Canonical Cartan connections associated to filtered G_0–structures

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- The topic of this talk is a recent construction of canonical Cartan connections associated to filtered G_0–structures, which is available as arXiv:1707.05627.
- The construction seems to cover essentially the same filtered G_0–structures as Morimoto's 1993 paper. It allows a broader class of homogeneous models, seems to be easier to apply, and leads to stronger uniqueness results (independent of the construction).
- I will concentrate on explaining what has to be verified in order to apply the construction to a specific structure. The construction itself can then be used as a "black box".
- A substantial example for an application is my joint article arXiv:1709.01130 with B. Doubrov and D. The, which deals with filtered G_0–structures equivalent to (systems of) ODEs of higher order. All verifications for this family of cases are carried out in that article.

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G_0–structures and Cartan connections

Recall that for a closed subgroup $G_0 \subset GL(n, \mathbb{R})$, the notion of a G_0–structure on n–manifolds is defined as a reduction of structure group of the linear frame bundle $\mathcal{P}M \to M$ to G_0. Viewing G_0 as defining a "structure" on $\mathbb{R}^n$, one simply endows each T_xM with such a structure (depending smoothly on x).

Associating a canonical Cartan connection means describing such structures as "curved analogs" of a *homogeneous model* G/P. Here G/P is described by the principal P–bundle $G \to G/P$ and the left Maurer Cartan form that P–equivariantly trivializes TG.

To get a candidate for a homogeneous model, one needs a G–invariant G_0–structure on G/P, which is not preserved by a larger group than G. Having that, one needs a normalization condition to select a canonical Cartan connection among the ones inducing the given G_0–structure.

filtered manifolds

Simple examples of G_0–structures are provided by distinguished subbundles in the tangent bundle. For such structures, the basic invariant ("intrinsic torsion") comes from tensorial maps induced by the Lie bracket of vector fields. Examples like contact structure show that it is important to impose conditions on these invariants.

A *filtered manifold* is a smooth manifold M together with a filtration $TM = T^{-\mu}M \supset \cdots \supset T^{-1}M$ by smooth subbundles, such that $[\Gamma(T^iM), \Gamma(T^jM)] \subset \Gamma(T^{i+j}M)$ for all i, j. The Lie bracket of vector fields then induces a tensorial map on the associated graded $\mathrm{gr}(TM)$, thus making each $\mathrm{gr}(T_xM)$ into a nilpotent graded Lie algebra, the *symbol algebra* at x.

For a nilpotent graded Lie algebra $\mathfrak{m} = \oplus_{i=-\mu}^{-1}\mathfrak{m}_i$, one says that a filtered manifold $(M, \{T^iM\})$ is *regular of type* $\mathfrak{m}$ if the symbol algebras form a locally trivial bundle of Lie algebras modelled on $\mathfrak{m}$.

Filtered G_0–structures

For $\mathfrak{m}$ as above, the automorphisms preserving the grading form a Lie group $\mathrm{Aut}_{gr}(\mathfrak{m})$. For a filtered manifold $(M, \{T^iM\})$, which is regular of type $\mathfrak{m}$, there is a natural frame bundle for $\mathrm{gr}(TM)$ with structure group $\mathrm{Aut}_{gr}(\mathfrak{m})$. Now consider a Lie group G_0 endowed with an infinitesimally injective homomorphism $G_0 \to \mathrm{Aut}_{gr}(\mathfrak{m})$.

A *filtered G_0–structure* on $(M, \{T^iM\})$ is a reduction of structure group of the natural frame bundle of $\mathrm{gr}(TM)$ to the group G_0. If $\mathfrak{m}$ is generated by $\mathfrak{m}_{-1}$ as a Lie algebra, then the filtration is determined by $T^{-1}M \subset TM$, which is a bracket generating distribution and the reduction can be expressed in terms of $T^{-1}M$.

Examples include equivalent descriptions of parabolic geometries (e.g. CR structures), many types of generic distributions, (systems of) ODEs, and standard G_0–structures (trivial filtration).

