Cartan connections and the deformation complex for CR structures

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- Applying this to the canonical Cartan connection associated to a (partially integrable almost) CR structure, one can use the algebraic machinery of BGG–sequences to obtain a description of infinitesimal automorphisms and deformations of this underlying structure in terms of higher order operators acting on sections of more traditional vector bundles.
- Restricting to (integrable) CR structures, one obtains a subcomplex, which governs the deformation theory in the integrable subcategory.

Structure

Recall that an almost CR structure of hypersurface type on a smooth manifold of real dimension $2n+1$ is given by a complex rank n subbundle $H \subset TM$. The complex structure J on H is equivalent to a decomposition of $H \otimes \mathbb{C} \subset TM \otimes \mathbb{C}$ into a direct sum $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$. The *Levi bracket* is the skew symmetric bilinear bundle map $\mathcal{L} : H \times H \to TM/H$ induced by the Lie bracket of vector fields. We will always assume that $\mathcal{L}$ is non–degenerate.

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Definition

(1) The almost CR structure (H, J) is called *partially integrable* if and only if $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in H$. Equivalently, the Lie bracket of two sections of $H^{0,1}$ has to be a section of $H \otimes \mathbb{C}$.
(2) The structure is called *integrable* or a *CR structure*, if and only if the subbundle $H^{0,1} \subset TM \otimes \mathbb{C}$ is involutive.

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Assuming that M is oriented and the almost CR structure is partially integrable, the Levi bracket can be considered as the imaginary part of a Hermitian form determined up to positive scale. Thus it has a well defined signature (p, q) with $n = p + q$. The description of CR structures as Cartan geometries is based on the Lie algebra $\mathfrak{g} := \mathfrak{su}(p+1, q+1)$. Starting with with the Hermitian form $z_0\bar{w}_{n+1} + z_{n+1}\bar{w}_0 + \sum_{j=1}^{n} \epsilon_j z_j \bar{w}_j$ on $\mathbb{C}^{n+1}$ (where $\epsilon_j = 1$ for $j = 1, \dots, p$ and $\epsilon_j = -1$ for $j = p+1, \dots, n$), $\mathfrak{g}$ consists of matrices of the form

$$\begin{pmatrix} a & Z & i\psi \\ X & A & -\mathbb{I}Z^* \\ i\varphi & -X^*\mathbb{I} & -\bar{a} \end{pmatrix}$$

with $a \in \mathbb{C}$, $\varphi, \psi \in \mathbb{R}$, $X \in \mathbb{C}^n$, $Z \in \mathbb{C}^{n*}$, and $A \in \mathfrak{u}(p, q)$ such that $a - \bar{a} + \mathrm{tr}(A) = 0$. Here $\mathbb{I}$ denotes the diagonal matrix with diagonal entries $\epsilon_1, \dots, \epsilon_n$.

The block form of the matrices in $\mathfrak{g}$ gives rise to a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ determined by

$$\begin{pmatrix} a & Z & i\psi \\ X & A & -\mathbb{I}Z^* \\ i\varphi & -X^*\mathbb{I} & -\bar{a} \end{pmatrix} \qquad \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$$

as well as a filtration $\mathfrak{g} = \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \cdots \supset \mathfrak{g}^2$, determined by $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_2$. These both are compatible with the Lie bracket in the sense that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$.

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This in particular implies that $\mathfrak{g}_0$ and $\mathfrak{p} := \mathfrak{g}^0$ are subalgebras in $\mathfrak{g}$, such that the grading $\{\mathfrak{g}_i\}$ is $\mathfrak{g}_0$–invariant and the filtration $\{\mathfrak{g}^i\}$ is $\mathfrak{p}$–invariant. In particular, $\mathfrak{p}_+ := \mathfrak{g}^1$ is an ideal in $\mathfrak{p}$ which is two step nilpotent by the grading property. Likewise, $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{g}$ is a Lie subalgebra, and both $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are complex Heisenberg algebras of signature (p, q).

