Line Patterns in Free Groups

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includes joint work with
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Basic Terminology

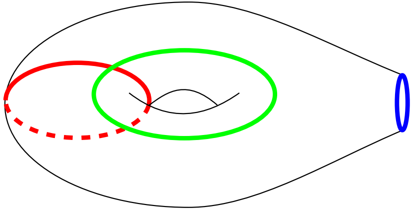
Two sets A and B in a metric space (X, d_X) are *coarsely equivalent* if there is an $\epsilon \geq 0$ so that every point of A is within ϵ of a point of B, and vice versa.

A map $\phi\colon (X, d_X) \to (Y, d_Y)$ is a *quasi-isometry* if there exist $\lambda \geq 1$ and $\epsilon \geq 0$ such that for all x, x' in X

$$\frac{1}{\lambda} d_X(x, x') - \epsilon \leq d_Y(\phi(x), \phi(x')) \leq \lambda d_X(x, x') + \epsilon$$

and $\phi(X)$ is coarsely equivalent to Y.

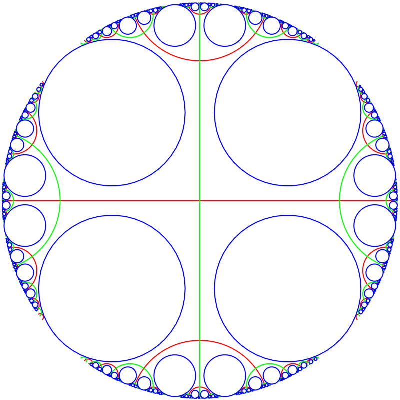
Pseudo-definition



Pick finitely many closed curves in a surface with boundary.

$$\pi_1(\Sigma) = F_2 = \langle a, b \rangle \qquad \underline{w} = \{{\color{red}a}, {\color{green}b}, {\color{blue}ab\bar{a}\bar{b}}\}$$

Pseudo-definition



The line pattern is the collection of all lifts of the closed curves to the universal cover.

Line Patterns

Let $F = F_n$ be a free group of rank $n \geq 2$.

Definition

The line pattern $\mathcal{L}$ generated by a word $w \in F$ is the collection of distinct coarse equivalence classes of sets of the form $\{gw^m \mid m \in \mathbb{Z}\}$ for some $g \in F$.

Similarly, if $\underline{w}$ is a multiword, the line pattern $\mathcal{L}$ generated by $\underline{w}$ is the line pattern generated by all of the $w \in \underline{w}$.

Line Patterns

If we choose a free generating set for F and look at the tree that is the corresponding Cayley graph, the set $\{gw^m \mid g \in F, m \in \mathbb{Z}\}$ looks coarsely like a geodesic in the tree.

Up to coarse equivalence, we might as well assume w is cyclically reduced and not a proper power of any other word. If w_1 and w_2 are words in $\underline{w}$ we may assume that w_1 and w_2 are not conjugates or inverses.

Quasi-Isometric Equivalence?

Given $\mathcal{L}$ a line pattern in F, and $\mathcal{L}'$ a line pattern in F', is there a quasi-isometry $\psi\colon F \to F'$ that takes each line of $\mathcal{L}$ within uniformly bounded distance of a line of $\mathcal{L}'$, and vice versa?

Related question: What is the group $\mathcal{QI}(F, \mathcal{L})$ of quasi-isometries of F that preserve $\mathcal{L}$?

Why We Might Care, I

Consider a group of the form

$$F *_{\mathbb{Z}} G = \left\langle F, G \mid w = w' \right\rangle$$

where $w \in F$ and $w' \in G$ are nontrivial, infinite order words.

If you can show that this splitting is invariant under quasi-isometries then the quasi-isometric equivalence class of the line pattern in F generated by w is a quasi-isometry invariant for the group.

Why We Might Care, II

A multiword $\underline{w}$ in F is *geometric* if it can be realized by an embedded multicurve on the boundary of a 3–dimensional handlebody H with $\pi_1(H) = F$.

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There are some multiwords that are virtually geometric but not geometric: they can be made geometric only after lifting to a finite index subgroup of F.

Why We Might Care, II

Line pattern techniques determine whether or not a multiword is virtually geometric.

Line Patterns and a Theorem of R. Schwartz

Let $\underline{w} = \{w_1, \ldots, w_k\}$ be a collection of words in $\pi_1 M$, where M is a compact hyperbolic orbifold of dimension $n \geq 3$.
Get a line pattern $\mathcal{L}$ in the universal cover $\mathbb{H}^n$ by lifting $\underline{w}$.

Line Patterns and a Theorem of R. Schwartz

Theorem (R. Schwartz)

$$\mathcal{QI}(\mathbb{H}^n, \mathcal{L}) \subset \mathrm{Isom}(\mathbb{H}^n)/\sim$$

Rigid!

