# RANDOM WALKS ON GROUPS, 2023 SS
# EXERCISES G

Assume for the first 4 problems that $X$ is a proper geodesic metric space, but **NOT NECESSARILY HYPEBOLIC**.

(1) Show that for $D \geq 0$ there is a $C$ such that if $Y \subset X$ is $D$–strongly contracting and $x \in X$ and $y \in Y$ then:

$$d(x, y) \geq d(x, \pi_Y(x)) + d(\pi_Y(x), y) - C$$

(2) Show the for $D \geq 0$ there exists $\chi$ such that if $Y \subset X$ is $D$–strongly contracting then it is $\chi$–quasi-geodesically quasi-convex.

(3) Show that given $D$, $L$, and $A$ there exist $D'$ and $H$ such that for every $D$–strongly contracting $(L, A)$–quasi-geodesic ray $\gamma$ there exists a geodesic ray $\alpha$ such that $d_{Haus}(\alpha, \gamma) \leq H$ and $\alpha$ is $D'$–strongly contracting.

(4) Suppose that $\phi\colon X \to Y$ is a quasi-isometry between proper geodesic metric spaces. Suppose that $Y$ has the property that given $\chi$ there exists $D$ such that $\chi$–quasi-geodesically quasi-convex sets in $Y$ are $D$–strongly contracting. Show that for all $D_1$ there exists $D_2$ such that $\phi$ sends any $D_1$–strongly contracting geodesic ray in $X$ to within bounded Hausdorff distance of a $D_2$–strongly contracting geodesic ray in $Y$. Remark: The statement is not true without the extra hypothesis on $Y$. The extra hypothesis is true, for example, if $Y$ is hyperbolic or CAT(0).

(5) Suppose $X$ is a proper geodesic space. Let $\mathcal{O}$ and $\mathcal{O}'$ be two different choices of basepoint. Show that the resulting boundaries of $X$ are homeomorphic.

(6) Suppose $X$ is a proper geodesic space. Show that an action $G \curvearrowright X$ by isometries induces an action of $G \curvearrowright \partial X$ by homeomorphisms.