RANDOM WALKS ON GROUPS, 2023 SS
EXERCISES F

(1) A subset $A \subset X$ is μ–*frilly* if for all $x, y \in X$ such that $d(x, A) = r$ and $d(y, A) = r$ and $\text{diam}\pi_A(x) \cup \pi_A(y) > 2r$, every path from x to y that does not enter the r–neighborhood of A has length at least $\mu(r) \cdot d(x, y)$.

Consider the graph that is the 1–skeleton of the $(5, 4)$–tiling of $\mathbb{H}^2$, as in Figure 1. Argue that geodesics

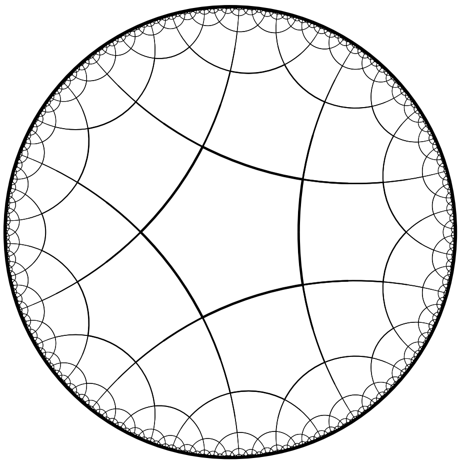

FIGURE 1. $(5, 4)$–tiling of $\mathbb{H}^2$

are μ–frilly for exponential μ. (In this example the projection diameter bound is not important.)

(2) Show that if every geodesic triangle in X is δ–thin then if α and β are geodesic rays based at o and $d_{Haus}(\alpha, \beta) < \infty$ then $d_{Haus}(\alpha, \beta) \leq \delta$.

Recall $d_{Haus}(A, B)$ is the infinum of distance r such that A is contained in the r–neighborhood of B and B is contained in the r–neighborhood of A.

(3) Show that if X has δ–thin triangles then geodesics in X are D–strongly contracting for some D depending only on δ. (Claim: $D = 16\delta$ works.)

Recall that γ is D–strongly contracting if for all $x, y \in X$ with $d(x, y) \leq d(x, \gamma)$ we have $\text{diam}\pi_\gamma(x) \cup \pi_\gamma(y) \leq D$, where $\pi_\gamma : x \mapsto \{p \in \gamma \mid d(x, p) = d(x, \gamma)\}$. Hint: Consider a geodesic square formed by x, y, $x' \in \pi_\gamma(x)$ and $y' \in \pi_\gamma(y)$. Subdivide it into geodesic triangles by adding a geodesic from x to y'.

(4) Suppose γ is D–strongly contracting. Show that for every $x, y \in X$, one of the following is true for $C := 4D$:

- $\text{diam}\pi_\gamma(x) \cup \pi_\gamma(y) \leq C$
- Every geodesic from x to y enters the C–neighborhood of γ.

Hint: Suppose α is a geodesic from x to y that does not enter the C–neighborhood of γ, and $\text{diam}\pi_\gamma(x) \cup \pi_\gamma(y) > C$. Show that $k = \lfloor \frac{d(x,y) - d(x,\gamma) - d(y,\gamma)}{C} \rfloor \geq 1$. Then there are $z_0, z_1, \ldots, z_{k+1}$ on α such that $d(x, \gamma) = d(x, z_0)$, $d(y, \gamma) = d(y, z_{k+1})$, and $d(z_i, z_{i+1}) \leq C$. Derive a contradiction.