Schwartz's Theorem from a Different Point of View

Choose a quasi-isometry $\phi\colon\ \pi_1 M \to \mathbb{H}^n$. Then we have:

$$\begin{tikzcd}
\pi_1 M,\, \mathcal{L} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \pi_1 M,\, \psi(\mathcal{L}) = \mathcal{L} \arrow[d, "\phi"'] \\
\mathbb{H}^n,\, \phi(\mathcal{L}) \arrow[r, "\phi\psi\phi^{-1}"] & \mathbb{H}^n,\, \phi(\mathcal{L})
\end{tikzcd}$$

So

$$\phi\, \mathcal{QI}(\pi_1 M,\, \mathcal{L})\phi^{-1} = \mathrm{Isom}(\mathbb{H}^n,\, \phi(\mathcal{L}))/\sim$$

Rigid Line Pattern

Definition

A line pattern $\mathcal{L}$ in F is *rigid* if there is some space X and a quasi-isometry $\phi\colon F \to X$, such that for all $\psi \in \mathcal{QI}(F, \mathcal{L})$ we have:

$$\phi\psi\phi^{-1} \in \mathrm{Isom}(X, \phi(\mathcal{L}))/\sim$$

(Note that the space X may depend on $\mathcal{L}$.)

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(Note that the space X may depend on $\mathcal{L}$.)

Theorem (C.-Macura)

There is a topological space $\mathcal{D}$ associated to a line pattern $\mathcal{L}$ such that $\mathcal{L}$ is rigid if and only if $\mathcal{D}$ is connected without cut points or cut pairs.

Boundary at Infinity

Pick some free generating set for F. The Cayley graph is a tree T. ∂T is a Cantor set. Quasi-isometries of T (equiv. of F) extend to homeomorphisms of ∂T.

In addition, if $\psi \in \mathcal{QI}(F, \mathcal{L})$ then ψ extends to a homeomorphism $\partial \psi$ of ∂T such that $\forall l \in \mathcal{L}\ \exists l' \in \mathcal{L}$ such that $\partial \psi$ takes the endpoints of l to the endpoints of l'.

The Decomposition Space

Definition

The *decomposition space* associated to a line pattern $\mathcal{L}$ in F is the topological space obtained by identifying the two endpoints of each line l in the line pattern.

$$\mathcal{D} = \partial T \big/ \{l^+ \sim l^- \mid l \in \mathcal{L}\}$$

For $\psi \in \mathcal{QI}(F, \mathcal{L})$, $\partial \psi$ descends to a homeomorphism of the decomposition space.

Relative Splittings

F *splits relative to* $\mathcal{L}$ if there is a free splitting $F = F' * F''$ such each generator of $\mathcal{L}$ is conjugate into F' or F''.

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F *splits over* $\mathbb{Z}$ *relative to* $\mathcal{L}$ if there is a splitting $F = F' *_{\mathbb{Z}} F''$ such each generator of $\mathcal{L}$ is conjugate into F' or F''.

The Decomposition Space Controls Relative Splittings

Fact: F splits relative to $\mathcal{L}$ if and only if $\mathcal{D}$ is disconnected.

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Theorem (C.)

When $\mathcal{D}$ is connected, F splits over $\mathbb{Z}$ relative to $\mathcal{L}$ if and only if $\mathcal{D}$ has cut points or cut pairs.

Relative JSJ Decomposition

Theorem (C.)

Let $\mathcal{L}$ be a line pattern in F such that $\mathcal{D}$ is connected. There is a canonical graph of groups decomposition of F relative to $\mathcal{L}$ satisfying:

1. *Vertex groups alternate between $\mathbb{Z}$ and non-cyclic free groups.*
2. *Edge groups are cyclic.*
3. *For every non-cyclic vertex group G, the induced line pattern in G gives decomposition space either a circle or connected with no cut points and no cut pairs.*

If F splits over $\langle g \rangle$ relative to $\mathcal{L}$ then either g is contained in a cyclic vertex group or a non-cyclic vertex group with circle decomposition space.

Application: Virtually Geometric Multiwords

Virtually geometric multiwords are made up of geometric pieces:

Theorem (C.)

A multiword $\underline{w}$ in F consisting of distinct indivisible words is virtually geometric if and only if the induced multiword in each non-cyclic vertex group of the relative JSJ decomposition is geometric.

