A relative version of Kostant's theorem

Andreas Čap[1]

University of Vienna
Faculty of Mathematics

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- This talk reports on joint work with Vladimir Souček (Prague).
- We first review the statement Kostant's theorem as well as the algebraic structures used for its proof and their role in the construction of Bernstein–Gelfand–Gelfand sequences for parabolic geometries.
- The main part of the talk will be the description of relative versions of these tools (associated to two nested parabolic subalgebras rather than one parabolic subalgebra) and of Kostant's theorem.
- Apart from providing the appropriate setup for a relative Version of the BGG–machinery (not discussed in detail), this also gives new insight into the absolute case. One obtains a new description of the Hasse–diagram of a non–maximal parabolic and (in regular infinitesimal character) a relation between absolute and relative Lie algebra (co)homology.

Contents

Let $\mathfrak{g}$ be a semi–simple Lie algebra. Then the choice of a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ can be viewed as a "coarser version" of the Cartan–decomposition of $\mathfrak{g}$ into a Cartan subalgebra and root spaces.

The subalgebra $\mathfrak{q}$ is equivalent to a $|k|$–grading of $\mathfrak{g}$, i.e. a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the positive part $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is generated by $\mathfrak{g}_1$. The subalgebra $\mathfrak{q}$ is the non–negative part of this grading.

Writing the grading as $\mathfrak{g} = \mathfrak{q}_- \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_+$, it follows from the grading property that $\mathfrak{q}_\pm$ are nilpotent subalgebras of $\mathfrak{g}$, which are graded modules over $\mathfrak{q}_0$ under the restriction of the adjoint action.

Kostant's theorem

Any representation $\mathbb{V}$ of $\mathfrak{g}$ is a representation of $\mathfrak{q}_0$ and of $\mathfrak{q}_-$ by restriction. Hence the standard complex $(C^*(\mathfrak{q}_-, \mathbb{V}), \partial)$ computing the Lie algebra cohomology $H^*(\mathfrak{q}_-, \mathbb{V})$ is a complex of $\mathfrak{q}_0$–modules and $\mathfrak{q}_0$–equivariant maps, so the cohomologies are $\mathfrak{q}_0$–modules.

For complex $\mathfrak{g}$ and irreducible $\mathbb{V}$, Kostant's theorem describes $H^*(\mathfrak{q}_-, \mathbb{V})$ as a representation of $\mathfrak{q}_0$ in terms of orbits of weights under an action of a subset $W^{\mathfrak{q}}$ of the Weyl group of $\mathfrak{g}$.

This may sound like a strange result, but there are important consequences:

- Even the version for the Borel subalgebra leads in a few lines to a proof of the Weyl character formula.
- Using the Peter–Weyl theorem, Kostant's result gives an alternative proof of the Bott–Borel–Weil theorem.

The relation to parabolic geometries

Viewing an irreducible representation $\mathbb{W}$ of $\mathfrak{q}_0$ as a representation of $\mathfrak{q}$, one obtains the *generalized Verma module* $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{W}^*$. These are infinite dimensional $\mathfrak{g}$–modules admitting a central character, which is a basic invariant. Combining Harish–Chandra's theorem on central character with Kostant's theorem, one gets

Theorem

$H^*(\mathfrak{q}_-, \mathbb{V})$ is a direct sum of different irreducible representations of $\mathfrak{q}_0$. The representations in this sum are exactly those, which lead to generalized Verma modules with the same central character as $\mathbb{V}$.

Hence the bundles induced by the representations in $H^*(\mathfrak{q}_-, \mathbb{V})$ are natural candidates for existence of invariant differential operators on the parabolic geometry determined by $(\mathfrak{g}, \mathfrak{q})$.

