Some constructions with parabolic geometries

Andreas Čap

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Parabolic geometries are a large class of geometric structures are equivalent to regular normal Cartan geometries with homogeneous model a generalized flag manifold.

Cartan geometries

Given a homogeneous space G/H, construct a geometric structure whose automorphisms are exactly the actions of elements of G.

Definition. A *Cartan geometry* of type (G, H) on a smooth manifold M is a principal $H-$bundle $p : \mathcal{G} \to M$ together with a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ (the *Cartan connection*) such that

- $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$.
- $\omega(\zeta_A) = A$ for all $A \in \mathfrak{h}$.
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a principal bundle homomorphism $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega} = \omega$.

A morphism is determined by the underlying map $M \to \tilde{M}$ up to a smooth function from M to the maximal normal subgroup of G which is contained in H. In all cases of interest, this subgroup is trivial or at least discrete.

The *homogeneous model* of the geometry is the principal bundle $G \to G/H$ together with the left Maurer–Cartan form ω^{MC}. The left action of $g \in G$ then defines an automorphism, and conversely, any smooth map $f : G \to G$ which pulls back ω^{MC} to itself is a left translation, so $G = \mathrm{Aut}(G \to G/H, \omega^{MC})$.

The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of $(\mathcal{G} \to M, \omega)$ is defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$, for $\xi, \eta \in \mathcal{X}(\mathcal{G})$. This form is horizontal and H–equivariant, so it may also be viewed as $\kappa \in \Omega^2(M, \mathcal{A}M)$, where $\mathcal{A}M = \mathcal{G} \times_H \mathfrak{g}$ is the *adjoint tractor bundle*.

General features of Cartan geometries:
• The curvature is a complete obstruction to local isomorphism with $(G \to G/H, \omega^{MC})$, and hence (at least in principle) solves the equivalence problem.

• For any Cartan geometry of type (G, H), the automorphism group is a Lie group of dimension $\leq \dim(G)$.
• Liouville type theorem: Any isomorphism between two open subsets of $(G \to G/H, \omega^{MC})$ uniquely extends to a global automorphism.
• General concepts of distinguished curves and normal coordinates.

These properties become particularly interesting if the Cartan geometry provides an equivalent description for some underlying geometric structure.

Such equivalences are often obtained via (fairly difficult) prolongation procedures. In such cases, there is the additional feature that one obtains new geometric objects for the underlying structure.

In the best cases, one obtains a categorical equivalence between some geometric structure and Cartan geometries of some type which satisfy additional normalization conditions (usually expressed via the curvature).

Generalized Flag manifolds

Let $\mathfrak{g}$ be a semisimple Lie algebra. A $|k|-$*grading* on $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$.

For given $\mathfrak{g}$ there is a simple complete description of such gradings in terms of the structure theory. For complex $\mathfrak{g}$, such gradings are in bijective correspondence with sets of simple roots of $\mathfrak{g}$ and hence are conveniently denoted by Dynkin diagrams with crosses. For real $\mathfrak{g}$ there is a similar description in terms of the Satake diagram.

Putting $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ we obtain a filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^k$ of $\mathfrak{g}$ such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. In particular, $\mathfrak{p} := \mathfrak{g}^0$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_+ := \mathfrak{g}^1$ is a nilpotent ideal in $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is a semidirect sum. The subalgebras $\mathfrak{p}$ obtained in that way are exactly the *parabolic* subalgebras of $\mathfrak{g}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, we define $P := N_G(\mathfrak{p}) \subset G$. It turns out that P has Lie algebra $\mathfrak{p}$ and for $g \in P$ the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is filtration preserving. Define $G_0 \subset P$ as the subgroup of those g, for which $\mathrm{Ad}(g)$ even preserves the grading. Then G_0 is reductive and has Lie algebra $\mathfrak{g}_0$. One shows that exp defines a diffeomorphism from $\mathfrak{p}_+$ onto a closed subgroup $P_+ \subset P$ and P is the semidirect product of G_0 and P_+.