Define $G := PSU(p+1, q+1)$, let $P \subset G$ be the stabilizer of the isotropic (complex) line spanned by the basis vector e_0, and $G_0 \subset P$ the stabilizer of the plane spanned by e_0 and e_{n+1}. Then all groups act on $\mathfrak{g}$ via the adjoint action, each $\mathfrak{g}^i \subset \mathfrak{g}$ is P–invariant, and each $\mathfrak{g}_i \subset \mathfrak{g}$ is G_0–invariant. The exponential map restricts to a diffeomorphism from $\mathfrak{p}_+ = \mathfrak{g}^1$ onto a closed subgroup $P_+ \subset P$ and $P \cong G_0 \ltimes P_+$. The subgroup G_0 is a conformal unitary group, i.e. generated by multiples of the identity and $U(p, q)$.

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Definition

A Cartan geometry of type (G, P) on a smooth manifold M of dimension $2n+1$ consists of a principal P–bundle $p : \mathcal{G} \to M$ and a one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is a *Cartan connection*, i.e. a P–equivariant trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ which reproduces the generators of fundamental vector fields.

Given a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ and a point $u \in \mathcal{G}$ lying over $x = p(u) \in M$, the linear isomorphism $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ descends to a linear isomorphism $T_xM \to \mathfrak{g}/\mathfrak{p}$, and changing to a different point $\hat{u}$, this isomorphism changes by the composition with the action of an element of P induced by the adjoint action. In particular, the P–invariant subspace $\mathfrak{g}^{-1}/\mathfrak{p}$ gives rise to a subspace $H_x \subset T_xM$, which inherits a complex structure. Hence M inherits an almost CR structure.

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Definition

(1) The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of the Cartan connection ω is defined by $K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$.
(2) A Cartan geometry $(p : \mathcal{G} \to M, \omega)$ is called *regular* if for all ξ, η such that $\omega(\xi), \omega(\eta) \in \mathfrak{g}^{-1}$ also $K(\xi, \eta) \in \mathfrak{g}^{-1}$.

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It is easy to see that in the regular case, the underlying almost CR structure on M is partially integrable and non–degenerate of signature (p,q).

We have just seen that a regular Cartan geometry of type (G, P) induces a partially integrable almost CR structure of signature (p, q). Conversely, starting from such a structure (H, J) on M, one can naturally construct a frame bundle for the vector bundle $H \to M$ with structure group $G_0 \cong CU(p, q)$. As a principal bundle, this can be trivially extended to a principal P–bundle, and making choices one can find a Cartan connection on this extension, which induces the given structure.

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There are many Cartan connections inducing a given underlying structure. The key to using the Cartan picture is that one can impose a *normalization condition* on the curvature of the Cartan connection ω in such a way, that a pair $(\mathcal{G}, \omega)$ which satisfies this condition is uniquely determined up to isomorphism by the underlying structure. This condition is due to N. Tanaka and in an equivalent form (for integrable structures only) to S.S. Chern and J. Moser. To formulate the condition, we need a bit of background.

- The properties of ω immediately imply that the curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ is P–equivariant and horizontal. Hence it can be viewed as $\kappa \in \Omega^2(M, \mathcal{A}M)$, where $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ is the *adjoint tractor bundle*.

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- The relation between $(\mathcal{G}, \omega)$ and TM discussed above can be more formally expressed as $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ (with the isomorphism depending on ω). In particular, TM is naturally a quotient of $\mathcal{A}M$.

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- The Killing form of $\mathfrak{g}$ induces a P–equivariant duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$. In particular, $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+$ naturally becomes a bundle of Heisenberg algebras.

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- The bundles of $\mathcal{A}M$–values differential forms are induced by the representations $\Lambda^k\mathfrak{p}_+ \otimes \mathfrak{g}$ of P. In particular, P–equivariant maps between these representations give rise to natural bundle maps.