The proof of Kostant's theorem relies on the observation that the Killing form of $\mathfrak{g}$ restricts to a duality between $\mathfrak{q}_-$ and $\mathfrak{q}_+$. Hence $C^k(\mathfrak{q}_-,\mathbb{V}) \cong \Lambda^k\mathfrak{q}_+ \otimes \mathbb{V}$, and in addition to the Lie algebra cohomology differential ∂, there also is a Lie algebra homology differential $\partial^* : C^k(\mathfrak{q}_-,\mathbb{V}) \to C^{k-1}(\mathfrak{q}_-,\mathbb{V})$.

Kostant proved that ∂^* and ∂ are adjoint with respect to a certain inner product. For $\Box = \partial\partial^* + \partial^*\partial$ acting on C^k one then obtains

$$\ker(\Box) \cong H^k(\mathfrak{q}_-,\mathbb{V}) \cong H_k(\mathfrak{q}_+,\mathbb{V}),$$

and this subspace can be analyzed using representation theory.

In applications to parabolic geometries, one focuses on the homology interpretation, which naturally consists of $\mathfrak{q}$–modules and $\mathfrak{q}$–equivariant maps. The full algebraic Hodge theory can be brought into the game by passing to the associated graded with respect to a natural $\mathfrak{q}$–invariant filtration.

To obtain the relative version, we first need a second parabolic subalgebra $\mathfrak{p}$ sitting between $\mathfrak{q}$ and $\mathfrak{g}$, with $\mathfrak{p} = \mathfrak{g}$ corresponding to the absolute case. Then $\mathfrak{p}_+$ is the annihilator of $\mathfrak{p}$ under the Killing form, whence $\mathfrak{p}_+ \subset \mathfrak{q}_+$. One obtains a decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{p}_- \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_-) \oplus \mathfrak{q}_0 \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_+) \oplus \mathfrak{p}_+$$

in a way compatible with the Killing form. From the purely algebraic point of view, the relative version of Kostant's theorem studies the decomposition of the reductive algebra $\mathfrak{p}_0$ given by the three middle summands.

For the use in geometry, a $\mathfrak{q}$–invariant formulation is more convenient: We have $\mathfrak{p}_+ \subset \mathfrak{q}_+$ and since $\mathfrak{p}_+$ is an ideal in $\mathfrak{p}$, we can form the quotient $\mathfrak{q}_+/\mathfrak{p}_+$. The Killing form induces a duality between this and the $\mathfrak{q}$–submodule $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$.

Now let $\mathbb{V}$ be a completely reducible representation of $\mathfrak{p}$. Then this is a representation of $\mathfrak{q}_+$ by restriction and by complete reducibility, $\mathfrak{p}_+$ acts trivially. Hence there is the standard complex for Lie algebra homology of $\mathfrak{q}_+/\mathfrak{p}_+$ with coefficients in $\mathbb{V}$, consisting of

- $C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) = \Lambda^k(\mathfrak{q}_+/\mathfrak{p}_+) \otimes \mathbb{V}$
- $\partial^*_\rho : C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) \to C_{k-1}(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ defined by

$$\partial^*_\rho(Z_1 \wedge \dots \wedge Z_k \otimes v) := \sum_i (-1)^i Z_1 \wedge \cdots \widehat{Z_i} \cdots \wedge Z_k \otimes Z_i \cdot v$$
$$+ \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \cdots \widehat{Z_i} \cdots \widehat{Z_j} \cdots \wedge Z_k \otimes v,$$

These are $\mathfrak{q}$–equivariant maps between $\mathfrak{q}$–modules, so the homology groups $H_k(\mathfrak{p}_+/\mathfrak{q}_+, \mathbb{V})$ are naturally representations of $\mathfrak{q}$. It turns out that they are completely reducible, so $\mathfrak{q}_+$ acts trivially, and it suffices to understand the $\mathfrak{q}_0$–module structure.

