Invariant quantizations

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- Then I will discuss a purely algebraic interpretation of a special case of the problem.
- In the last part of the talk, I will describe the setting of AHS–structures and outline our results.

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Structure

1. Natural Quantizations
2. Algebraic formulations
3. AHS–structures and results

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More conceptually, one may use jet bundles. Given a vector bundle $E \to M$ over a smooth manifold, an integer k and a point $x \in M$, one says that two sections s_1 and s_2 of E have the same k–jet in x if and only if $s_1(x) = s_2(x)$ and their representations in local coordinates have the same partial derivatives up to order k. This defines an equivalence relation whose equivalence classes are called k–jets at x, and one shows

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There is a vector bundle $J^k E \to M$, whose fiber over x consists of all k–jets of smooth sections of E at x.

principal symbol

For a section $s \in \Gamma(E)$ one denotes its k–jet at x by $j^k s(x)$. Then one obtains a tautological operator $j^k : \Gamma(E) \to \Gamma(J^k E)$. This leads to a coordinate free definition of differential operators. Namely, $D : \Gamma(E) \to \Gamma(F)$ is a differential operator of order at most k if and only if there is a vector bundle map $\tilde{D} : J^k E \to F$ such that $D(s)(x) = \tilde{D}(j^k s(x))$ for all s and x.

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There is an obvious projection $\pi : J^k E \to J^{k-1} E$, and it can be shown that the kernel of π can be naturally identified with $S^k T^* M \otimes E$. Defining kth order symbol of D to be the restriction of $\tilde{D}$ to this kernel, one obtains:

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For a differential operator $D : \Gamma(E) \to \Gamma(F)$ of order $\leq k$, there is a well defined (kth order) principal symbol $\sigma(D) = \sigma_k(D) : T^* M \otimes E \to F$, which vanishes if and only if D is of order $\leq k - 1$.

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Definition

Let M, E, and F be as before and fix an order k. Then a *quantization for kth order symbols from E to F* is a rule assigning to each smooth bundle map $\tau : S^kT^*M \otimes E \to F$ a differential operator $A_\tau : \Gamma(E) \to \Gamma(F)$ of order $\leq k$ such that $\sigma_k(A_\tau) = \tau$.

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Observe that given such quantizations for all $k \leq N$, we can actually identify the space of differential operators $\Gamma(E) \to \Gamma(F)$ of order $\leq N$ with the direct sum of the spaces of bundle maps $S^kT^*M \otimes E \to F$ for $k = 0, \dots, N$.

An example

In general, there is no canonical way to construct a quantization, unless one chooses some additional structures. The most important possibility is to choose linear connections on E and the tangent bundle TM. (We will denote these and all induced connections by ∇.) In particular, if $M = \mathbb{R}^n$ then E is trivial, and we can just use the ordinary iterated derivatives of $\mathbb{R}^m$–valued functions on $\mathbb{R}^n$.

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Suppose we have chosen connections ∇ on TM and E. Given a section $s \in \Gamma(E)$, one can look at the symmetrized k–fold covariant derivative $\nabla^{(k)}s \in \Gamma(S^kT^*M \otimes E)$. Given a symbol $\tau : S^kT^*M \otimes E \to F$, we can thus define $A_\tau(s) := \tau(\nabla^{(k)}s) \in \Gamma(F)$, and for any choice of ∇, $\tau \mapsto A_\tau$ defines a quantization.

the relation to geometry

The last example leads to geometry. Suppose that we choose a Riemann metric g on M. Then there is the concept of natural vector bundles (basically tensor and spinor bundles in this case). On any such bundle there is a canonical linear connection, the Levi-Civita connection, and we conclude

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Now the basic question that we will ask is given some geometric structure and some class of natural bundles, is there a quantization map for symbols acting between these bundles which is natural, i.e. intrinsic to the given geometric structure.

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- In the same spirit, replace a single Riemannian metric by a conformal equivalence class of such metrics (i.e. allow rescalings by positive smooth functions). Then look for quantizations on natural bundles which only depend on this class and not on individual representatives.

Structure

Both projective structures and conformal structures admit a homogeneous model. In the first case, this is $\mathbb{R}P^n$ as a homogeneous space of $PGL(n+1,\mathbb{R})$. The corresponding connections are those whose unparametrized geodesics are the projective lines.

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In both cases, the homogeneous model has the form G/P, where G is semisimple and $P \subset G$ is a parabolic subgroup. Any natural bundle E is a homogeneous vector bundle over G/P, i.e. the G action on G/P naturally lifts to an action on E. Such bundles are in bijective correspondence with representations of P. This gives rise to a natural representation of G on the space $\Gamma(E)$, defined by $(g \cdot s)(x) := g \cdot (s(g^{-1} \cdot x))$. If E is irreducible, these are principal series representations of G.