Rigidity

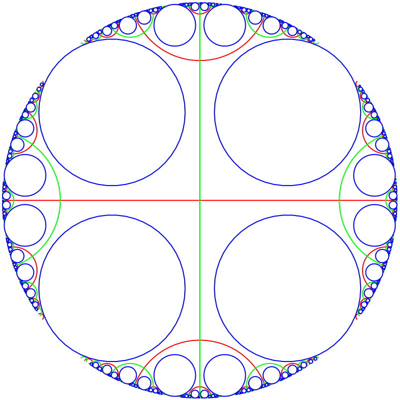
Rigidity fails if $\mathcal{D}$ is disconnected or a circle or if the relative JSJ decomposition is non-trivial. In these cases there is a "Dehn twist" quasi-isometry.

Theorem (C.-Macura)

If $\mathcal{L}$ is a line pattern in F such that $\mathcal{D}$ is connected with no cut points and no cut pairs then $\mathcal{L}$ is rigid. There exists a finite dimensional, locally finite cube complex X and a quasi-isometry $\phi\colon F \to X$ such that for any $\psi \in \mathcal{QI}(F, \mathcal{L})$ we have $\phi\psi\phi^{-1} \in \mathrm{Isom}(X, \phi(\mathcal{L}))$.

Decomposition space of $ab\bar{a}\bar{b}$

Consider the pattern generated by $ab\bar{a}\bar{b}$ in F_2 (blue curves only).
The decomposition space is a circle, same as $\partial\,\mathbb{H}^2$.



Decomposition space of $ab\bar{a}\bar{b}$

In fact, this is the only way to get a circle:

Theorem (Otal, C.-Macura)

The decomposition space of a line pattern generated by $\underline{w}$ is a circle if and only if the words of $\underline{w}$ are the boundary curves of a surface with fundamental group F.

Decomposition space of $\{ab\bar{a}\bar{b},\, a\}$

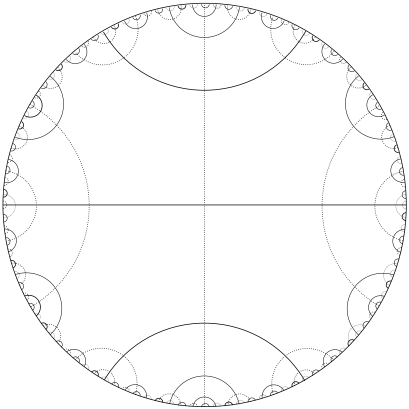


Figure: Line Pattern

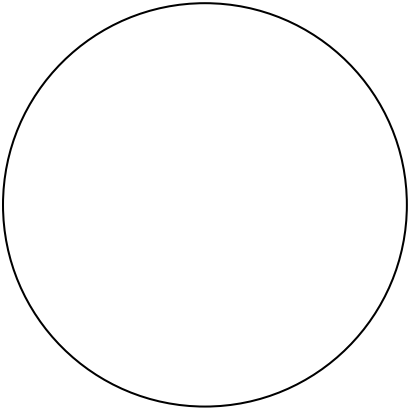


Figure: Decomposition Space

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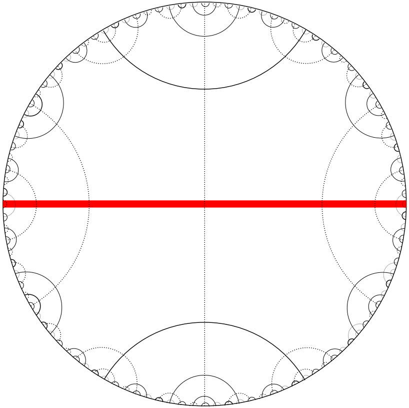

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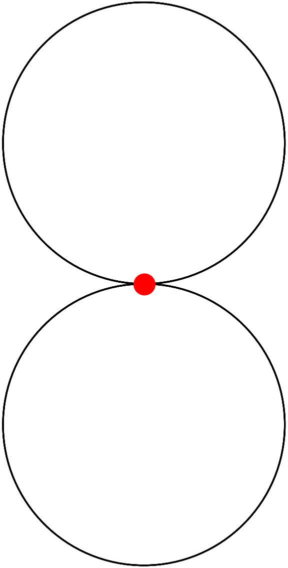

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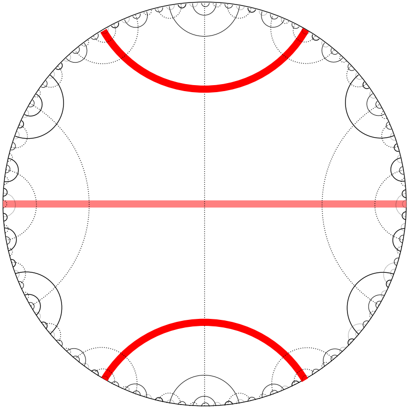