A *generalized flag variety* is a homogeneous space G/P for G and P as above. These homogeneous spaces are always compact and for complex G they are the only compact homogeneous spaces of G.

Parabolic geometries are Cartan geometries of type (G, P) for G and P as above. Under the conditions of regularity and normality (to be discussed below), such a Cartan geometry is equivalent to an underlying structure that we will discuss next.

Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P). Then we define the *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$. The P–invariant filtration $\{\mathfrak{g}^i\}$ of $\mathfrak{g}$ give rise to a filtration

$$\mathcal{A}M = \mathcal{A}^{-k}M \supset \mathcal{A}^{-k+1}M \supset \cdots \supset \mathcal{A}^k M$$

by smooth subbundles, and the Lie bracket on $\mathcal{G}$ induces a tensorial map $\{\ ,\ \} : \mathcal{A}M \times \mathcal{A}M \to \mathcal{A}M$. In particular, each fiber of $\mathcal{A}M$ is a filtered Lie algebra isomorphic to $\mathfrak{g}$.

The Cartan connection ω gives us an identification $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, with the action coming from the adjoint action. Hence $TM \cong \mathcal{A}M/\mathcal{A}^0M$, and we obtain an induced filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle. The associated graded bundle is

$$\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \cdots \oplus \mathrm{gr}_{-1}(TM),$$

where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$. The Killing form of $\mathfrak{g}$ induces a duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$, so $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+ = \mathcal{A}^1M$. Hence T^*M is a bundle of nilpotent Lie algebras.

The subgroup $P_+ \subset P$ acts freely on $\mathcal{G}$, so the quotient $\mathcal{G}_0 := \mathcal{G}/P_+$ is a principal bundle over M with structure group $P/P_+ = G_0$. The Cartan connection ω induces a bundle map from $\mathcal{G}_0$ to the frame bundle of $\mathrm{gr}(TM)$, hence defining a reduction of structure group. In particular, $\mathrm{gr}_i(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$.

Digression on filtered manifolds

A *filtered manifold* is a smooth manifold M together with a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of its tangent bundle such that for sections ξ of T^iM and η of T^jM the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$. Since T^iM and T^jM are contained in $T^{i+j+1}M$, the map which sends ξ and η to the class of $[\xi, \eta]$ in $\mathrm{gr}_{i+j}(TM)$ is bilinear over smooth functions, and depends only on the classes of ξ in $\mathrm{gr}_i(TM)$ and η in $\mathrm{gr}_j(TM)$.

For each $x \in M$, this makes $\mathrm{gr}(T_xM)$ into a nilpotent graded Lie algebra called the *symbol algebra* of the filtration at x. If the symbol algebras are all isomorphic to some fixed $\mathfrak{a}$, then we have an obvious frame bundle for $\mathrm{gr}(TM)$ with structure group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{a})$.

The equivalence to underlying structures

A parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) is called *regular*, if its curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ has the property that

$$\kappa(T^iM, T^jM) \subset \mathcal{A}^{i+j+1}M.$$

Proposition. Let $(p : \mathcal{G} \to M, \omega)$ be a regular parabolic geometry. Then the induced filtration $\{T^iM\}$ of the tangent bundle makes M into a filtered manifold, such that each symbol algebra is isomorphic to $\mathfrak{g}_-$. Moreover, the brackets of the symbol algebras are induced by the bracket $\{\ ,\ \}$ on $\mathcal{A}M$.

Now we have all the underlying structure at hand: A filtration compatible with the Lie bracket such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ and a reduction of the structure group of gr(TM) to the group $G_0 \subset \text{Aut}_{\text{gr}}(\mathfrak{g}_-)$. Fixing this underlying structure still leaves lots of freedom for the Cartan connection ω, so we need an additional *normalization condition*.

Define $\partial^* : \Lambda^2 T^*M \otimes \mathcal{A}M \to T^*M \otimes \mathcal{A}M$ by

$$\partial^*(\alpha \wedge \beta \otimes s) := \\ -\beta \otimes \{\alpha, s\} + \alpha \otimes \{\beta, s\} - \{\alpha, \beta\} \otimes s.$$

This gives rise to a tensorial operator on $\mathcal{A}M$–valued two forms denoted by the same symbol.

A parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is called *normal* if its curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ satisfies $\partial^*(\kappa) = 0$.

Theorem. Let M be a filtered manifold such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ and let $\mathcal{G}_0 \to M$ be a reduction of $\text{gr}(TM)$ to the structure group G_0. Then there is a regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ inducing the given data. If $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in non–positive homogeneous degrees, then the pair $(\mathcal{G}, \omega)$ is unique up to isomorphism.

- one obtains an equivalence of categories in that way.
- $H_*(\mathfrak{p}_+, \mathfrak{g})$ is algorithmically computable.

Harmonic curvature

One may pass from the full Cartan curvature κ to the harmonic curvature κ_H, which is much easier to handle, but as powerful as κ. Similarly as above, one defines

$$\partial^* : \Lambda^3 T^*M \otimes \mathcal{A}M \to \Lambda^2 T^*M \otimes \mathcal{A}M$$

and shows that $\partial^* \circ \partial^* = 0$. Hence $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^2 T^*M \otimes \mathcal{A}M$ are natural subbundles, and we put $\mathcal{K}M := \ker(\partial^*)/\operatorname{im}(\partial^*)$. For normal geometries, $\kappa \in \Gamma(\ker(\partial^*))$ by definition, and we define $\kappa_H \in \Gamma(\mathcal{K}M)$ to be the corresponding section of the quotient bundle. This quotient bundle can be described as $\mathcal{G}_0 \times_{G_0} H_2(\mathfrak{p}_+, \mathfrak{g})$ so it is algorithmically computable can be directly interpreted in terms of the underlying structure. Moreover, it splits into a direct sum of subbundles according to the splitting of $H_2(\mathfrak{p}_+, \mathfrak{g})$ into G_0–irreducible components.

Theorem.(1) (Tanaka) κ_H is a complete obstruction to local isomorphism with G/P.
(2) (Calderbank–Diemer) There is a natural differential operator L such that $L(\kappa_H) = \kappa$.

Examples

(1) Let us first consider the case of $|1|$–gradings. In this case, the filtration of the tangent bundle is trivial and the regularity condition is vacuous. One obtains an equivalence between classical first order G_0–structures and regular normal parabolic geometries of type (G, P). The most important examples are conformal, almost quaternionic, and almost Grassmannian structures.

(2) Suppose that $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in negative homogeneous degrees. Then it turns out that choosing $G = \text{Aut}(\mathfrak{g})$ one obtains $G_0 = \text{Aut}_{\text{gr}}(\mathfrak{g}_-)$. Hence there is no reduction of structure group of $\text{gr}(TM)$ and one obtains an equivalence between filtered manifolds such that each symbol algebra is isomorphic to $\mathfrak{g}_-$ and regular normal parabolic geometries. This class of examples contains the quaternionic contact structures introduced by O. Biquard, generic distributions of rank 2 in dimension 5, rank 3 in dimension 6, and rank 4 in dimension 7.

(3) *Parabolic contact structures*: These correspond to $|2|$–gradings such that $\mathfrak{g}_-$ is a (real) Heisenberg algebra. With a few exceptions, any simple $\mathfrak{g}$ admits a unique grading of this form. The filtration consists only of one subbundle $T^{-1}M \subset TM$, which defines a contact structure on M. The reduction to the structure group G_0 can be expressed by some additional structure on $T^{-1}M$.

This class contains partially integrable almost CR structures, Lagrangian (or Legendrean) contact structures, and Lie sphere structures.

(4) As an example for more general structures, we discuss *(generalized) path geometries*. These correspond to the $|2|$–grading on $\mathfrak{sl}(n+2, \mathbb{R})$ corresponding to the first and second simple root. They are defined on manifolds of dimension $2n+1$. The geometry is given by two subbundles $L, R \subset TM$ of rank 1 and n, respectively, such that for $\xi, \eta \in \Gamma(R)$ we have $[\xi, \eta] \in \Gamma(L \oplus R)$ while the Lie bracket induces an isomorphism $L \otimes R \to TM/(L \oplus R)$.