We have already mentioned that the Killing form of $\mathfrak{g}$ induces a duality between $\mathfrak{q}_+/\mathfrak{p}_+$ and $\mathfrak{p}/\mathfrak{q}$. The associated graded of $\mathfrak{p}/\mathfrak{q}$ can be identified with the nilpotent Lie subalgebra $\mathfrak{p}_0 \cap \mathfrak{q}_-$ of $\mathfrak{p}_0$, which also acts on $\mathbb{V}$ by restriction. This leads to

- $C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) \cong L(\Lambda^k(\mathfrak{p}_0 \cap \mathfrak{q}_-)^*, \mathbb{V})$
- $\partial_\rho : C_k(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}) \to C_{k+1}(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$, defined by

$$\begin{aligned}\partial_\rho\varphi(X_0, \dots, X_k) := \textstyle\sum_{i=0}^k (-1)^i X^i \cdot \varphi(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ + \textstyle\sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k),\end{aligned}$$

Similarly to the classical case, one proves that ∂_ρ^* and ∂_ρ are adjoint, so one introduces $\Box_\rho = \partial_\rho^*\partial_\rho + \partial_\rho\partial_\rho^*$ and obtains a Hodge–decomposition.

As in the absolute case, this implies that $\ker(\Box_\rho)$ is isomorphic to the homology as a $\mathfrak{q}_0$–module. The action of $\Box_\rho$ on $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ can be described in representation theory terms. In in the complex case, this can then be analyzed in terms of weights.
Denoting by $\delta_\mathfrak{p}$ half the sum of those positive roots whose root spaces are contained in $\mathfrak{p}_0$, this implies

Proposition

If $-\lambda$ is the lowest weight of $\mathbb{V}$, then $\ker(\Box_\rho)$ is the direct sum of those isotypical components of $C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ whose lowest weight $-\nu$ has the property that $\|\nu + \delta_\mathfrak{p}\| = \|\lambda + \delta_\mathfrak{p}\|$, where the norm is induced by the Killing form of $\mathfrak{g}$.

Let W be the Weyl group of $\mathfrak{g}$ and for $w \in W$ define $\Phi_w := \{\alpha \in \Delta^+ : w^{-1}(\alpha) \in -\Delta^+\}$. According to the decomposition of $\mathfrak{g}$, we can also decompose Δ^+, and then define several subsets of W via properties of the sets Φ_w.

The Hasse-diagram of $\mathfrak{q}$ (which occurs in the absolute version of Kostant's theorem) is $W^{\mathfrak{q}} := \{w : \Phi_w \subset \Delta^+(\mathfrak{q}_+)\}$, and likewise one defines $W^{\mathfrak{p}}$. On the other hand, $W_{\mathfrak{p}} := \{w : \Phi_w \subset \Delta^+(\mathfrak{p}_0)\}$ is the Weyl group of (the semisimple part of) $\mathfrak{p}_0$, and likewise for $W_{\mathfrak{q}}$.

Definition

The *relative Hasse diagram* associated to $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ is $W^{\mathfrak{q}}_{\mathfrak{p}} = W^{\mathfrak{q}} \cap W_{\mathfrak{p}} = \{w \in W : \Phi_w \subset \Delta^+(\mathfrak{p}_0 \cap \mathfrak{q}_+)\}$.

These have similar properties as the usual Hasse–diagrams. In particular

- $W^{\mathfrak{q}}_{\mathfrak{p}}$ can be determined by computing the orbit of an appropriate weight under $W_{\mathfrak{p}}$.
- If λ is a $\mathfrak{p}$–dominant weight and $w \in W^{\mathfrak{q}}_{\mathfrak{p}}$, then $w(\lambda)$ is $\mathfrak{q}$–dominant.

Theorem

If $\mathbb{V}$ has lowest weight $-\lambda$, then $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ is the direct sum of one copy of each of the irreducible representations of $\mathfrak{q}_0$ with lowest weight $-(w(\lambda+\delta)-\delta)$ with $w \in W_\mathfrak{p}^\mathfrak{q}$, and such a component occurs in degree $\ell(w)$.

Example: For $\mathfrak{p} = \times\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$, $\mathfrak{q} = \times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\circ$, one gets $W_\mathfrak{p}^\mathfrak{q} = \{e, \sigma_2, \sigma_2\sigma_3\}$ with elements of length 0, 1, and 2.

