Lie algebra cohomology and overdetermined systems

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This can be used as a replacement for Se–Ashi’s theory in proofs of Fubini–Griffiths–Harris rigidity in the style of Hwang–Yamaguchi and Landsberg–Robles.
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This can be used as a replacement for Se–Ashi’s theory in proofs of Fubini–Griffiths–Harris rigidity in the style of Hwang–Yamaguchi and Landsberg–Robles.

Via prolongation procedures, it also leads to results on first BGG operators for non–projective AHS–structures.
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1. The basic setup
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Consider a semisimple Lie algebra $\mathfrak{g}$ endowed with a grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that no simple ideal is contained in $\mathfrak{g}_0$. Then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a parabolic subalgebra with nilradical $\mathfrak{g}_1$, which leads to a complete classification of such gradings.
1–graded Lie algebras

Consider a semisimple Lie algebra \( \mathfrak{g} \) endowed with a grading of the form \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) such that no simple ideal is contained in \( \mathfrak{g}_0 \). Then \( \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a parabolic subalgebra with nilradical \( \mathfrak{g}_1 \), which leads to a complete classification of such gradings.

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), \( P \subset G \) a subgroup corresponding to \( \mathfrak{p} \). For \( g \in P \), we have \( \text{Ad}(g)(\mathfrak{p}) \subset \mathfrak{p} \) and \( \text{Ad}(g)(\mathfrak{g}_1) \subset \mathfrak{g}_1 \). Let \( G_0 \subset P \) be the subgroup consisting of those \( g \) for which \( \text{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \) for all \( i = -1, 0, 1 \).
Consider a semisimple Lie algebra $\mathfrak{g}$ endowed with a grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that no simple ideal is contained in $\mathfrak{g}_0$. Then $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a parabolic subalgebra with nilradical $\mathfrak{g}_1$, which leads to a complete classification of such gradings.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, $P \subset G$ a subgroup corresponding to $\mathfrak{p}$. For $g \in P$, we have $\text{Ad}(g)(\mathfrak{p}) \subset \mathfrak{p}$ and $\text{Ad}(g)(\mathfrak{g}_1) \subset \mathfrak{g}_1$. Let $G_0 \subset P$ be the subgroup consisting of those $g$ for which $\text{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ for all $i = -1, 0, 1$.

It then turns out that $\text{Ad} : G_0 \to GL(\mathfrak{g}_{-1})$ is infinitesimally effective. Hence on manifolds of dimensions $\dim(\mathfrak{g}_{-1})$ the notion of first order $G_0$–structures makes sense. Further, $P$ is the semidirect product of $G_0$ and $\mathfrak{g}_1 \cong \mathfrak{g}^*_{-1}$. 
Under a cohomological condition, first order $G_0$–structures as discussed above are equivalent to normal Cartan geometries of type $(G, P)$. This means that the principal $G_0$–bundle and the soldering form defining a $G_0$–structure canonically extend to a principal $P$–bundle and a normal Cartan connection.
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**Examples**

- $G = SO(p + 1, q + 1)$, $G_0 = CO(p, q)$ conformal
- $G = PSL(p + q, \mathbb{R})$, $G_0 = S(GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$ almost Grassmannian
- $G = SL(n + 1, \mathbb{H})$, $G_0 = Sp(1)GL(n, \mathbb{H})$ almost quaternionic
- $G = PGL(n + 1, \mathbb{R})$, $G_0 = GL(n, \mathbb{R})$ Cartan geometries are equivalent to projective structures, but not to $G_0$–structures
Let $\mathbb{V}$ be a representation of $G$ (or simplicity assumed to be effective with values in $SL(\mathbb{V})$), and let $\rho : g \rightarrow sl(\mathbb{V})$ the infinitesmial representation.
Let $V$ be a representation of $G$ (or simplicity assumed to be effective with values in $SL(V)$), and let $\rho: g \to sl(V)$ the infinitesmial representation.

- Via the action of the grading element, $V$ splits as $V = V_0 \oplus \cdots \oplus V_N$ in such a way that $g_i \cdot V_j \subset V_{i+j}$. This splitting is $G_0$–invariant.
Let $\mathbb{V}$ be a representation of $G$ (or simplicity assumed to be effective with values in $SL(\mathbb{V})$), and let $\rho : \mathfrak{g} \to \mathfrak{sl}(\mathbb{V})$ the infinitesimal representation.