Examples come from path geometries. Let N be a manifold, dim$(N) = n + 1$ and put $M := \mathcal{P}TN$, the space of lines in TN. Take R to be the vertical bundle of $\mathcal{P}TN \to N$. Then R is contained in the tautological subbundle H of TM. A *path geometry* on N is a decomposition $H = L \oplus R$. Such a geometry is equivalent to a family of unparametrized curves in N, with exactly one curve through each point in each direction. In particular, a system of second order ODE's in Y can be viewed as a path geometry on $Y \times \mathbb{R}$.

For $n \neq 2$, any normal parabolic geometry (M, L, R) as before is locally isomorphic to a path geometry. In this case the subbundle $R \subset TM$ is integrable, and one defines N to be a local leaf space. So for $U \subset M$ open, there is a surjective submersion $\psi : U \to N$ such that ker$(T_x\psi) = R_x$ for all $x \in U$. Under $T_x\psi$, the line L_x gives rise to a line in $T_{\psi(x)}N$, hence defining a lift $\tilde{\psi} : U \to \mathcal{P}TN$. Possibly shrinking U, $\tilde{\psi}$ is an open embedding. By construction, $T\tilde{\psi}$ maps R to the vertical subbundle and $L \oplus R$ to the tautological subbundle.

Correspondence spaces

The first relation between geometries of different type we discuss deals with the case of nested parabolic subgroups $Q \subset P \subset G$. For the homogeneous models, this is simply the observation that G/Q naturally fibers over G/P. Moreover, $G/Q = G \times_P (P/Q)$, so this is a natural fiber bundle. It turns out that the fiber P/Q is always a generalized flag manifold. In the Dynkin (or Satake) diagram notation, $\mathfrak{q}$ is obtained from $\mathfrak{p}$ by adding crosses, and the fiber P/Q can be read off the diagram.

Carrying this over to curved Cartan geometries is easy. Given a geometry $(p : \mathcal{G} \to N, \omega)$ of type (G, P) the subgroup $Q \subset P$ acts freely on $\mathcal{G}$, so $\mathcal{C}N := \mathcal{G}/Q$ is a smooth manifold, and the obvious map $\mathcal{G} \to \mathcal{C}N$ is a Q–principal bundle. Moreover, $\mathcal{C}N = \mathcal{G} \times_P (P/Q)$, so $\pi : \mathcal{C}N \to N$ is a natural fiber bundle with compact fibers. By definition, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ can also be viewed as a Cartan connection on $\mathcal{G} \to \mathcal{C}N$.

Now the uniform algebraic construction of the normalization condition pays off:

Proposition. If $(\mathcal{G} \to N, \omega)$ is a normal parabolic geometry of type (G, P) then the parabolic geometry $(\mathcal{G} \to \mathcal{C}N, \omega)$ of type (G, Q) is normal, too.

To obtain an interpretation in terms of underlying structures it thus remains to check regularity, which is easy in each case.

Example. Let $Q \subset G := SL(n+2, \mathbb{R})$ be the parabolic describing generalized path geometries. Then $Q = P_1 \cap P_2$ for parabolics P_1 and P_2 (the stabilizers of a line respectively a plane). Let us start by analyzing $Q \subset P_1 \subset G$. Parabolic geometries of type (G, P_1) form one of the two exceptional cases, which we have not described yet. If $(\mathcal{G} \to Z, \omega)$ is such a geometry, then $\mathcal{G}_0 \to Z$ is the full linear frame bundle of Z. The geometry is given by a *projective structure* on Z, i.e. the choice of a projective class $[\nabla]$ of torsion free linear connections on TZ.