There are some subtleties in the notion of a filtered G_0–structure, which concern popular structures. Let us consider the case of a corank 1 subbundle $H \subset TM$ for odd–dimensional M. This defines a G_0–structure and its intrinsic torsion is the tensorial map $\mathcal{L} : H \times H \to TM/H$ induced by the Lie bracket.

- H is contact if each $\mathcal{L}_x$ is non–degenerate.
- Equivalently, (M, H) is a filtered manifold which is regular of type $\mathfrak{m}$ for a Heisenberg algebra $\mathfrak{m}$.
- Adding an inner product or a complex structure on a general H, one again obtains a (usual) G_0–structure.
- If H is contact, then additional compatibility conditions have to be satisfied in order to get a filtered G_0–structure.
- For inner products these conditions amount to requiring continuously varying invariants to be constant.

What is needed to make a homogeneous space G/P into a filtered manifold in a G–invariant way?

- a filtration of $\mathfrak{g}/\mathfrak{p}$ compatible with the bracket or equivalently
- a filtration $\mathfrak{g} = \mathfrak{g}^{-\mu} \supset \cdots \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 = \mathfrak{p}$ such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ for all $i, j \leq 0$
- Then G/P is automatically regular of type $\mathfrak{m}$ with $\mathfrak{m}_i = \mathfrak{g}^i/\mathfrak{g}^{i+1}$ for $i = -\mu, \dots, -1$ and the induced bracket.

This filtration can be extended to positive degrees by defining $\mathfrak{g}^1 := \{A \in \mathfrak{g}^0 : \forall j < 0 : \mathrm{ad}_A(\mathfrak{g}^j) \subset \mathfrak{g}^{j+1}\}$ and similarly defining $\mathfrak{g}^i$ for all $i > 0$. In particular, $\mathfrak{g}^1$ corresponds to a closed normal subgroup $P_+ \subset P$ and putting $G_0 := P/P_+$, we obtain a G–invariant filtered G_0–structure on G/P. Now we take an abstract version of this. It turns out that not all conditions have to be imposed at this stage.

Admissible pairs

An *admissible pair* $(\mathfrak{g}, P)$ consists of

- a filtered Lie algebra $(\mathfrak{g}, \{\mathfrak{g}^i\}_{i=-\mu}^{\nu})$ with $\mu \geq 1$, $\nu \geq 0$
- a Lie group P with Lie algebra $\mathfrak{p} := \mathfrak{g}^0$
- a homomorphism $\mathrm{Ad} : P \to \mathrm{Aut}_f(\mathfrak{g})$ with derivative $\mathrm{ad}\,|_{\mathfrak{g}^0}$

such that

- there is no ideal in $\mathfrak{g}$ which is contained in $\mathfrak{g}^0$
- if $A \in \mathfrak{g}^0$ is such that for all $i = -\mu, \dots, -1$, we have $\mathrm{ad}(A)(\mathfrak{g}^i) \subset \mathfrak{g}^{i+1}$, then $A \in \mathfrak{g}^1$

The ideal $\mathfrak{g}^1 \subset \mathfrak{g}^0$ corresponds to a closed normal subgroup $P_+ \subset P$ and we put $G_0 = P/P_+$ and $\mathfrak{m} = \mathrm{gr}_-(\mathfrak{g})$. Now the concept of a regular Cartan geometry of type $(\mathfrak{g}, P)$ makes sense, and one proves directly that such a geometry determines an underlying filtered G_0–structure.

Consider an admissible pair $(\mathfrak{g}, P)$, the associated graded Lie algebra $\mathrm{gr}(\mathfrak{g}) = \oplus_{i=-\mu}^{\nu} \mathrm{gr}_i(\mathfrak{g})$, and put $\mathfrak{m} := \mathrm{gr}_-(\mathfrak{g})$ and $G_0 := P/P_+$. In terms of G/P, this gives us a G–invariant filtered G_0–structure, but we also need that G is the full automorphism group to have a chance for canonical Cartan connections.