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- According to homogeneity, we get an induced $G_0$–invariant splitting $\mathfrak{sl}(\mathbb{V}) = \mathfrak{sl}_-N(\mathbb{V}) \oplus \cdots \oplus \mathfrak{sl}_N(\mathbb{V})$. 
Let $\mathbb{V}$ be a representation of $G$ (or simplicity assumed to be effective with values in $SL(\mathbb{V})$), and let $\rho : g \rightarrow sl(\mathbb{V})$ the infinitesimal representation.

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- According to homogeneity, we get an induced $G_0$-invariant splitting $sl(\mathbb{V}) = sl_{-N}(\mathbb{V}) \oplus \cdots \oplus sl_N(\mathbb{V})$.

- Via $\rho$, we view $g$ as a Lie subalgebra in $sl(\mathbb{V})$ and $g_i \subset sl_i(\mathbb{V})$. This $g$-invariant subspace admits a $g$-invariant complement $g^\perp \subset sl(\mathbb{V})$, which decomposes as $g^\perp = \oplus g_i^\perp$. 
Let $\mathcal{V}$ be a representation of $G$ (or simplicity assumed to be effective with values in $SL(\mathcal{V}))$, and let $\rho : \mathfrak{g} \to \mathfrak{sl}(\mathcal{V})$ the infinitesimal representation.

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- According to homogeneity, we get an induced $G_0$–invariant splitting $\mathfrak{sl}(\mathcal{V}) = \mathfrak{sl}_{-N}(\mathcal{V}) \oplus \cdots \oplus \mathfrak{sl}_{N}(\mathcal{V})$.

- Via $\rho$, we view $\mathfrak{g}$ as a Lie subalgebra in $\mathfrak{sl}(\mathcal{V})$ and $\mathfrak{g}_i \subset \mathfrak{sl}_i(\mathcal{V})$. This $\mathfrak{g}$–invariant subspace admits a $\mathfrak{g}$–invariant complement $\mathfrak{g}^\perp \subset \mathfrak{sl}(\mathcal{V})$, which decomposes as $\mathfrak{g}^\perp = \bigoplus \mathfrak{g}_i^\perp$.

We put $\mathcal{V}^i := \bigoplus_{j \geq i} \mathcal{V}_j$ and similarly for all the other spaces. This endows each of them with a $P$–invariant filtration.
For each $i \geq 0$, there is a natural subgroup $SL^i(\mathcal{V}) \subset SL(\mathcal{V})$ with Lie algebra $\mathfrak{sl}^i(\mathcal{V})$. The group $SL^0(\mathcal{V})$ consists of all filtration preserving automorphisms of $\mathcal{V}$. For $i > 0$, the maps in $SL^i(\mathcal{V})$ are congruent to the identity modulo maps which move up in the filtration by $i$ degrees.
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The crucial ingredients for our purposes are subgroups $P^\# \subset G_0^\# \subset SL^0(\mathbb{V})$. We define $G_0^\#$ as the subgroup generated by $G_0$ and $SL^1(\mathbb{V})$ and $P^\#$ as the subgroup generated by $P$ and $SL^2(\mathbb{V})$. The Lie algebras of these subgroups are $\mathfrak{g}_0 \oplus \mathfrak{sl}^1(\mathbb{V})$ and $\mathfrak{p} \oplus \mathfrak{sl}^2(\mathbb{V})$, respectively.
Structure

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3. Applications
Let $M$ be a smooth manifold of dimension $\dim(g_{-1})$ and let $\mathcal{V} \to M$ be a vector bundle modelled on $\mathcal{V}$ with a preferred volume form. Then we naturally get a frame bundle $SL(\mathcal{V})$ for $\mathcal{V}$ with structure group $SL(\mathcal{V})$. Principal connections on this frame bundle are equivalent to volume preserving linear connections on $\mathcal{V}$. 

Proposition
Suppose that $\gamma$ is a principal connection on $SL(\mathcal{V})$ and that $j: G\# \to SL(\mathcal{V})$ is a reduction to the structure group $G\#$ such that $j^*\gamma$ is injective on each tangent space and has values in $\mathfrak{g}_{-1} \oplus \mathfrak{sl}(\mathcal{V})$. Then the $\mathfrak{g}_{-1}$–component of $j^*\gamma$ descends to a soldering form on $G_0 := G\# / SL(\mathcal{V}) \to M$, making it into a first order $G_0$–structure.
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**Proposition**

Suppose that $\gamma$ is a principal connection on $SL(\mathcal{V})$ and that $j : G_0^\# \to SL(\mathcal{V})$ is a reduction to the structure group $G_0^\#$ such that $j^* \gamma$ is injective on each tangent space and has values in $g_{-1} \oplus sl^0(\mathcal{V})$.

