RANDOM WALKS ON GROUPS, 2023 SS
EXERCISES B

(1) Compute the spectral radius of the simple random walk on the $\mathbb{Z}^d$ grid.

(2) Show the following equivalent formulations of recurrence/transience:

- recurrent $\iff$ $\mathbb{P}$[random walk starting at x never returns to x] $= 0$
- recurrent $\iff$ $\forall x, y \in X,\ F(x,y) = 1$
- recurrent $\iff$ $\forall x, y \in X,\ \mathbb{P}_x[Z_n = y \text{ for infinitely many } n] = 1$
- transient $\iff$ $\forall x \in X\ \forall$ finite $A \subset X,\ \mathbb{P}_x[Z_n \in A \text{ for infinitely many } n] = 0$

(3) Construct an example of a non-recurrent graph in which there exist vertices $x \neq y$ with $F(x,y) = 1$.

(4) Show that if $Cay(G,S)$ is homeomorphic to $\mathbb{R}$ then G is either $\mathbb{Z}$ or the infinite dihedral group D_∞.

$$(x|y)_z := \frac{1}{2}(d(x,z) + d(y,z) - d(x,y))$$

(5) Show that if X is a tree then $(x|y)_z$ is the distance from z to the unique geodesic $[x,y]$ between x and y.

(6) Show that if X is a tree then given any x, y, z there is a unique point $m = m(x,y,z) = [x,y] \cap [y,z] \cap [x,z]$, where $[x,y]$ is the unique geodesic from x to y.

(7) Show the same is true in the grid $\mathbb{Z}^2$, where this time $[x,y]$ denotes the set of all possible geodesics from x to y.