Definition

- $(\mathfrak{g}, P)$ is an *infinitesimal homogeneous model for filtered G_0–structures* if $\mathrm{gr}(\mathfrak{g})$ is the full (Tanaka) prolongation of $\mathfrak{m} \oplus \mathrm{gr}_0(\mathfrak{g}) \subset \mathrm{gr}(\mathfrak{g})$.
- If in addition $\mathrm{gr}_0(\mathfrak{g}) = \mathfrak{der}_{\mathrm{gr}}(\mathfrak{m})$, then $(\mathfrak{g}, P)$ is an *infinitesimal homogeneous model for filtered manifolds of type* $\mathfrak{m}$.

This is most easily expressed in terms of $H^1(\mathfrak{m}, \mathrm{gr}(\mathfrak{g}))$ being concentrated in certain homogeneous degrees. If there are no better choices, one can algebraically construct a candidate for an infinitesimal homogeneous model starting from $(\mathfrak{m}, G_0)$.

A Cartan geometry of type $(\mathfrak{g}, P)$ over M is given by a principal P–bundle $p : \mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. In their study, horizontal $\mathfrak{g}$–valued differential forms on $\mathcal{G}$ are of particular importance. Using ω, such k–forms can be identified with smooth functions with values in $\Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. These spaces naturally are P–modules and carry a P–invariant filtration.

The associated graded vector space to this filtered vector space can be naturally identified with the graded vector space $C^k(\mathfrak{m}, \mathrm{gr}(\mathfrak{g}))$ of k–cochains. The projection to the associated graded is given by choosing representatives in $\mathfrak{g}^i/\mathfrak{p}$ for elements in $\mathrm{gr}_i(\mathfrak{g})$ (with $i < 0$), then apply the given map and project the result to $\mathrm{gr}(\mathfrak{g})$.

The cochain spaces naturally are G_0–modules, but not P–modules. But we have a G_0–equivariant Lie algebra cohomology differential ∂ defined on them. This involves the Lie algebra structures on $\mathfrak{m}$ and on $\mathrm{gr}(\mathfrak{g})$, not the one on $\mathfrak{g}$! Using this, we can define an abstract version of a normalization condition.

A *normalization condition* is a P–submodule $\mathcal{N} \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ such that for each $i > 0$ the image of $\mathcal{N}^i$ (homogeneity $\geq i$) under the projection to the associated graded $C^2(\mathfrak{m}, \mathrm{gr}(\mathfrak{g}))_i$ is complementary to $\mathrm{im}(\partial) \cap C^2(\mathfrak{m}, \mathrm{gr}(\mathfrak{g}))_i$.

To get an analog of harmonic curvature, a second concept is needed. A P–submodule $\tilde{\mathcal{N}} \subset \mathcal{N}$ is called *negligible* if for each $i > 0$, $\mathrm{gr}_i(\tilde{\mathcal{N}}^i)$ has trivial intersection with $\ker(\partial)$. It is called *maximally negligible* if this image is even complementary to $\ker(\partial)$.

For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type $(\mathfrak{g}, P)$, the curvature function has values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. Thus we can simply define such a geometry to be *normal* if the values lie in $\mathcal{N}$.

If we have also fixed a negligible submodule $\tilde{\mathcal{N}} \subset \mathcal{N}$, then we can define the *essential curvature* via projecting the curvature function to $\mathcal{N}/\tilde{\mathcal{N}}$. If $\tilde{\mathcal{N}}$ is maximal, then $\mathrm{gr}(\mathcal{N}/\tilde{\mathcal{N}}) \cong H^2(\mathfrak{m}, \mathrm{gr}(\mathfrak{g}))_+$

Codifferentials

There is a general notion of a *codifferential* that can be used to obtain normalization conditions, e.g. for parabolics and (systems of) ODEs. Such a codifferential consists of P–equivariant maps $\partial^* : \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ for $k = 2, 3$ such that $\partial^* \circ \partial^* = 0$. In order to ensure that $\ker(\partial^*) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is a normalization condition and $\operatorname{im}(\partial^*) \subset \ker(\partial^*)$ is a maximally negligible submodule, one has to require in addition that

- ∂^* is compatible with homogeneities, and if an element of homogeneity $\ell > 0$ is in $\operatorname{im}(\partial^*)$ then it can be written as the image of an element of the same homogeneity.
- The induced maps $\operatorname{gr}_0(\partial^*)$ on the associated graded are disjoint to ∂ (in the appropriate degrees) in the usual sense of Kostant.