Then the $g_{-1}$–component of $j^* \gamma$ descends to a soldering form on $G_0 := G_0^\#/SL^1(\mathcal{V}) \to M$, making it into a first order $G_0$–structure.
In the reduction theorem, we need assumptions on the Lie algebra cohomology group $H^1(g_{-1}, g^\perp)$. This space is a subquotient of $L(g_{-1}, g^\perp)$ and hence decomposes according to homogeneous degrees of linear maps. In particular, we get spaces $H^1(g_{-1}, g^\perp)_i$ and $H^1(g_{-1}, g^\perp)^i$ for all $i$. 
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Although the $g$–representations $g^{\perp}$ quickly become very complicated, Kostant’s version of the BBW–theorem can be used to obtain general information.
Lie algebra cohomology

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Although the $g$–representations $g^\perp$ quickly become very complicated, Kostant’s version of the BBW–theorem can be used to obtain general information.

Lemma

If none of the simple ideals of $g$ is of projective type, then $H^1(g_{-1}, g^\perp)^2 = 0$. If in addition none of the simple ideals is of conformal type, then also $H^1(g_{-1}, g^\perp)_1 = 0$. 
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The reduction theorem

Step 1

Suppose that $\gamma$ is a principal connection on $SL(V)$ and that $j : G_0^\# \to SL(V)$ is a reduction to the structure group $G_0^\#$ such that $j^* \gamma$ is injective and has values in $g_{-1} \oplus sl^0(V)$. If $\gamma$ is flat and $H^1(g_{-1}, g^\perp)_1 = 0$, then there is a reduction $P^\# \hookrightarrow G_0^\#$ to the structure group $P^#$ with the same underlying $G_0$–structure, such that $\tilde{j}^* \gamma$ is injective and has values in $g_{-1} \oplus g_0 \oplus sl^1(V)$. 

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Step 2

Suppose that we have a reduction as in the result of step 1 and that $H^1(\mathfrak{g}_{-1}, \mathfrak{g}_{\perp})_2 = 0$. Then there is a reduction $\mathcal{G} \hookrightarrow P^\#$ to the structure group $P$ with the same underlying $G_0$–structure, such that $\hat{j}^* \gamma$ has values in $\mathfrak{g}$ and defines a flat Cartan connection on $\mathcal{G}$.
We sketch the proof of step 1, the second step is proved similarly.

- Let \( \mathcal{G}_0 \rightarrow M \) be the underlying \( G_0 \)–structure of \( \mathcal{G}_0^\# \). Since \( SL^1(V) \) is a contractible group, there is a global \( G_0 \)–equivariant section \( \sigma : \mathcal{G}_0 \rightarrow \mathcal{G}_0^\# \).
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- Let $G_0 \to M$ be the underlying $G_0$–structure of $G_0^\#$. Since $SL^1(V)$ is a contractible group, there is a global $G_0$–equivariant section $\sigma : G_0 \to G_0^\#$.

- The main aim is to modify $\sigma$ to a section $\hat{\sigma}$ in such a way that $\hat{\sigma}^* j^* \gamma$ has values in $g_{-1} \oplus g_0 \oplus sl^1(V)$. Since this subspace is normalized by $P^\#$ we get a reduction with the required properties for $P^\# := G_0 \times G_0 \, P^\#$. 

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- Let $G_0 \to M$ be the underlying $G_0$–structure of $G_0^{\#}$. Since $SL^1(V)$ is a contractible group, there is a global $G_0$–equivariant section $\sigma : G_0 \to G_0^{\#}$.

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- To modify $\sigma$, we first decompose

\[
\sigma^* j^* \gamma = \gamma_{-1} + (\gamma_0 + \gamma_0^\perp) + (\gamma_1 + \gamma_1^\perp) + \gamma_2^\perp + \ldots
\]

according to the decomposition of $g_{-1} \oplus sl^0(V)$.
Since \(\gamma\) is flat, \(\sigma^*j^*\gamma\) satisfies the Maurer–Cartan equation. Looking at the component in \(sl_{-1}(\mathbb{V}) = g_{-1} \oplus g_{-1}^\perp\), we get for all \(\xi, \eta \in \mathfrak{X}(G_0)\)

\[
0 = d\gamma_{-1}(\xi, \eta) + [\gamma_0(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_0(\eta)]
\]

\[
0 = [\gamma_0^\perp(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_0^\perp(\eta)]
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In a point, $\gamma_0^\perp(\xi) = \varphi(\gamma_{-1}(\xi))$ for some linear map $\varphi : \mathfrak{g}_{-1} \to \mathfrak{g}_0^\perp$, and the second equation shows that $\partial \varphi = 0$. 