Definition

(1) The *codifferential* is the natural bundle map $\partial^*:\Lambda^k T^*M\otimes\mathcal{A}M\to\Lambda^{k-1}T^*M\otimes\mathcal{A}M$ induced by the P–equivariant map $\Lambda^k\mathfrak{p}_+\otimes\mathfrak{g}\to\Lambda^{k-1}\mathfrak{p}_+\otimes\mathfrak{g}$ defined by

$$Z_1\wedge\cdots\wedge Z_k\otimes A\mapsto\sum_i(-1)^i Z_1\wedge\cdots\widehat{Z_i}\cdots\wedge Z_k\otimes[Z_i,A]+$$
$$\sum_{i<j}(-1)^{i+j}[Z_i,Z_j]\wedge Z_1\wedge\cdots\widehat{Z_i}\cdots\widehat{Z_j}\cdots\wedge Z_k\otimes A$$

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$$Z_1 \wedge \cdots \wedge Z_k \otimes A \mapsto \sum_i (-1)^i Z_1 \wedge \cdots \widehat{Z_i} \cdots \wedge Z_k \otimes [Z_i, A] + \sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \cdots \widehat{Z_i} \cdots \widehat{Z_j} \cdots \wedge Z_k \otimes A$$

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Theorem (Tanaka)

Mapping a regular normal Cartan geometries of type (G, P) to the underlying partially integrable almost CR structure defines an equivalence of categories.

Structure

Via the equivalence between partially integrable almost CR structures and regular normal Cartan geometries, one can study (infinitesimal) automorphisms and deformations in the Cartan setting, taking into account regularity and normality. In the setting of general Cartan geometries, these concepts are rather easy:

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- It is a classical result that any smooth deformation of a principal bundle is trivial, i.e. the bundles in a smooth family of principal bundles are always isomorphic. Hence to deform a Cartan geometry, one may assume that the bundle is fixed and only the Cartan connection varies.
- The difference $\hat{\omega} - \omega$ of two Cartan connections on $\mathcal{G}$ is an element of $\Omega^1(\mathcal{G}, \mathfrak{g})$ which by definition is horizontal and P–equivariant, while being fiberwise a linear isomorphism is an open condition. Hence the space of infinitesimal deformations of a Cartan connection ω can be identified with $\Omega^1(M, \mathcal{A}M)$.

We have already noted that the curvature κ of a Cartan connection can be interpreted as an element of $\Omega^2(M, \mathcal{A}M)$. Hence also the infinitesimal change of curvature caused by an infinitesimal deformation has values in that space.

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It remains to understand trivial (infinitesimal) deformations, i.e. those induced by automorphisms of the principal bundle. Since such an automorphism is just a P–equivariant diffeomorphism, it corresponds infinitesimally to a P–invariant vector field $\xi \in \mathfrak{X}(\mathcal{G})^P$.

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Observation

A vector field $\xi \in \mathfrak{X}(\mathcal{G})$ is P–invariant if and only if the function $\omega(\xi) : \mathcal{G} \to \mathfrak{g}$ is P–equivariant and thus corresponds to a section s of $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. Hence $\Gamma(\mathcal{A}M) = \Omega^0(M, \mathcal{A}M) \cong \mathfrak{X}(\mathcal{G})^P$, and $\Pi : \mathcal{A}M \to TM$ is given by the projection of vector fields in this picture.

The infinitesimal deformation of a Cartan connection ω caused by $\xi \in \mathfrak{X}(\mathcal{G})^P$ is of course given by the Lie derivative $L_\xi\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. For $\xi, \eta \in \mathfrak{X}(\mathcal{G})$, one computes

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$$(L_\xi\omega)(\eta) = d\omega(\xi, \eta) + \eta \cdot \omega(\xi) = K(\xi, \eta) + \eta \cdot \omega(\xi) - [\omega(\eta), \omega(\xi)].$$

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Since $\omega(\xi)$ is P–equivariant, the last two terms cancel if η is vertical and they are P–equivariant if η is P–invariant. Hence one can use them to define a linear connection ∇ on $\mathcal{A}M$. This is called the *tractor connection* and turns out to be equivalent to ω.