Two linear connections ∇ and $\hat{\nabla}$ on TZ are *projectively equivalent* if there is a one form $\Upsilon \in \Omega^1(Z)$ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$$

for all vector fields ξ and η on Z. Evidently, projectively equivalent connections have the same torsion. Alternatively, projective equivalence can be characterized as having the same torsion and the same geodesics up to parametrization. The harmonic curvature is the projective Weyl curvature.

Since ω is a Cartan connection on $\mathcal{G} \to Z$, we have $TZ = \mathcal{G} \times_{P_1} (\mathfrak{g}/\mathfrak{p}_1)$. One easily verifies that $Q \subset P_1$ can be described as the stabilizer of a line in $\mathfrak{g}/\mathfrak{p}_1$. Since P_1 acts transitively on $\mathcal{P}(\mathfrak{g}/\mathfrak{p}_1)$ we conclude that $\mathcal{C}Z = \mathcal{P}TZ$. Moreover, ω is always regular as a Cartan connection on $\mathcal{G} \to \mathcal{C}Z$. Hence $(\mathcal{G} \to \mathcal{C}Z, \omega)$ can be interpreted as a classical path geometry on Z. One verifies that the paths described in that way are exactly the unparametrized geodesics of the connections from the projective class.

Let us analyze $Q \subset P_2 \subset G$. A normal parabolic geometry $(\mathcal{G} \to N, \omega)$ of type (G, P_2) exists only for $\dim(N) = 2n$ and is equivalent to an almost Grassmannian structure. Essentially, this means that we have vector bundles E and F over N of rank 2 and n, respectively, and an isomorphism $E \otimes F \to TN$. The subgroup $Q \subset P_2$ can be characterized as the stabilizer of a line in the representation inducing E, so $\mathcal{C}N \cong \mathcal{P}E$.

Here ω is not regular as a Cartan connection on $\mathcal{G} \to \mathcal{C}N$ in general. Regularity turns out to be equivalent to the fact that the structure on N is Grassmannian, i.e. admits a torsion free connection. In that case we obtain a generalized path geometry on $\mathcal{P}E$. The corresponding subbundles L and $L \oplus R$ can be characterized as the vertical respectively the tautological subbundle in $T\mathcal{P}E$. In particular, the manifold N is the space of all paths of the induced path geometry. The splitting of the tautological bundle as $L \oplus R$ comes from the torsion free connections compatible with the Grassmannian structure.

Starting from a Grassmannian structure on N, we obtain a generalized path geometry on $\mathcal{C}N := \mathcal{P}E$. We know that the resulting subbundle $R \subset T\mathcal{C}N$ is involutive, so for sufficiently small open subsets $U \subset \mathcal{C}N$ we can form a local leaf space $\psi : U \to Z$. With a bit more work one shows that one may take $U = \pi^{-1}(V)$, for sufficiently small and convex open subsets $V \subset N$, where $\pi : \mathcal{C}N \to N$ is the natural projection. One then obtains a correspondence

$$Z \xleftarrow{\psi} \pi^{-1}(V) \xrightarrow{\pi} V,$$

which is the basis for *twistor theory* for Grassmannian structures.

Returning to the general case $Q \subset P \subset G$, we now turn to the question when a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, Q) is locally isomorphic to a correspondence space. There is an obvious necessary condition: The subspace $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is Q–invariant, thus giving rise to a subbundle $\mathcal{V} \subset TM$. If $M \cong \mathcal{C}N$, then $\mathcal{V}$ is the vertical subbundle of $\mathcal{C}N \to N$. Hence vectors from $\mathcal{V}$ must hook trivially into the Cartan curvature.

Using principal bundle geometry one proves that this necessary condition is also sufficient:

Theorem. Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, Q) with Cartan curvature κ, and let $\mathcal{V} \subset TM$ be the distribution corresponding to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. Then $(\mathcal{G} \to M, \omega)$ is locally isomorphic to a correspondence space $\mathcal{C}N$ of a parabolic geometry of type (G, P) if and only if $i_\xi \kappa = 0$ for all $\xi \in \mathcal{V}$.