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0 = d\gamma_{-1}(\xi, \eta) + [\gamma_0(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_0(\eta)]
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0 = [\gamma_{0}^\perp(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_{0}^\perp(\eta)]
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In a point, $\gamma_{0}^\perp(\xi) = \varphi(\gamma_{-1}(\xi))$ for some linear map $\varphi : \mathfrak{g}_{-1} \to \mathfrak{g}_{0}^\perp$, and the second equation shows that $\partial \varphi = 0$.

Since $H^1(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}^\perp)_1 = 0$, there is an element $Z \in \mathfrak{g}_{1}^\perp$ such that $\varphi = \partial Z$ (easily seen to be unique). A direct computation then shows that $\hat{\sigma}(u) = \sigma(u) \exp(Z(u))$ does the job.
Since \( \gamma \) is flat, \( \sigma^* j^* \gamma \) satisfies the Maurer–Cartan equation. Looking at the component in \( \mathfrak{sl}_1(\mathbb{V}) = \mathfrak{g}_1 \oplus \mathfrak{g}^\perp_1 \), we get for all \( \xi, \eta \in \mathfrak{X}(G_0) \)

\[
0 = d\gamma(-1)(\xi, \eta) + [\gamma_0(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_0(\eta)]
\]

\[
0 = [\gamma_0^\perp(\xi), \gamma_{-1}(\eta)] + [\gamma_{-1}(\xi), \gamma_0^\perp(\eta)]
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In a point, \( \gamma_0^\perp(\xi) = \varphi(\gamma_{-1}(\xi)) \) for some linear map \( \varphi : \mathfrak{g}_{-1} \to \mathfrak{g}_0^\perp \), and the second equation shows that \( \partial \varphi = 0 \).

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The further steps are similar, and the first line in the equation shows that in the end one obtains a flat Cartan connection.
Structure

1. The basic setup
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Assume that $G$ is complex, none of the simple ideals of $\mathfrak{g}$ is of projective or conformal type and that $\mathbb{V}$ is a complex representation such that $\dim(\mathbb{V}_N) = 1$. Then $P$ is the stabilizer of the line $\mathbb{V}_N$ in $G$, and the representation induces an embedding of the generalized flag variety $G/P$ into the projectivization $\mathbb{P}\mathbb{V}$. Via $T(G/P) \cong G \times_P \mathfrak{g}_{-1}$, the kernel of the Fubini form $F_{2,2}$ for this embedding corresponds to a cone in $\mathfrak{g}_{-1}$, and $G_0 \subset GL(\mathfrak{g}_{-1})$ is the subgroup of maps preserving this cone.
Assume that $G$ is complex, none of the simple ideals of $\mathfrak{g}$ is of projective or conformal type and that $\mathbb{V}$ is a complex representation such that $\dim(\mathbb{V}_N) = 1$. Then $P$ is the stabilizer of the line $\mathbb{V}_N$ in $G$, and the representation induces an embedding of the generalized flag variety $G/P$ into the projectivization $\mathbb{P}\mathbb{V}$. Via $T(G/P) \cong G \times_P \mathfrak{g}_{-1}$, the kernel of the Fubini form $F_{2,2}$ for this embedding corresponds to a cone in $\mathfrak{g}_{-1}$, and $G_0 \subset GL(\mathfrak{g}_{-1})$ is the subgroup of maps preserving this cone.

