RANDOM WALKS ON GROUPS, 2023 SS
EXERCISES A

(1) Show the asymptotic distribution for the simple random walk on $\mathbb{Z}$ is, for $k \leq n/2$, given by:

$$P[Z_{2n} = 2k] \sim \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot e^{-\frac{k^2}{n}}$$

Fact: For $b, p > 0$, $\lim_{x\to\infty}(1+\frac{b}{x^p})^{x^p} = e^b$

(2) Show the return probability for the simple random walk on the 3–dimensional grid is:

$$p^{(2n)}(\vec{0},\vec{0}) \sim \frac{1}{2} \cdot (\frac{3}{\pi})^{3/2} \cdot n^{-3/2}$$

A walk that starts and ends at $\vec{0}$ takes ℓ steps left and ℓ steps right, h steps up and h steps down, and b steps back and b steps forward, where ℓ, h, and b are non-negative numbers such that $n = \ell + h + b$. The following will be helpful:

- Stirling's approximation
- $\ell!h!b! \geq \frac{n}{3}!\frac{n}{3}!\frac{n}{3}!$
- $3^n = \sum_{\ell+h\leq n} \frac{n!}{\ell!h!b!}$

(3) The Soulmates Problem: Ingrid and Hagrid are simple random walkers starting at different points in the grid $\mathbb{Z}^d$. For which d do they almost surely eventually find one another? That is, for which d is the probability that there exists n such that $I_n = H_n$ equal to 1?

(4) Soulmates II, the Perfume Problem: As above, but additionally Ingrid wears a perfume that lingers at each vertex she visits. For which d does Hagrid almost surely catch a whiff of Ingrid's perfume; that is, for which d is the probability that there exists n such that for some $m \leq n$, $H_n = I_m$ equal to 1?

(5) Try drawing some Cayley graphs, say for the symmetric groups on 2, 3, and 4 elements.

(6) Show that different word metrics on a finitely generated group are biLipschitz equivalent; that is, if G is a group with finite generating sets S_1 and S_2 then there exists L such that for all $g, h \in G$ we have:

$$\frac{1}{L}d_{S_1}(g,h) \leq d_{S_2}(g,h) \leq Ld_{S_1}(g,h)$$

A *quasiisometry* is a map $\phi\colon X \to Y$ between metric spaces such that there exist L and A such that:

- (coarsely biLipschitz) $\forall x_1, x_2 \in X$
$$\frac{1}{L}d_X(x_1,x_2) - A \leq d_Y(\phi(x_1),\phi(x_2)) \leq Ld_X(x_1,x_2) + A$$
- (coarsely surjective) $\forall y \in Y\, \exists x \in X, \quad d_Y(y,\phi(x)) \leq A$

Two maps $\phi, \psi\colon X \to Y$ are *coarsely equivalent* if $\exists A\, \forall x \in X$, $d_Y(\phi(x),\psi(x)) \leq A$.

(7) Show the integer grid $\mathbb{Z}^2$ (with its edge length metric!!) is quasiisometric to the Euclidean plane.

(8) Show that if $\phi\colon X \to Y$ is a quasiisometry then there is a map $\bar{\phi}\colon Y \to X$ such that $\bar{\phi} \circ \phi$ is coarsely equivalent to Id_X and $\phi \circ \bar{\phi}$ is coarsely equivalent to Id_Y. Show that $\bar{\phi}$ is well-defined up to coarse equivalence. Show that $\bar{\phi}$ is a quasiisometry. $\bar{\phi}$ is called a *quasiisometry inverse* of ϕ.

(9) Show that if G is a finitely generated group and H is a finite index subgroup of G then H is finitely generated and for any choice of finite generating sets S of G and T of H the inclusion of H into G is a quasiisometry between $Cay(G,S)$ and $Cay(H,T)$. Describe a quasiisometry inverse.