Affine holonomies and parabolic geometries

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- More recently, it was notice that almost all the possible non–Riemannian affine holonomies can be produced by two constructions starting from the canonical parabolic geometries on generalized flag manifolds.
- Based on work of M. Cahen and L. Schwachhöfer (arXiv:math/0402221), and myself (arXiv:0804.1838, PAMS **137** (2009), 1073-1080).

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2. Holonomies related to $|1|$–gradings
3. Holonomies related to contact gradings

Structure

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Definition of Holonomy

Let M be a smooth manifold of dimension n and let ∇ be a linear connection on the tangent bundle TM. For a point $x \in M$, the *holonomy of* ∇ *at* x is the closed subgroup of $GL(T_xM)$ generated by parallel transports with respect to ∇ along piecewise smooth closed loops based at x.

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Identifying $GL(T_xM)$ with $GL(n, \mathbb{R})$, the subgroups obtained from different points in M are all conjugate. Thus one may view the holonomy group $\mathrm{Hol}(\nabla)$ of ∇ as a subgroup of $GL(n, \mathbb{R})$ defined up to conjugation, or equivalently as a Lie group endowed with a representation on $\mathbb{R}^n$ defined up to isomorphism.

In many cases it is desireable to eleminate the influence of the fundamental group of M. To do this, one passes to the *restricted holonomy group* $\mathrm{Hol}_0(\nabla)$ is the group obtained from parallel transports along null–homotopic loops. It is the connected component of the identity of $\mathrm{Hol}(\nabla)$.

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By the Ambrose–Singer theorem, the holonomy Lie algebra at x is generated by parallely transporting the values of the curvature and its iterated covariant derivatives at arbitrary points of M to x.

Geometric meaning of the holonomy group

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- In this sense $\mathrm{Hol}(\nabla)$ can be thought of as the smallest structure group of a geometric structure compatible with ∇.
- The various compatible geometric structures can be used to define subclasses of holonomy groups, e.g.(pseudo–)Riemannian and symplectic holonomies (compatible with a symplectic form).

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Without the first condition, the problem becomes meaningless and the second is rather harmless. Irreducibility is not a real restriction for Riemannian holonomies by the de–Rham theorem, but in general it is a purely technical assumption.

Berger's lists

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Over many years many people, in particular R. Bryant, corrected an extend Berger's second list, adding (infinitely many) so–called exotic holonomies. Proving existence of holonomy groups in many cases needed lots of effort and case–by–case considerations. The problem was finally solved by S. Merkulov and L. Schwachhöfer in 2001. Similarities of the final list with other classification results were noted, but not used systematically.

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- This exhaust all entries except $\mathfrak{sp}(2n)$ on Berger's original second list, as well as some exotic holonomies.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there are natural subgroups $G_0 \subset P \subset G$ corresponding to the Lie subalgebras $\mathfrak{g}_0$, and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, respectively. Moreover, there is an Abelian normal vector subgroup $P_+ \subset P$ such that $P = G_0 \ltimes P_+$. The quotient $M := G/P$ is a compact Hermitian symmetric space, and $E := G/P_+ \to M$ is a first order structure with structure group G_0, with TM and T^*M corresponding to the representations $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ of G_0.

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The theory of Weyl structures for parabolic geometries leads to the class of *Weyl connections* on the principal bundle E. Moreover, there always is the subclass of *exact Weyl connections* which come from a further reductions with structure group the semisimple part of G_0.

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$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta - \{\{\Upsilon, \xi\}, \eta\}$$
$$\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) + \nabla_\xi \Upsilon + \tfrac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}$$

First one constructs an exact Weyl connection ∇ on M, such that locally around $o = eP \in M$, the Rho tensor vanishes identically. Choosing a change $\Upsilon \in \Omega^1(M)$ with vanishing k–jet in o, we conclude that the curvature $\hat{R}$ of has vanishing $(k-1)$–jet in o and that

$$\hat{\nabla}^k \hat{R}(o) = (\mathrm{id} \otimes \partial)(\nabla^{k+1}\Upsilon(o)),$$

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Now $\nabla^{k+1}\Upsilon(o)$ can be prescribed arbitrarily, and with a bit of Lie theory one shows that for $k = 5$, one can find a find a one form (respectively an exact one–form) Υ such that that the values of $\hat{\nabla}^k \hat{R}(o)$ exhaust all of $\mathfrak{g}_0$ (respectively its semisimple part).

Main result for |1|–gradings

Since the values of the covariant derivatives of the curvature are well known to lie in the holonomy Lie algebra, we conclude:

Theorem

For any |1|–graded Lie algebra $\mathfrak{g}$, the Lie algebra $\mathfrak{g}_0$ and its semisimple part both are realized as the holonomy Lie algebra of non–locally symmetric affine connections on the compact Hermitian symmetric space G/P.

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Contact gradings

A contact grading of a simple Lie algebra $\mathfrak{g}$ is a grading of the form $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_{-2}$ has dimension one and the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non–degenerate.

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- The list of all contact gradings is also well known from its relation to the classification of quaternionic symmetric spaces.

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- As a module over $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{h}$, the subspace $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ decomposes as $\mathbb{R}^2 \otimes V$ and V carries a natural, $\mathfrak{h}$–invariant symplectic form.

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The algebras $\mathfrak{h} \subset \mathfrak{sp}(V)$ obtained in this way are called *special symplectic* subalgebras.

special symplectic connections

Using the structure theory of simple Lie algebras, one can show directly that the Ricci–type contraction induces a surjection from the module $K(\mathfrak{h})$ of formal curvatures onto $\mathfrak{h}$. This leads to a subspace $\mathcal{R}(\mathfrak{h}) \subset K(\mathfrak{h})$ which is isomorphic to $\mathfrak{h}$ and complementary to Ricci flat formal curvatures.

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Definition (Cahen–Schwachhöfer)

Let (M, ω) be a symplectic manifold. A *special symplectic connection* on M is a torsion free symplectic connection ∇ on TM whose curvature in each point is contained in $\mathcal{R}(\mathfrak{h})$ for some special symplectic subalgebra $\mathfrak{h}$.

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Using the classification results, this definition can be turned into something much more concrete:

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Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $P \subset G$ a parabolic subgroup corresponding to $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then G/P carries a natural geometric structure, which includes a contact subbundle $H \subset T(G/P)$ coming from $\mathfrak{g}_{-1}$. For this parabolic geometry, there also is the notion of Weyl structures and Weyl connections. The construction of Cahen–Schwachhöfer uses this setting (but not the language of Weyl structures).

For any $A \in \mathfrak{g}$, the right invariant vector field R_A generates a one–parameter group of automorphisms of the parabolic geometry on G/P. We assume that in some point, the image of R_A in $\mathfrak{X}(G/P)$ is transverse to the contact subbundle. This is satisfied for generic elements A, and then it holds on an open subset $U \subset G/P$. On U we obtain from A

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The flow lines of ξ_A foliate U and for any open subset $V \subset U$ for which the space of leaves is a smooth manifold, Cahen–Schwachhöfer prove:

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The results of Cahen–Schwachhöfer actually go much further. They prove that up to local isomorphism any special symplectic connection is obtained in this way. This implies that special symplectic connections are automatically real analytic, and it leads to a description of the moduli space of all such connections, as well as a complete classification of homogeneous examples on simply connected manifolds.