Suppose that $X \subset \mathbb{P}\mathbb{V}$ is a subvariety of dimension $\dim(G/P)$ which, in a generic point $x$, has the same Fubini form $F_{2,2}$ as $G/P$. Restrict the canonical principal bundle $p : SL(\mathbb{V}) \to \mathbb{P}\mathbb{V}$ to $X$, and let $\omega$ be the Maurer–Cartan form of $SL(\mathbb{V})$. Consider the set of points in this restrictions in which $\omega$ has values in $\mathfrak{g}_{-1} \oplus \mathfrak{sl}^0(\mathbb{V})$. 
For each such point $g$ we get an induced isomorphism $T_{p(g)}X 	o \mathfrak{g}_{-1}$, and we further restrict to those $g$, for which this isomorphism maps the kernel of the Fubini form $F_{2,2}$ to the distinguished cone. Multiplication from the right defines an action of $G_0^\#$ on this subset, which is easily seen to be free and transitive on each fiber.
For each such point $g$ we get an induced isomorphism $T_{p(g)}X \to \mathfrak{g}_{-1}$, and we further restrict to those $g$, for which this isomorphism maps the kernel of the Fubini form $F_{2,2}$ to the distinguished cone. Multiplication from the right defines an action of $G_0^#$ on this subset, which is easily seen to be free and transitive on each fiber.

Hence we get a principal bundle $G_0^#$ over a dense open subset of $X$, which defines a reduction to the structure group $G_0^#$. Denoting by $\tilde{P}$ the stabilizer of $V_N$ in $SL(V)$, the extension $\tilde{E} := SL(V) \times_{\tilde{P}} SL(V) \to \mathbb{P}V$ carries a flat principal connection induced by $\omega$. Now $G_0^# \hookrightarrow \tilde{E}$ satisfies the assumptions of the reduction theorem, so we get a reduction to $P$ for which $\omega$ pulls back to a flat Cartan connection. This gives a local isomorphism to (the natural embedding of) $G/P$. 
Let $p_0 : G_0 \to M$ be a first order $G_0$–structure. As discussed before, this canonically extends to a normal Cartan geometry $(G \to M, \omega)$ of type $(G, P)$. 
Let \( p_0 : G_0 \to M \) be a first order \( G_0 \)-structure. As discussed before, this canonically extends to a normal Cartan geometry \((G \to M, \omega)\) of type \((G, P)\).

The representation \( V \) gives rise to natural vector bundle \( V := G \times_P \mathbb{V} \to M \). Since \( V \) is the restriction of a representation of \( G \), the Cartan connection \( \omega \) induces a linear connection \( \nabla^V \) on \( V \). These are the so-called *tractor bundles* and *tractor connections*. 
Let $p_0 : G_0 \to M$ be a first order $G_0$–structure. As discussed before, this canonically extends to a normal Cartan geometry $(G \to M, \omega)$ of type $(G, P)$.

The representation $\nabla$ gives rise to natural vector bundle $\mathcal{V} := G \times_P \nabla \to M$. Since $\mathcal{V}$ is the restriction of a representation of $G$, the Cartan connection $\omega$ induces a linear connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}$. These are the so–called tractor bundles and tractor connections.

The Kostant codifferential induces natural bundle maps $\partial^* : \Lambda^k T^* M \otimes \mathcal{V} \to \Lambda^{k-1} T^* M \otimes \mathcal{V}$ for all $k$. The subquotients $\ker(\partial^*)/\text{im}(\partial^*)$ turn out to be isomorphic to $G_0 \times G_0 H^k(g_{-1}, \mathcal{V}) =: \mathcal{H}^k$ and the cohomologies are computable using Kostant’s theorem.
The $P$–invariant filtration $\{V^i\}$ of $V$ induces a filtration $\{\mathcal{V}^i\}$ of $\mathcal{V}$ by smooth subbundles. In particular, $\mathcal{V}/\mathcal{V}^1 \cong G_0 \times G_0 \mathcal{V}_0$. The BGG machinery can be used to construct a differential operator $L : \Gamma(\mathcal{V}/\mathcal{V}^1) \to \Gamma(\mathcal{V})$ such that $\partial^* \circ \nabla^\mathcal{V} \circ L = 0$ and hence $\nabla^\mathcal{V} \circ L$ induces an invariant operator $D : \Gamma(\mathcal{V}/\mathcal{V}^1) \to \Gamma(H^1)$, which is always overdetermined. For example, these include all the conformal Killing operators.
The $P$–invariant filtration $\{\mathcal{V}^i\}$ of $\mathcal{V}$ induces a filtration $\{\mathcal{V}^i\}$ of $\mathcal{V}$ by smooth subbundles. In particular, $\mathcal{V}/\mathcal{V}^1 \cong G_0 \times_{G_0} \mathcal{V}_0$. The BGG machinery can be used to construct a differential operator $L : \Gamma(\mathcal{V}/\mathcal{V}^1) \to \Gamma(\mathcal{V})$ such that $\partial^* \circ \nabla^\mathcal{V} \circ L = 0$ and hence $\nabla^\mathcal{V} \circ L$ induces an invariant operator $D : \Gamma(\mathcal{V}/\mathcal{V}^1) \to \Gamma(\mathcal{H}^1)$, which is always overdetermined. For example, these include all the conformal Killing operators.

