Parabolic geometries and geometric compactifications lecture 3

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Recap / program

- In the second lecture we have seen that Poincaré-Einstein metrics can be described in terms of conformal structures that admit a parallel section of the standard tractor bundle. This provides a relation to holonomy reductions of Cartan connections which will be discussed in the first part of today's talk.
- In the second part of the talk, I will discuss how the holonomy perspective leads to the concepts of projective compactness that has been developed in several recent articles of R. Gover and myself. I will also make some remarks on c-projective compactness.
- In the last part of the talk, I will sketch applications of the holonomy theory to the study of compactifications of symmetric spaces and more general homogeneous spaces.

Contents

- Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P). Since ω trivializes $T\mathcal{G}$, it has no holonomy in a naive sense.
- As we have seen, there is an induced principal connection $\tilde{\omega}$ on $\tilde{\mathcal{G}} := \mathcal{G} \times_P G$, which has a holonomy group $\subset G$.

For conformal Cartan connections, this was studied since the early 2000's under the name "conformal holonomy". There were some early results, including classification results on possible holonomy groups (S. Armstrong), but some basic geometric features remained unnoticed for a longer time.

The fact that a parallel standard tractor is equivalent to an Einstein metric on a dense open subset was fundamental for the classification results. But the relation to Poincaré-Einstein was first noticed in a 2009 article of R. Gover, and the general holonomy theory appeared in a 2014 article of R. Gover, M. Hammerl and myself:

Let $\mathcal{O}$ be a homogeneous space of G. For a geometry $(\mathcal{G}, \omega)$ of type (G, P), one gets $\mathcal{G} \times_P \mathcal{O} \cong \tilde{\mathcal{G}} \times_G \mathcal{O}$, so this inherits an Ehresmann connection. One defines a *holonomy reduction of type* $\mathcal{O}$ to be a parallel section of that bundle. This corresponds to a P-equivariant smooth function $f : \mathcal{G} \to \mathcal{O}$.

Let $\mathcal{O} = \sqcup \mathcal{O}_i$ be the decomposition of $\mathcal{O}$ into P-orbits. Then for each $x \in M$, $f(\mathcal{G}_x) \subset \mathcal{O}$ is one of these orbits and if this is $\mathcal{O}_i$, we say that $i \in \mathcal{O}/P$ is the *P-type of x*. Correspondingly, we get a decomposition $M = \sqcup M_i$ according to P-types.

For G/P, one verifies that there is a unique holonomy reduction of type $\mathcal{O}$ for each $\alpha \in \mathcal{O}$, corresponding to the P-equivariant function $G \to \mathcal{O}$ given by $f(g) = g^{-1} \cdot \alpha$. Putting $H := G_\alpha$ (so we can identify $\mathcal{O}$ with G/H) we conclude that the decomposition of G/P according to P-types coincides with the decomposition into H-orbits. Thus P-types are indexed by $H \backslash G/P$ in general.

It is a general result that H-orbits in G/P are initial submanifolds, and of course they are homogeneous spaces of H. In the curved case, one uses a version of normal coordinates to prove:

Let $x \in M$ be a point of P-type i, fix a holonomy reduction of G/P corresponding to H for which eP has P-type i. Let L_i be the stabilizer of eP in H. Then

1. There are open neighborhoods U of x and V of eP and a diffeomorphism $\varphi : U \to V$ that is compatible with the decomposition into P-types. In particular, $M_i \subset M$ is an initial submanifold.
2. The initial submanifold M_i inherits a Cartan geometry of type (H, L_i) from the holonomy reduction.

This motivates the terminology "curved orbit decomposition". The curved orbits $M_i \subset M$ cannot look worse than the H-orbits in G/P.

For the case relevant to Poincaré-Einstein, $G = SO_0(n+1,1)$, P is the stabilizer of a null ray and $H \cong SO(n,1)$ is the stabilizer of a positive vector in $\mathbb{R}^{n+1,1}$. This gives

- $M = M_- \cup M_0 \cup M_+$, with $M_\pm$ open and M_0 (if non-empty) a separating hypersurface according to the inner product of the vector with the ray.
- $L_\pm \cong SO(n)$, so $H/L_\pm$ is hyperbolic space and on $M_\pm$ obtains induced Riemannian metrics. Einstein follows from normality of the conformal Cartan connection.
- L_0 is the stabilizer of a null line in $\mathbb{R}^{n,1}$, H/L_0 is the conformal $(n-1)$-sphere and on M_0 one obtains the normal Cartan geometry associated to a conformal structure.

So this not only provides the picture for Poincaré-Einstein we had, but it also leads to the canonical Cartan geometries associated to all the geometric structures on the curved orbits.

Projective structures are one of two examples of parabolic geometries that are not determined by the underlying structure discussed in lecture 1. Here $G = SL(n+1,\mathbb{R})$ and P is the stabilizer of a ray in $\mathbb{R}^{n+1}$, so $G_0 = GL^+(n,\mathbb{R})$.

For a Cartan geometry of type (G, P), the underlying bundle $\mathcal{G}_0 \to N$ is the full oriented frame bundle of N. But one can use G_0-equivariant sections $\mathcal{G}_0 \to \mathcal{G}$ to pull back the $\mathfrak{g}_0$-component of ω to a principal connection on $\mathcal{G}_0$.

This defines a family of linear connections ∇ on TN, that turn out to be torsion free and have the same geodesics up to parametrization. Thus they form a "projective equivalence class" and it turns out that the Cartan geometry equivalently encodes this class respectively the family of geodesic paths.
There are again density bundles $\mathcal{E}(w)$ and non-vanishing densities select connections in the class.

