{V}^i/\mathbb{V}^{i+1}$.

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- For a conformal manifold M, the bundle VM induced by $\mathbb{V}$ is naturally filtered by smooth subbundles V^iM. By naturality, $\mathcal{C}$ respects the sections of any of these subbundles.

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- For a conformal manifold M, the bundle VM induced by $\mathbb{V}$ is naturally filtered by smooth subbundles V^iM. By naturality, $\mathcal{C}$ respects the sections of any of these subbundles.
- Now define $L_j := (\mathcal{C} - \alpha_j^1) \circ \dots \circ (\mathcal{C} - \alpha_j^{n_j})$. Then L_j preserves each of the subspaces $\Gamma(V^iM)$. Restricting to $\Gamma(V^jM)$, the induced operator on $\Gamma(V^jM/V^{j+1}M)$ vanishes by construction, so $L_j(\Gamma(V^jM)) \subset \Gamma(V^{j+1}M)$.

The basic construction principle II

- Now fix indices $i < j$, and let π_j be the bundle map induced by the projection $\mathbb{V}^i \to \mathbb{V}^i/\mathbb{V}^{j+1}$. Consider the composition $\pi_{j+1} \circ L_j \circ \dots \circ L_{i+1}$.

The basic construction principle II

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- This defines an operator $\Gamma(V^iM) \to \Gamma(V^iM/V^{j+1}M)$. A moment of thought shows that it vanishes on $\Gamma(V^{i+1}M)$ and hence factorizes to $\Gamma(V^iM/V^{i+1}M)$. Choosing an irreducible component $\mathbb{W}$ in $\mathbb{V}^i/\mathbb{V}^{i+1}$, we can then restrict to $\Gamma(WM)$ to obtain an operator $L : \Gamma(WM) \to \Gamma(V^iM/V^{j+1}M)$, which is natural by construction.

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Let α be the scalar by which $\mathcal{C}$ acts on $\Gamma(WM)$. Then the behavior of L depends on whether α is different from all the α_ℓ^k or not. Let us first assume that $\alpha \neq \alpha_\ell^k$ for all $i < \ell \leq j$ and all k.

Splitting operators

The natural projection $\mathbb{V}^i \to \mathbb{V}^i/\mathbb{V}^{i+1}$ induces a bundle map $\pi : V^i M \to V^i M/V^{i+1} M$, which also makes sense on $V^i M/V^{j+1} M$.

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$$\pi \circ L = \pi \circ \prod_{i<\ell\leq j} \prod_{1\leq k\leq n_j} (\mathcal{C} - \alpha_\ell^k),$$

is obtained by first applying π and then the given polynomial in $\mathcal{C}$, but for the curved Casimir operator on $\Gamma(WM)$. Hence this acts by the scalar

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Up to this nonzero factor, the operator L defines a differential splitting of the tensorial operator on sections induced by π. Such splitting operators are the basis for the curved translation principle and for the machinery of BGG sequences.

Now let us assume that (with appropriate numeration) $\alpha = \alpha_j^1$. Let $\tilde{\mathbb{W}} \subset \mathbb{V}^j/\mathbb{V}^{j+1}$ be the sum of the irreducible components corresponding to this scalar, and let $\tilde{W}M$ be the induced bundle. Then we claim that the operator L automatically has values in $\Gamma(\tilde{W}M)$, so we directly obtain an invariant operator between irreducible bundles.

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This follows readily by writing $L_j = (\mathcal{C} - \alpha_j^1) \circ L'_j$ and

$$L = \pi_{j+1} \circ L'_j \circ L_{j-1} \circ \ldots \circ L_{i+1} \circ (\mathcal{C} - \alpha_j^1)$$

By naturality, $\mathcal{C} - \alpha_j^1 = \mathcal{C} - \alpha$ maps sections of V^iM whose projection to $V^iM/V^{i+1}M$ has values in WM to sections of $V^{i+1}M$. These are then mapped to V^jM by $L_{j-1} \circ \ldots \circ L_{i+1}$, and those components whose projection to $V^jM/V^{j+1}M$ does not have values in $\tilde{W}M$ are mapped to sections of $V^{j+1}M$ by L'_j.