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Any linear connection on $\mathcal{A}M$ extends to the *covariant exterior derivative* on $\mathcal{A}M$–valued forms, so in particular, we obtain $d^{\tilde{\nabla}} : \Omega^k(M, \mathcal{A}M) \to \Omega^{k+1}(M, \mathcal{A}M)$ defined by

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$$(d^{\tilde{\nabla}}\varphi)(\xi_0, \dots, \xi_k) = \sum_i (-1)^i \tilde{\nabla}_{\xi_i}\varphi(\xi_0, \dots, \widehat{\xi_i}, \dots, \xi_k) + \sum_{i<j} (-1)^{i+j} \varphi([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi_i}, \dots, \widehat{\xi_j}, \dots, \xi_k).$$

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With a bit more effort than before, one shows that $\tilde{\nabla}$ also governs the next step in the deformation process:

Proposition

The infinitesimal change of curvature caused by the infinitesimal deformation $\varphi \in \Omega^1(M, \mathcal{A}M)$ of a Cartan geometry $(\mathcal{G}, \omega)$ is given by $d^{\tilde{\nabla}}\varphi \in \Omega^2(M, \mathcal{A}M)$.

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Proposition

Let $(\mathcal{G}, \omega)$ be a Cartan geometry of type (G, P).
(1) The space of infinitesimal automorphisms of the geometry is given by $\{s \in \Gamma(\mathcal{A}M) : \tilde{\nabla} s = 0\}$.
(2) The quotient of infinitesimal deformations by trivial ones can be identified with the quotient of $\Omega^1(M, \mathcal{A}M)$ by the image of $\tilde{\nabla} : \Gamma(\mathcal{A}M) \to \Omega^1(M, \mathcal{A}M)$.

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Much more can be done for locally flat Cartan geometries. By definition this means that ω has vanishing curvature.

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- For an infinitesimal deformation given by $\varphi \in \Omega^1(M, \mathcal{A}M)$ the condition $d^{\tilde{\nabla}}\varphi = 0$ means that infinitesimally there is no change of curve. Hence we obtain

Theorem

For a locally flat Cartan geometry $(\mathcal{G}, \omega)$, $(\Omega^*(M, \mathcal{A}M), d^{\tilde{\nabla}})$ is a complex which governs the deformation theory in the category of locally flat geometries, i.e. its cohomologies in degree zero an one are given by the space of infinitesimal automorphisms respectively the quotient of infinitesimal deformations by trivial deformations.

Structure

To study deformations of partially integrable almost CR structures, we have to study the deformation theory in the category of regular normal Cartan geometries.

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It turns out in the end that regularity follows automatically, so we focus on normality. Starting from a normal geometry, an infinitesimal deformation defined by $\varphi \in \Omega^1(M, \mathcal{A}M)$ clearly is normal if and only if $\partial^*(d^{\tilde{\nabla}}\varphi) = 0$.

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We have the bundle maps $\partial^* : \Lambda^k T^*M \otimes \mathcal{A}M \to \Lambda^k T^*M \otimes \mathcal{A}M$ and by construction, these satisfy $\partial^* \circ \partial^* = 0$. In particular, for each k we can form the natural quotient bundle $\mathcal{H}_k := \ker(\partial^*)/\operatorname{im}(\partial^*)$ and we denote by $\pi_H : \Omega^k(M, \mathcal{A}M) \supset \ker(\partial^*) \to \mathcal{H}_k$ the natural projection. It turns out that $\mathcal{H}_0 = TM/HM$, $\mathcal{H}_1$ is the subbundle of $L(HM, HM)$ consisting of conjugate linear maps, while $\mathcal{H}_2$ is a direct sum of two bundles which we will describe later.