There are two relevant holonomy reductions here: First, we can take the stabilizer H_1 of a linear functional α on $\mathbb{R}^{n+1}$. On the homogeneous model, this decomposes S^n into two open hemispheres which inherit flat connections and the totally geodesic equator.

On a curved geometry, such a reduction is given by a parallel sections of $\mathcal{G} \times_P \mathbb{R}^{(n+1)*}$ and provides

- $M = M_- \cup M_0 \cup M_+$ with $M_\pm$ open and M_0 a totally geodesic separating hypersurface.
- Ricci flat connections in the projective class on $M_\pm$.
- The normal Cartan geometry associated to a projective structure on M_0.

If the connections on $M_\pm$ are Levi-Civita for some metric, there is an induced holonomy reduction for the Cartan geometry on M_0.

For the stabilizer H_2 of a non-degenerate bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$, the orbit structure on G/P is more complicated:

- Two copies of Riemannian hyperbolic space (negative rays)
- two copies of a Riemannian conformal sphere (null rays)
- One copy of Lorentzian de Sitter space (positive rays)

On curved geometries, such holonomy reductions are equivalent to *Klein-Einstein metrics*. The setup looks similar to the PE case: Open curved orbits inherit Einstein metrics and there are separating hypersurfaces that inherit the normal Cartan geometries associated to a conformal structure.

From a geometric point of view, there are several remarkable differences to the Poincaré-Einstein case:

- Here the metrics on the open orbits have different signature (Riemannian and Lorentzian). Crossing a boundary, the signature changes.
- What one actually gets on the open orbits are is a connection ∇ with non-degenerate, parallel Schouten-tensor (which then is an Einstein metric).
- The induced conformal structure on M_0 can be described in terms of the projective second fundamental form (which is well defined up to scale).
- The conformal tractor bundle on M_0 is the projective standard tractor bundle endowed with the the parallel bundle metric defining the holonomy reduction.

Trying to weaken the two types of holonomy reductions to obtain an analog of conformal compactness, it turns out that they both fit into a broader picture:

A projective modification of ∇ is described by a one-form Υ via $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. One defines projective compactness by the requirement that certain modifications admit a smooth extension, but this allows for an additional parameter.

Fix $\alpha \in (0, 2]$. For $\overline{M} = M \cup \partial M$ a connection ∇ on M is called *projectively compact of order* α if for a local defining function ρ for ∂M, the projective modification of ∇ determined by $\Upsilon := \frac{d\rho}{\alpha\rho}$ admits a smooth extension to $\overline{M}$.

- α is related to the rate of volume growth, $\alpha \in (0, 2]$ is needed for the boundary to be at infinity.
- The holonomy reductions from before are special cases for $\alpha = 1$ (Ricci flat) and $\alpha = 2$ (Einstein), respectively.
- If ∇ preserves a volume form, projective compactness of order α is equivalent to a defining density in $\mathcal{E}(\alpha)$.

Selected results on projective compactness

- Asymptotic forms for metrics that are sufficient for projective compactness of any order α, thus many local examples.
- For $\alpha = 2$, the asymptotic form is equivalent to projective compactness.
- If the projective structure $[\nabla]$ determined by a metric g on M admits a smooth extension to $\overline{M}$, then $Scal(g)$ extends to $\overline{M}$. Locally around boundary points in which this extension is non-zero, g is projectively compact of order 2.
- Suppose that g is Einstein on M, $[\nabla]$ extends to $\overline{M}$ but ∇ does not extend to any open subset of ∂M. Then g is projectively compact of order 1 if $Ric(g) = 0$ and of order 2 if $Ric(g) \neq 0$ (different rates of volume growth are forced).
- Explicit description of the conformal boundary tractor bundle and connection from the projective data in the interior.

There is an almost complex analog of projective compactness of order 2, which involves c-projective geometry ("c-projective compactness"). In its metric version, this involves (quasi-)Kähler metrics in the interior and (almost) CR structures on the boundary.

Results

- The complete Kähler metrics on smoothly bounded domains constructed from defining functions are c-projectively compact.
- Equivalent characterization via an asymptotic form.
- Holonomy reductions to $U(p,q) \subset SL(n,\mathbb{C})$ are contained as a special case. These lead to Kähler-Einstein metrics in the interior.

Suppose that G/P is a generalized flag manifold and $H \subset G$ is a subgroup that acts with finitely many orbits on G/P. Then there are open H-orbits H/L which inherit a (locally flat) parabolic geometry of type (G, P) and the closure in G/P defines a compactification of H/L. This is not a rare situation:

Theorem (J. Wolf, 1976)

Let G be a real simple Lie group and θ an involutive automorphism of G with fixed point group $H \subset G$. Then for each parabolic subgroup $P \subset G$, H acts on G/P with finitely many orbits.

Example: For $p \leq q$ put $n = p + q$, take $G := SL(n, \mathbb{R})$ and P such that $G/P = Gr(p, \mathbb{R}^n)$. Let $H := SO(p, q) \subset G$ defined by a bilinear form b on $\mathbb{R}^n$. H-orbits in G/P are determined by rank and signature of the restriction of b, and for positive definite restriction, one gets the Riemannian symmetric space H/K, $K = S(O(p) \times O(q))$.

Hence we get a compactification of H/K, in which all positive semi-definite subspaces are added as boundary components. The space H/K carries an invariant Grassmannian structure (decomposition of TM as a tensor product), which admits a smooth extension across the boundary.

Results

- b can be used to construct parallel sections of tractor bundles that project to solutions of invariant operators.
- Description of orbit-closures as zero sets of such solutions.
- Construction of defining sections for orbits, which lead to slice theorems that describe the neighborhood of each orbit in the compactification.
- Local models turn out to be symmetric matrices decomposed according to rank.