While this result is very satisfactory from a conceptual point of view, it is difficult to use in concrete cases, since the Cartan curvature is hard to handle. Using BGG sequences, one constructs a differential operator, which reconstructs κ from the harmonic curvature κ_H. The algebraic properties of this operator can be controlled very well, and one proves

Theorem. Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, Q) with Cartan curvature κ and harmonic curvature κ_H, and let $\mathcal{V} \subset TM$ be as above. If $i_\xi \kappa_H = 0$ for all $\xi \in \mathcal{V}$, then $i_\xi \kappa = 0$ for all $\xi \in \mathcal{V}$.

Let us return to the example of generalized path geometries. For a regular normal geometry $(\mathcal{G} \to M, \omega)$ of type (G, Q) the harmonic curvature κ_H consists of two irreducible components:

$$T : L \wedge TM/(L \oplus R) \to R \qquad \text{Torsion}$$
$$\rho : R \wedge TM/(L \oplus R) \otimes R \to R \qquad \text{Curvature}$$

Looking at $Q \subset P_1 \subset G$ we have $\mathcal{V} = R \subset TM$, and the last two theorems show that M is locally isomorphic to the correspondence space of some projective structure if and only if $\rho = 0$. Rephrased in terms of second order ODE's this gives

Theorem. A system of second order ODE's is locally equivalent to a geodesic equation if and only if the curvature ρ of the associated path geometry vanishes identically.

If $\rho = 0$, then the torsion T is directly related to the projective Weyl curvature of the projective structures on the local leaf spaces. In particular, the path geometry is locally flat if and only if the induced projective structures on all local leaf spaces are projectively flat.

For the other “side” $Q \subset P_2 \subset G$ we get $\mathcal{V} = L \subset TM$. The statement then is that a path geometry descends to a Grassmannian structure on that space of all paths if and only if $T = 0$. This is equivalent to torsion freeness in the sense of Cartan geometries. Then the curvature ρ is an equivalent encoding of the Grassmannian curvature downstairs. In particular, the twistor space of a Grassmannian structure inherits a projective structure only in the locally flat case.

Now one defines torsion freeness for systems of second order ODE's as torsion freeness of the associated path geometries. Since a torsion free path geometry is obtained as a pull back from the space of paths, the curvature descends to the space of paths, and hence is constant along each path. Using this, D. Grossman proved:

Theorem. For generic torsion free systems of second order ODE's, the curvature of the associated path geometry can be used to solve the equation explicitly.

The case $n = 2$. This corresponds to three dimensional projective structures, generalized path geometries in dimension 5 and almost Grassmannian structures on 4–manifolds. In dimension 4, an almost Grassmannian structure is equivalent to a split signature conformal spin structure. The two components in the harmonic curvature of such a structure correspond to the self dual and anti self dual part of the Weyl curvature.

Likewise, on the level of path geometries, the harmonic curvature now splits into three components, the additional being a torsion $\tau : \Lambda^2 R \to L$. For the correspondence space of a conformal structure, this encodes the self dual part of the Weyl curvature. The tensor τ is the obstruction to integrability of the subbundle R and hence we see that for split signature conformal four manifolds a twistor space exists exactly in the anti self dual case.

The Fefferman construction

Let $\Omega \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with smooth boundary $M := \partial\Omega$. Then M naturally inherits a hypersurface type CR structure of signature $(n, 0)$, see below. Using the ambient metric, Ch. Fefferman constructed a conformal structure of signature $(2n+1, 1)$ on $M \times S^1$, which depends only on the CR structure on M.

For $p \geq q$ put $G := SU(p+1, q+1)$ and let $P \subset G$ be the stabilizer of an isotropic complex line in $\mathbb{V} := \mathbb{C}^{p+1,q+1}$. Then regular normal parabolic geometries of type (G, P) are equivalent to *partially integrable* (p.i.) *almost CR structures* of signature (p, q). Such structures exist on manifolds M of dimension $2(p+q)+1$. They are given by a contact structure $H \subset TM$ and an almost complex structure J on H, such that the tensor $\mathcal{L} : \Lambda^2 H \to TM/H$ induced by the Lie bracket of vector fields satisfies $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$. Then $\mathcal{L}$ is the imaginary part of a Hermitian form (the *Levi form*) and (p, q) is the signature of this form.