The main tool for relating to the underlying structure is an application of the machinery of BGG sequences (originally developed for the tractor connection ∇) to the connection $\tilde{\nabla}$. The idea is that $d^{\tilde{\nabla}}$ and ∂^* are close enough to being adjoint to admit an analog of a Hodge decomposition. Technically, one has to prove that the operator $\partial^* \circ d^{\tilde{\nabla}}$ is invertible on $\operatorname{im}(\partial^*)$ and the inverse is differential, too. Using this, one shows:

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- For any $\alpha \in \Gamma(\mathcal{H}_k)$ there exists a unique $L(\alpha) \in \Omega^k(M, \mathcal{A}M)$ such that $\partial^*(L(\alpha)) = 0$, $\pi_H(L(\alpha)) = \alpha$, and $\partial^*(d^{\tilde{\nabla}} L(\alpha)) = 0$.

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- This defines (higher order) invariant differential operators $L = L_k : \Gamma(\mathcal{H}_k) \to \Omega^k(M, \mathcal{A}M)$ ("splitting operators") and $D = D_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1})$ ("BGG operators") by $D(\alpha) = \pi_H(d^{\tilde{\nabla}} L(\alpha))$. These can be made as explicit as needed.

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Proposition

(1) π_H and L_0 induce inverse isomorphisms between $\{s \in \Gamma(\mathcal{A}M) : \tilde{\nabla} s = 0\}$ and $\ker(D_0) \subset \Gamma(\mathcal{H}_0)$. Hence this kernel describes infinitesimal automorphisms of the structure.

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(2) For $s \in \Gamma(\mathcal{A}M)$ we have $\partial^*(d^{\tilde{\nabla}} \tilde{\nabla} s) = 0$, so any trivial deformation of a normal geometry is normal.

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(3) π_H and L_1 induce inverse isomorphisms between the quotients $\{\varphi \in \Omega^1(M, \mathcal{A}M) : \partial^*(d^{\tilde{\nabla}}\varphi) = 0\} / \operatorname{im}(\tilde{\nabla})$ and $\Gamma(\mathcal{H}_1) / \operatorname{im}(D_0)$. Hence the latter quotient describes infinitesimal normal deformations modulo trivial deformations.

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Proposition

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(3) π_H and L_1 induce inverse isomorphisms between the quotients $\{\varphi \in \Omega^1(M, \mathcal{A}M) : \partial^*(d^{\tilde{\nabla}} \varphi) = 0\} / \operatorname{im}(\tilde{\nabla})$ and $\Gamma(\mathcal{H}_1) / \operatorname{im}(D_0)$. Hence the latter quotient describes infinitesimal normal deformations modulo trivial deformations.
(4) Starting with a locally flat geometry, $(\Gamma(\mathcal{H}_*), D)$ is a complex, which governs the deformations theory in the category of spherical CR structures.

It turns out that the bundles $\mathcal{H}_k$ for $k = 0, \dots, n$ split into direct sums of irreducible bundles according to $\mathcal{H}_k = \oplus_{i+j=k; i \geq j} \mathcal{H}_{i,j}$, then the sequence continues symmetrically. In particular, $\mathcal{H}_0$ and $\mathcal{H}_1$ are irreducible (as we have seen), while $\mathcal{H}_2 = \mathcal{H}_{2,0} \oplus \mathcal{H}_{1,1}$.

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Starting from a CR structure, one can now look at infinitesimal deformations in the subcategory of (integrable) CR structures. In the Cartan picture, this means that $\varphi \in \Omega^1(M, \mathcal{A}M)$ representing a normal infinitesimal deformation should in addition satisfy that $\pi_H(d^{\tilde{\nabla}}\varphi)$ has vanishing component in $\Gamma(\mathcal{H}_{2,0})$.

Restricting D_i to $\Gamma(\mathcal{H}_{i,0}) \subset \Gamma(\mathcal{H}_i)$ and composing with the projection $\mathcal{H}_{i+1} \to \mathcal{H}_{i+1,0}$, one obtains a sequence of differential operators

$$0 \to \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1) \to \Gamma(\mathcal{H}_{2,0}) \to \cdots \to \Gamma(\mathcal{H}_{n,0}) \to 0$$

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Using a theory of subcomplexes in BGG sequences developed jointly with V. Souček one proves

Theorem

Starting with a CR structure, the above sequence is a complex, which governs the deformation theory in the subcategory of (integrable) CR structures.