For a weight $\lambda = \overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!-\!\!\!-\!\!\overset{c}{\circ}$ with $a, b, c \in \mathbb{Z}$, to be $\mathfrak{p}$–dominant, we need $b, c > 0$. The cohomology in degree zero then corresponds to $\overset{a}{\times}\!\!-\!\!\!-\!\!\overset{b}{\times}\!\!-\!\!\!-\!\!\overset{c}{\circ}$, while in degree one and two, we obtain $\overset{a+b+1}{\times}\!\!-\!\!\!-\!\!\overset{-b-2}{\times}\!\!-\!\!\!-\!\!\overset{b+c+1}{\circ}$ $\overset{a+b+c+2}{\times}\!\!-\!\!\!-\!\!\overset{-b-c-3}{\times}\!\!-\!\!\!-\!\!\overset{b}{\circ}$.

For $a = -1$, $a = -b-2$ and $a = -b-c-3$ one obtains a pattern of representations for which the corresponding generalized Verma modules have the same singular infinitesimal character.

In the case of regular infinitesimal character, the representations $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ also occur in $H_*(\mathfrak{q}_+, \tilde{\mathbb{V}})$ for some representation $\tilde{\mathbb{V}}$ of $\mathfrak{g}$. Now one proves:

Theorem

(1) The multiplication in W induces a bijection $W^{\mathfrak{q}}_{\mathfrak{p}} \times W^{\mathfrak{p}} \to W^{\mathfrak{q}}$ such that $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.
(2) For an irreducible representation $\tilde{\mathbb{V}}$ of $\mathfrak{g}$, one has $H_k(\mathfrak{q}_+, \tilde{\mathbb{V}}) = \oplus_{i+j=k} H_i(\mathfrak{q}_+/\mathfrak{p}_+, H_j(\mathfrak{p}_+, \tilde{\mathbb{V}}))$.

Part (1) exhibits a product structure of $W^{\mathfrak{q}}$ which was not known before. Moreover, to determine the affine Weyl orbit of a weight under $W^{\mathfrak{q}}$, one can first determine the affine orbit under $W^{\mathfrak{p}}$ and then the orbits of each of the resulting weights under $W^{\mathfrak{q}}_{\mathfrak{p}}$.
For (2) one has to observe that $H_*(\mathfrak{p}_+, \tilde{\mathbb{V}})$ is completely reducible.

For the algebraic considerations one is only interest in the $\mathfrak{q}_0$–module structures, and the isomorphism $H_k(\mathfrak{q}_+,\tilde{\mathbb{V}})=\oplus_{i+j=k}H_i(\mathfrak{q}_+/\mathfrak{p}_+,H_j(\mathfrak{p}_+,\tilde{\mathbb{V}}))$ is proved by coincidence of irreducible components. For applications to geometry, additional considerations are necessary:

- For any irreducible representation $\mathbb{V}$ of $\mathfrak{p}$, a $\mathfrak{q}$–invariant version of the full Hodge theory is available on the associated graded of $C_*(\mathfrak{q}_+/\mathfrak{p}_+,\mathbb{V})$.
- For a $\mathfrak{g}$–irreducible representation $\tilde{\mathbb{V}}$, there is a $\mathfrak{q}$–invariant filtration $\mathcal{F}^\ell$ of $C_*(\mathfrak{q}_+,\tilde{\mathbb{V}})$ such that the restriction of the projection to homology to $\mathcal{F}^\ell\cap\ker(\partial^*)$ induces an isomorphism of $\mathfrak{q}$–modules $\frac{\mathcal{F}^\ell\cap\ker(\partial^*)}{\mathcal{F}^{\ell+1}\cap\ker(\partial^*)+\mathcal{F}^\ell\cap\operatorname{im}(\partial^*)}\to H_*(\mathfrak{q}_+/\mathfrak{p}_+,H_\ell(\mathfrak{p}_+,\tilde{\mathbb{V}}))$.