The construction also implies that the obvious projection and $L$ induce inverse bijections

$$\ker(D) \leftrightarrow \{ s \in \Gamma(\mathcal{V}) : \nabla^\mathcal{V} s \in \Gamma(\text{im}(\partial^*)) \}$$

If $G_0 \to M$ is locally flat, then the connection $\nabla^\mathcal{V}$ is flat and one shows that $\nabla^\mathcal{V} s \in \Gamma(\text{im}(\partial^*))$ is only possible if $\nabla^\mathcal{V} s = 0$. Hence in these cases, the machinery provides a system in closed form which is equivalent to $D(\varphi) = 0$ and $\dim(\ker(D)) = \dim(\mathcal{V})$. 

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Cohomology and PDE's
In the general case, one can use prolongation procedures by [BCEG] or [HSSS] to construct a vector bundle map $C : \mathcal{V} \to T^*M \otimes \mathcal{V}$ such that the linear connection $\hat{\nabla} := \nabla^\mathcal{V} + C$ has the property that $\hat{\nabla} s = 0$ is equivalent to $\nabla^\mathcal{V} s \in \Gamma(\text{im}(\partial^*)).$ Thus, $\dim(\ker(D)) \leq \dim(\mathcal{V})$ always holds.
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By construction, the (volume preserving) frame bundle of $\mathcal{V}$ can be written as $SL(\mathcal{V}) := G \times_P SL(\mathcal{V})$ and the tractor connection $\nabla^\mathcal{V}$ is induced by a principal connection $\gamma$ on $\mathcal{P}$ which under the natural map $G \to SL(\mathcal{V})$ pulls back to $\omega$. Putting $G_0^\# = G \times_P G_0^\#$ and $\mathcal{P}^\# := G \times_P \mathcal{P}^\#$, we get a reductions $\mathcal{P}^\# \hookrightarrow G_0^\# \hookrightarrow SL(\mathcal{V})$. The pullback of $\gamma$ under these has values in $\mathfrak{g}_{-1} \oplus \mathfrak{sl}^0(\mathcal{V})$ and $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{sl}^1(\mathcal{V})$, respectively.
The prolongation procedures are set up in such a way that $C(\mathcal{V}^i)$ is always contained in $T^*M \otimes \mathcal{V}^i$ and for torsion free geometries, it is even contained in $T^*M \otimes \mathcal{V}^{i+1}$. Otherwise put, $C$ can be viewed as a one–form on $M$ with values in $\mathfrak{gl}^0(\mathcal{V})$ respectively $\mathfrak{sl}^1(\mathcal{V})$. 
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The upshot of this is that the principal connection $\hat{\gamma}$ corresponding to $\hat{\nabla} = \nabla^V + C$ always admits a reduction to $G_0^#$ and in the torsion free case even to $P^#$. If $\dim(\ker(D)) = \dim(V)$, then of course the connection $\hat{\nabla}$ must be flat. It is easy to see that replacing $C$ by its tracefree part (which has values in $\mathfrak{sl}^0(V)$) the resulting connection is still flat. If the assumptions of the reduction theorem are satisfied, then we obtain a flat Cartan geometry of type $(G, P)$, whose underlying $G_0$–structure is the same as for $G$, so the original structure must have been flat. Hence we conclude
The basic setup
A reduction theorem
Applications

Theorem

Consider a parabolic geometry \((p : \mathcal{G} \to M, \omega)\) of type \((G, P)\) corresponding to a \(|1|\)–grading of \(\mathfrak{g}\) such that none of the simple ideals of \(\mathfrak{g}\) is contained in \(\mathfrak{g}_0\) or of projective type. For an infinitesimally faithful representation \(\mathcal{V}\) of \(\mathfrak{g}\) assume that the kernel of the corresponding first BGG–operator \(D\) has dimension \(\dim(\mathcal{V})\). If either none of the simple ideals of \(\mathfrak{g}\) is of conformal type or the geometry is torsion free, then \((p : \mathcal{G} \to M, \omega)\) must be locally flat.