In addition, one has to choose an $(n+2)$nd root of the so–called canonical bundle, a complex line bundle on M. For $M = \partial\Omega$ one defines H_x as the maximal complex subspace of $T_xM \subset T_x\mathbb{C}^{n+1}$ and J_x by restriction. The complexification of H splits as $H^{1,0} \oplus H^{0,1}$, and the integrability of the complex structure on $\mathbb{C}^{n+1}$ implies that the subbundle $H^{0,1} \subset TM \otimes \mathbb{C}$ is involutive. This conditions is called *integrability* and structures satisfying it are called *CR structures*. It implies compatibility of $\mathcal{L}$ and J and is equivalent to torsion freeness of the associated parabolic geometry. The canonical bundle is trivial in this case, so the choice of a root makes no problem.

Forgetting the complex structure on $\mathbb{V}$ and looking at the real part of the Hermitian inner product gives an inclusion $G \hookrightarrow \tilde{G} := SO(2p+2, 2q+2)$. Fix a real line in the chosen isotropic complex line and let $\tilde{P} \subset \tilde{G}$ be the stabilizer of this real line. Evidently, $G \cap \tilde{P} \subset P$ and $P/(G \cap \tilde{P}) \cong \mathbb{R}P^1$. The space $\tilde{G}/\tilde{P}$ is the projectivized null cone, and thus the homogeneous model for conformal structures of signature $(2p+1, 2q+1)$.

Elementary linear algebra shows that G acts transitively on $\tilde{G}/\tilde{P}$. Thus $\tilde{G}/\tilde{P} \cong G/(G\cap\tilde{P})$ and the latter space is a circle bundle over G/P. This is Fefferman's construction for the homogeneous model.

Consider a p.i. almost CR manifold M and let $(\mathcal{G}\to M,\omega)$ be the corresponding regular normal parabolic geometry. Then we define $\tilde{M} := \mathcal{G}/(G\cap\tilde{P})$, which is a circle bundle over M. Let $\mathbb{V}^1\subset\mathbb{V}$ be the isotropic line stabilized by P and put $\mathcal{E}(-1,0) := \mathcal{G}\times_P\mathbb{V}^1$. (This bundle is dual to the chosen root of the canonical bundle.) Since $G\cap\tilde{P}$ is the stabilizer of a real line in $\mathbb{V}^1$, $\tilde{M}$ is the space of real lines in $\mathcal{E}(-1,0)$. By construction, $\mathcal{G}\to\tilde{M}$ is a principal bundle with structure group $G\cap\tilde{P}$ and ω can be viewed as a Cartan connection on that bundle. In particular, $T\tilde{M}\cong\mathcal{G}\times_{G\cap\tilde{P}}\mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}})$.

The inclusion $G\hookrightarrow\tilde{G}$ induces an equivariant isomorphism $\mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}})\to\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. This can be used to pull back the $\tilde{P}$–invariant conformal inner product on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ to a $(G\cap\tilde{P})$–invariant conformal inner product on $\mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}})$, which induces a conformal structure on $\tilde{M}$.

This can be easily reformulated in terms of Cartan connections: Define a $\tilde{P}$–principal bundle $\tilde{\mathcal{G}} \to \tilde{M}$ as $\tilde{\mathcal{G}} := \mathcal{G} \times_{G \cap \tilde{P}} \tilde{P}$. Then $\mathcal{G} \subset \tilde{\mathcal{G}}$, and it is elementary to show that there is a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $\tilde{\omega}|_{T\mathcal{G}} = \omega$. Since any Cartan connection on $\tilde{\mathcal{G}}$ is automatically regular, $\tilde{\omega}$ induces a conformal structure on $\tilde{M}$, which is exactly the one described above.

Using this observation, one can explicitly describe the conformal structure on $\tilde{M}$ in terms of a Webster–Tanaka connection on M, its torsion and curvature.

