The (relative) BGG machinery
lecture 1

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- This series of lectures is based on recent joint work with Vladimir Souček (Prague), in which we found a relative version of BGG sequences and a simpler construction for the original sequences.
- In the first lecture I will first discuss background on invariant differential operators, using conformally invariant operators as a guideline.
- Next, I will talk about Cartan geometries and geometric objects and differential operators associated to them.
- In the second lecture, I will describe a construction of BGG sequences based on these ideas. The third lecture will discuss the relative analog.

Contents

"Invariant differential operator" should be understood as (linear) differential operator intrinsically associated to a given geometric structure. Such operators have to act on "geometric objects" which are intrisically associated to the given structure. Let us start discuss this for Riemannian (spin) manifolds, for which everything is well known:

- Geometric objects are built up from tensors and spinors. Some of them exist on general manifolds, some (like trace–free tensors) depend on the Riemannian metric g.
- All these objects can be differentiated using the Levi–Civita connection ∇ of g, which can be iterated to obtain higher derivatives.
- This can be mixed with expressions built up from the Riemann curvature and its iterated covariant derivatives, then one applies tensorial operations.

This can be formulated more systematically via the orthonormal frame bundle or its Spin extension. This is a principal bundle $P \to M$ with structure group $H = O(n)$ (or $SO(n)$ in the oriented case or $Spin(n)$ in the spin case). This bundle is endowed with a canonical principal connection, the Levi–Civita connection. Thus

- Representations of H give rise to natural (associated) bundles, equivariant maps between such representations induce natural vector bundle homomorphisms.
- Via induced connections, all these geometric objects can be differentiated covariantly. Since T^*M is associated to H, this again produces sections of an associated bundle. Hence iterated derivatives are no problem.

Since there are so many invariant operators in the Riemannian setting, it is a natural idea to look at stronger invariance properties to identify particularly robust operators. The simplest idea in this direction is conformal invariance.

Recall that two metrics g and $\hat{g}$ on a manifold M are *conformally equivalent* if and only if $\hat{g} = f^2 g$ for a smooth, non–vanishing function f on M. The basic definition of a conformally invariant operator then is

Given g, form a Riemannian–invariant operator D in the form described above (using ∇, R, vol_g and tensorial operations). The requirement then is that replacing all quantities associated to g by those corresponding to $\hat{g}$, the resulting operator $\hat{D}$ should be the same.

- Trying to construct such operators directly works well in very low orders, but then quickly gets out of hand.
- One has to involve density bundles ("conformal weights"). A choice of metric identifies densities with smooth functions, but changing the metrics changes the function.
- Several well known operators (conformal Killing operators, twistor operator on spinors) are conformally invariant.

The fact that finding conformally invariant operators is difficult, is explained by a surprising relation to representation theory. Since the round metric on the sphere S^n is homogeneous, also the group G of conformal isometries acts transitively, so $S^n \cong G/P$ for some subgroup $P \subset G$.

- $G \cong O(n+1,1)$ is semisimple
- $P \subset G$ is the stabilizer of a null line in the standard representation of G. This is a parabolic subgroup of G.
- P is a semi–direct product of $CO(n)$ and a nilpotent normal subgroup $P_+ \cong \mathbb{R}^{n*}$. The second factor corresponds to transformations fixing the base point to first order.
- The natural bundles we discussed before are exactly the homogeneous bundles induced by completely reducible representations of P on which P_+ acts trivially.

Hence the space of sections of any of the natural bundles over S^n carries a representation of G (principal series representations). Any conformally invariant operator defines an intertwining operator between the corresponding representations. These can be analyzed via homomorphisms of generalized Verma modules, and existence of such a homomorphism requires the modules to have the same infinitesimal character. This leads to:

- Starting from a fixed bundle E (with fixed conformal weight), there are only finitely many bundles F, for which conformally invariant operators $\Gamma(E) \to \Gamma(F)$ or $\Gamma(F) \to \Gamma(E)$ can exist.
- The representations inducing these bundle can be determined explicitly and algorithmically (affine action of the Weyl group).
- The orders of the potential operators in the pattern can be determined from the representations and can be arbitrarily high (for appropriate E).