Representation theory formulation

The representations of G on $\Gamma(E)$ and $\Gamma(F)$ give rise to a representation on the space of linear operators $\Gamma(E) \to \Gamma(F)$, defined by $(g \cdot D)(s) = g \cdot (D(g^{-1} \cdot s))$. By construction, the subspace $\mathrm{Diff}_k(E, F)$ of differential operators of order $\leq k$ is invariant, so this naturally carries a representation of G.

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Finding a natural quantization for the canonical structure on G/P is equivalent to finding a G–equivariant splitting of the quotient projection $\mathrm{Diff}_k(E, F) \to \mathrm{Diff}_k(E, F)/\mathrm{Diff}_{k-1}(E, F)$ of G–representations.

Lie algebra formulation

There is a further reduction of the problem as follows. The generalized flag manifold G/P contains an open dense subset (the big Schubert cell) which is naturally diffeomorphic to $\mathbb{R}^n$. This gives rise to the canonical projective structure (given by the straight lines) respectively to the flat conformal class on $\mathbb{R}^n$.

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The action of G on G/P infinitesimally gives rise to a realization of its Lie algebra $\mathfrak{g}$ as a subalgebra of the Lie algebra of polynomial vector fields on $\mathbb{R}^n$, and these (finite dimensional) Lie subalgebras turn out to be maximal subalgebras.
The Lie algebra of polynomial vector fields then naturally acts on sections of natural bundles and on the spaces of differential operators. Similarly as before, finding natural quantizations for the canonical structure on $\mathbb{R}^n$ becomes equivalent to finding maps which are equivariant for the action of the subalgebra $\mathfrak{g}$.

Structure

earlier results

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In November 2008, a preprint with a construction of a conformally natural quantization for symbols between arbitrary bundles appeared.

AHS–structures

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Our approach to the quantisation problem is to imitate the construction of a quantization from a linear connection, but using these invariant differential operators.

A central ingredient in our approach is the *adjoint tractor bundle* $\mathcal{A}M$, a natural vector bundle for any AHS–structure, which is constructed from the Cartan bundle and connection. This contains the cotangent bundle T^*M as a natural subbundle and has the tangent bundle TM as a natural quotient. Using this, the first bit is easy:

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For any natural bundle V, there is a simple invariant operator $D : \Gamma(V) \to \Gamma(\mathcal{A}^*M \otimes V)$ called the fundamental derivative. Like a covariant derivative, this can be iterated and then the result can be symmetrized in the $\mathcal{A}^*M$–entries. For a section $s \in \Gamma(E)$ we can thus form $D^{(k)}s \in \Gamma(S^k\mathcal{A}^*M \otimes E)$. It is rather easy to show that the principal part of $D^{(k)}$ is a symmetrized k–fold covariant derivative.

The second part is much more subtle. The symbols are sections of $S^K TM \otimes E^* \otimes F$ and this bundle is naturally a quotient of $S^K \mathcal{A}M \otimes E^* \otimes F$. A section of the latter bundle can be naturally paired with $D^{(k)}s$ for $s \in \Gamma(E)$ to produce a section of F. To construct a natural quantization, we thus want to construct a natural differential operator splitting this quotient projection.

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It follows from existing results that this will not always be possible. One common feature of AHS–structures is that they always admit a family $\{\mathcal{E}[w] : w \in \mathbb{R}\}$ of natural line bundles ("density bundles", "bundles of conformal weights"). The problem is then formulated for symbols mapping E to $F[w] := F \otimes \mathcal{E}[w]$, and all available results have to exclude certain values for w ("critical weights" or "resonant weights").

The main result

To construct a splitting operator as described above, we use that machinery of curved Casimir operators. Using these operators one constructs an invariant differential operator

$$L : \Gamma(S^k TM \otimes E^* \otimes F[w]) \to \Gamma(S^k \mathcal{A}M \otimes E^* \otimes F[w]),$$

which has the property that projecting back $L(\tau)$ to $S^k TM \otimes E^* \otimes F[w]$ one obtains $p(w)\tau$ for some polynomial p depending only on k, E, and F (and not on M or its geometric structure). Then one shows

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Theorem

Suppose that w has the property that $p(w) \neq 0$. Then mapping a symbol τ to the operator $A_\tau(f) := \frac{1}{p(w)} L(\tau)(D^{(k)} f)$ defines a natural quantization.

Results on critical weights

Natural bundles for AHS–structures correspond to representations of the parabolic subgroup P. Irreducible representations of P come from the reductive Levi–factor $G_0 \subset P$ (which is $GL(n, \mathbb{R})$ in the projective case and $CO(n)$ in the conformal case). General representations of P have composition series with completely reducible subquotients.

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To determine the operators L and hence the corresponding polynomial p and the critical weights (for which $p(w) = 0$), one has to understand the composition series of $S^k\mathfrak{g} \otimes V$, where V is the representations inducing E. This is a problem of finite dimensional representation theory (which quickly becomes complicated with growing k).

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One can apply finite dimensional representation theory methods to get structural information on the set of critical weights and determine this set completely in simple cases.