Example: Weighted standard tractors

Here $\mathbb{V} = \mathbb{R}^{n+1,1}[w]$ (tensor product with densities of weight w), which corresponds to the weighted standard tractor bundle $\mathcal{E}_A[w]$. This has a filtration $\mathbb{V} = \mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1$ with $\mathbb{V}^1 = \mathbb{R}[w-1]$, $\mathbb{V}^0/\mathbb{V}^1 \cong \mathbb{R}^{n*}[w+1]$ and $\mathbb{V}/\mathbb{V}^0 \cong \mathbb{R}[w+1]$. Hence there are just three relevant scalars α, α_0, and α_1 (depending on w). In particular, $\alpha = \alpha_0$ for $w = -1$ and $\alpha = \alpha_1$ for $w = -\frac{n}{2}$.

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Up to a constant, the operator $L : \mathcal{E}[w+1] \to \mathcal{E}_A[w]$ induced by $(\mathcal{C} - \alpha_1) \circ (\mathcal{C} - \alpha_0)$ is exactly the tractor D–operator. If $w \neq -1, -\frac{n}{2}$, then this is a splitting operator. For $w = -1$, it specializes to the exterior derivative, while for $w = -\frac{n}{2}$ it gives the Yamabe–operator.

Similarly, one constructs an operator $L : \mathcal{E}[w+2] \to \mathcal{E}_{(AB)_0}[w]$ which is a splitting operator for generic w, while for $w = -\frac{n}{2}$ it specializes to a conformally invariant square of the Laplacian in dimensions $n \neq 4$.

Similarly, one constructs an operator $L : \mathcal{E}[w+2] \to \mathcal{E}_{(AB)_0}[w]$ which is a splitting operator for generic w, while for $w = -\frac{n}{2}$ it specializes to a conformally invariant square of the Laplacian in dimensions $n \neq 4$.

In the same way, one obtains $L : \mathcal{E}[w+3] \to \mathcal{E}_{(ABC)_0}[w]$ which is a splitting operator for generic w, while for $w = -\frac{n}{2}$ it specializes to a conformally invariant cube of the Laplacian in dimensions $n \neq 4, 6, 10$.

A general theorem

For $k > 0$ consider $\mathbb{V} = S_0^k\mathbb{R}^{n+1,1}[w]$. This has an invariant filtration of the form $\mathbb{V} = \mathbb{V}^{-k} \supset \cdots \supset \mathbb{V}^k$ such that $\mathbb{V}^{-k}/\mathbb{V}^{-k+1} \cong \mathbb{R}[w+k]$ and $\mathbb{V}^k \cong \mathbb{R}[w-k]$. For any irreducible representation $\mathbb{W}$, consider the tensor product $\mathbb{V} \otimes \mathbb{W}$. This has a similar composition series starting with $\mathbb{W}[w+k]$ and ending with $\mathbb{W}[w-k]$, which both are irreducible. Denoting by E the irreducible bundle corresponding to $\mathbb{W}$, there is a unique weight w_0 such that $\mathcal{C}$ acts by the same scalar on sections of $E[w_0+k]$ and $E[w_0-k]$.

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For $k > 0$ consider $\mathbb{V} = S_0^k \mathbb{R}^{n+1,1}[w]$. This has an invariant filtration of the form $\mathbb{V} = \mathbb{V}^{-k} \supset \cdots \supset \mathbb{V}^k$ such that $\mathbb{V}^{-k}/\mathbb{V}^{-k+1} \cong \mathbb{R}[w+k]$ and $\mathbb{V}^k \cong \mathbb{R}[w-k]$. For any irreducible representation $\mathbb{W}$, consider the tensor product $\mathbb{V} \otimes \mathbb{W}$. This has a similar composition series starting with $\mathbb{W}[w+k]$ and ending with $\mathbb{W}[w-k]$, which both are irreducible. Denoting by E the irreducible bundle corresponding to $\mathbb{W}$, there is a unique weight w_0 such that $\mathcal{C}$ acts by the same scalar on sections of $E[w_0+k]$ and $E[w_0-k]$.

Theorem

For any irreducible bundle E, the general construction produces an operator $\Gamma(E[w_0+k]) \to \Gamma(E[w_0-k])$ of order $2k$ which has nonzero symbol of that order except in finitely many dimensions.