For a (regular) Cartan geometry $(p : \mathcal{G} \to M, \omega)$, the basic constructions all work in the language of P–equivariant, horizontal $\mathfrak{g}$–valued forms on $\mathcal{G}$ or the corresponding P–equivariant functions. The main tool is the covariant exterior derivative d^ω defined by

$$d^\omega\varphi(\xi_0, \dots, \xi_k) := d\varphi(\xi_0, \dots, \xi_k) + \sum_{i=0}^{k}(-1)^i[\omega(\xi_i), \varphi(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k)]$$

- d^ω preserves the subspace of horizontal, equivariant forms, and is compatible with the natural notion of homogeneity on such forms.
- Thus it induces a well defined operation on the associated graded, which turns out to be tensorial and induced by the Lie algebra cohomology differential ∂.
- The curvature K of the Cartan connection ω satisfies the Bianchi identity $d^\omega K = 0$.

The key property of d^ω for our purposes is its role in computing the change of curvature resulting from a change of Cartan connections:

- If ω and $\hat{\omega}$ are Cartan connections on $\mathcal{G}$, then $\varphi := \hat{\omega} - \omega$ lies in $\Omega^1_{\text{hor}}(\mathcal{G}, \mathfrak{g})_P$.
- If ω is regular, then $\hat{\omega}$ is regular iff φ is homogeneous of degree ≥ 0 and it induces the same underlying filtered G_0–structure iff φ is homogeneous of degree ≥ 1.
- Conversely, if ω is regular then for any $\varphi \in \Omega^1_{\text{hor}}(\mathcal{G}, \mathfrak{g})^1_P$, $\hat{\omega} = \omega + \varphi$ is a regular Cartan connection inducing the same underlying filtered G_0–structure as ω.
- If for $\ell > 1$, we have $\hat{\omega} = \omega + \varphi$ with φ homogeneous of degree $\geq \ell$, then for the curvatures $\hat{K}$ and K, we get that $\hat{K} - K$ is homogeneous of degree $\geq \ell$ and $\hat{K} - K - d^\omega\varphi$ is homogeneous of degree $\geq \ell + 1$.

The two main technical theorems

Theorem

Let $(\mathfrak{g}, P)$ be an admissible pair and let $\mathcal{N}$ be a normalization condition for $(\mathfrak{g}, P)$. Then given any regular Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type $(\mathfrak{g}, P)$, there is a normal Cartan connection $\hat{\omega}$ on $\mathcal{G}$ that induces the same underlying filtered G_0–structure as ω.

One also easily proves that for any negligible submodule $\tilde{\mathcal{N}} \subset \mathcal{N}$, the curvature of a normal Cartan geometry vanishes if and only if its essential curvature vanishes.

Theorem

Let $(\mathfrak{g}, P)$ be an infinitesimal homogeneous model for filtered G_0–structures and let $\mathcal{N}$ be a normalization condition. Then any two regular normal Cartan connections, which induce the same underlying filtered G_0–structure are related by an isomorphism.

To get from these results to canonical Cartan connections, another technical condition on the group P is needed:

The group P is of *split exponential type* iff for $G_0 = P/P_+$ there is a homomorphism $i : G_0 \to P$ such that the map $G_0 \times \mathfrak{g}^1 \to P$ defined by $(g_0, Z) \mapsto i(g_0)\exp(Z)$ is a diffeomorphism.

Assume that $(\mathfrak{g}, P)$ is an infinitesimal homogeneous model with P of split exponential type and that $\mathcal{N}$ is a normalization condition. Then one shows that any filtered G_0 structure is induced by some Cartan geometry and for normal Cartan geometries, any morphism of the underlying filtered G_0–structures lifts to a morphism of Cartan geometries. This then implies

Under these assumptions, passing from a regular normal Cartan geometry to the underlying filtered G_0–structure defines an equivalence of categories.