Surprisingly, $\tilde{\omega}$ is *not* the normal Cartan connection associated to the conformal structure on $\tilde{M}$ in general, but one has:

Theorem. Let M be a p.i. almost CR manifold with Fefferman space $\tilde{M}$. Then $\tilde{\omega}$ is the normal Cartan connection associated to the canonical conformal structure on $\tilde{M}$ if and only if the structure on M is integrable.

The relation between the Cartan connection implies much more than just the existence of a canonical conformal structure. This can be most easily expressed in terms of the *standard tractor bundles*. The CR standard tractor bundle of M is $\mathcal{T} := \mathcal{G} \times_P \mathbb{V}$. This is a complex vector bundle with a Hermitian metric of signature $(p+1, q+1)$ and a Hermitian connection. We can also form $\tilde{\mathcal{T}} = \mathcal{G} \times_{G \cap \tilde{P}} \mathbb{V} \to \tilde{M}$, which comes with the same data. The theorem can be rephrased as the fact that, ignoring the complex structure, $\tilde{\mathcal{T}}$ is the normal conformal standard tractor bundle of $\tilde{M}$.

Applications

• Chern Moser chains on M are the projections of null geodesics on $\tilde{M}$.
• Relation between conformal tractor calculus on $\tilde{M}$ and CR tractor calculus on M.
• Conformally invariant differential operators on $\tilde{M}$ descend to families of CR invariant differential operators on M.
• Conformal interpretation of solutions of certain CR invariant operators.

Fefferman spaces form a very nice and interesting class of conformal structures:

- Fefferman spaces have a parallel orthogonal complex structure on the standard tractor bundle and are locally characterized by that.
- Existence of twistor spinors and odd degree Killing forms on Fefferman spaces.
- Interpretation of infinitesimal conformal isometries of $\tilde{M}$ via the CR geometry of M.

Analogs of the Fefferman construction

To obtain such an analog one needs an inclusion $G \hookrightarrow \tilde{G}$ between semisimple Lie groups, and a parabolic $\tilde{P} \subset \tilde{G}$ such that the G–orbit of $e\tilde{P} \in \tilde{G}/\tilde{P}$ is open. Equivalently, the inclusion $\mathfrak{g} \to \tilde{\mathfrak{g}}$ has to induce a linear isomorphism $\mathfrak{g}/(\mathfrak{g}\cap\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Finally, one needs a parabolic P containing $G\cap\tilde{P}$. For a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), one then defines

$$\tilde{M} := \mathcal{G}/(G \cap \tilde{P}) = \mathcal{G} \times_{G\cap\tilde{P}} (P/G \cap \tilde{P}) \to M$$

As before, one obtains $\tilde{\mathcal{G}} \to \tilde{M}$ and a Cartan connection $\tilde{\omega}$ on this bundle, and thus a geometry of type $(\tilde{G}, \tilde{P})$ on $\tilde{M}$.

To obtain the underlying geometric structure on $\tilde{M}$, one only has to check regularity, which usually is very easy. The harder part (which has not been carried out in all the examples below) is to determine the precise conditions under which the Cartan connection on $\tilde{\omega}$ is the normal Cartan connection determined by this underlying structure.

Examples

- $Sp(p+1,q+1) \hookrightarrow SU(2p+2,2q+2)$, stabilizers of a quaternionic respectively complex isotropic line. This leads to twistor theory for quaternionic contact structures.
- $SO(p+1,q+1) \hookrightarrow SO(p+2,q+1)$, stabilizers of an isotropic line respectively an isotropic plane. Conformal structure on M induces Lie sphere structures on open subsets of T^*M.
- $Sp(2n,\mathbb{R}) \hookrightarrow SL(2n,\mathbb{R})$, stabilizers of a line. Contact projective structure on M extends to a projective structure.
- $G_2 \hookrightarrow SO(4,3)$, stabilizers of a null line. Generic rank two distribution on M^5 induces a conformal class of signature $(3,2)$.