More explicitly, in even dimensions, conformally invariant operators can only occur in patterns, which generically have the form of the de–Rham complex.

$$\begin{tikzcd}[column sep=small]
& & & & & \bullet \arrow[dr] & & & & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \dots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[r] & \bullet \arrow[r] & \dots \arrow[r] & \bullet \arrow[r] & \bullet \\
& & & & & \bullet \arrow[ur] & & & & &
\end{tikzcd}$$

- Each bundle E occurs in exaclty on such pattern.
- The place in which it occurs, the other bundles in the pattern, and the orders of the potential operators in the pattern can all be computed algorithmically in representation theory terms.

So while it would be highly desireable to work in a conformally invariant way from the beginning, this is impossible without involving more general geometric objects.

A classical construction of E. Cartan from 1923 provides a description of a manifold endowed with a conformal structure, which is formally looks like the homogeneous space $S^n = G/P$.

- The conformal structure is equivalent to a princial bundle with structure group $CO(n) = P/P_+$ enodowed with a soldering form.
- Extend the structure group to obtain a principal P–bundle $\mathcal{G} \to M$.
- This can be canonically endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. This extends the lift of the soldering form and statisfies a normalization condition on its curvature.

The Cartan connection should be thought of as an analog of the left Maurer–Cartan form of G. It trivializes the tangent bundle $T\mathcal{G}$ in a P–equivariant way and reproduces the generators of fundamental vector fields.

The identification of $P/P_+ \cong CO(n)$ is obtained via the action of P on $\mathfrak{g}/\mathfrak{p} \cong \mathbb{R}^n$ induced by the adjoint action. This allows us to view all natural bundles considered so far as associated bundles to $\mathcal{G}$. Sections of $E := \mathcal{G} \times_P \mathbb{W}$ are in bijective correspondence with equivariant smooth functions $\mathcal{G} \to \mathbb{W}$. This shows a way to differentiate such sections:

- For a P–invariant vector field $\xi \in \mathfrak{X}(\mathcal{G})$ and a P–equivariant function $f : \mathcal{G} \to \mathbb{W}$, also $\xi \cdot f$ is P–equivariant.
- Via ω, P–invariant vector fields are in bijective correspondence with P–equivariant functions $\mathcal{G} \to \mathfrak{g}$, and hence with sections of $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$.
- Hence we obtain $D : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$, written as $(s, \sigma) \mapsto D_s\sigma$, which has strong naturality properties.
- One can view $\sigma \mapsto D\sigma$ as an operator $\Gamma(E) \to \Gamma(\mathcal{A}^*M \otimes E)$ and in this form, the operator can be iterated.

The bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ is called the *adjoint tractor bundle*, the operator D is the *fundamental derivative*. In general, *tractor bundles* are natural bundles associated to the restriction of G–representations to P. From the definition, one can deduce several properties of $\mathcal{A}M$.

- The P–invariant subspaces $\mathfrak{p}_+ \subset \mathfrak{p} \subset \mathfrak{g}$ give rise to smooth subbundles $\mathcal{A}^1M \subset \mathcal{A}^0M \subset \mathcal{A}M$ such that $\mathcal{A}M/\mathcal{A}^0M \cong TM$ and $\mathcal{A}^1M \cong T^*M$, and $\mathcal{A}^0M/\mathcal{A}^1M \cong \mathfrak{co}(TM)$.
- The Lie bracket on $\mathfrak{g}$ is P–equivariant, thus making $\mathcal{A}M$ into a bundle of Lie algebras.
- The Killing form of $\mathfrak{g}$ gives rise to a non–degenerate (but indefinite) bundle metric on $\mathcal{A}M$.

Tractor bundles always encode second order information, but a choice of metric gives an explicit identification with a direct sum of tensor–spinor bundles, see A. Waldron's lectures.

Conformal structures are one of the simplest instances of *parabolic geometries*. For the purpose of these lectures, we can view theses as Cartan geometries of type (G, P) with G semisimple and $P \subset G$ a parabolic subgroup. There is a general theory which shows that such geometries equivalently encode underlying structure. This includes examples like

- classical projective structures
- almost quaternionic structures
- hypersurface–type CR structures
- path geometries
- quaternionic contact structutures
- generic distributions of rank k on manifolds of dimension n for $(k, n) = (2, 5)$, $(4, 7)$, $(4, 8)$, $(k, k(k+1)/2)$ (which includes $(3, 6)$ and $(4, 10)$).