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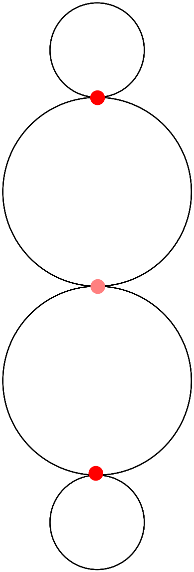

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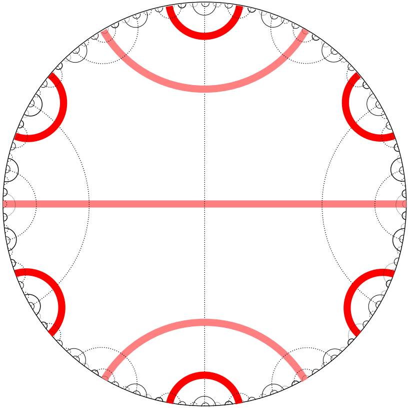

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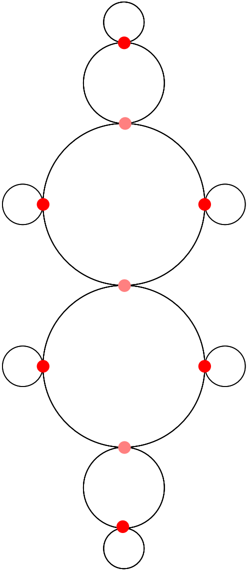

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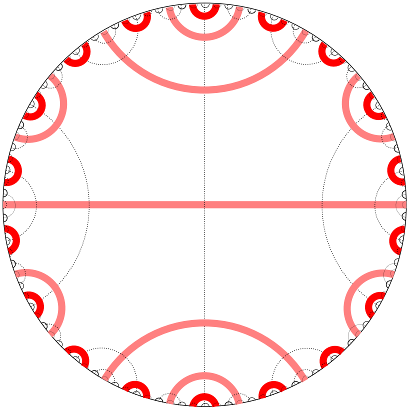

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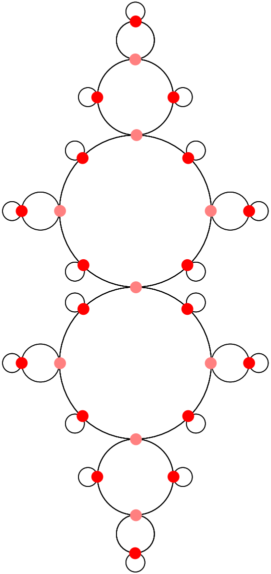

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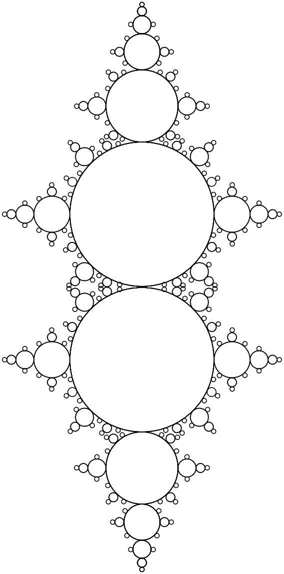
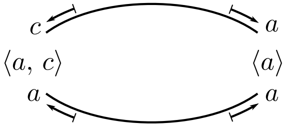


rJSJ Decomposition

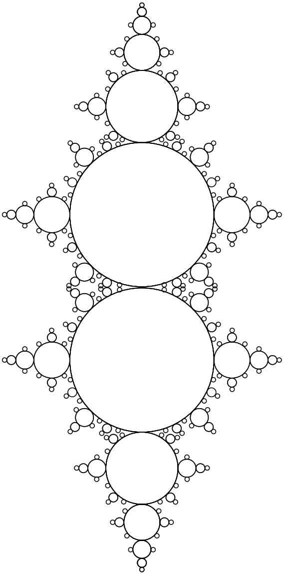
The JSJ Decomposition for $F = \langle a, b \rangle$ relative to $\{ab\bar{a}\bar{b},\, a\}$ is



where $c = b\bar{a}\bar{b}$.

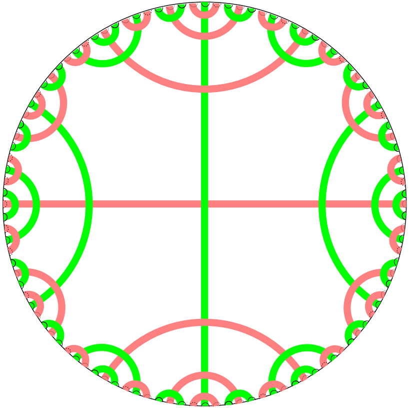
The induced line pattern in $\langle a, c\rangle$ is generated by $\{a, c, ac\}$ and gives a circle for the decomposition space.

Decomposition Space of $\{ab\bar{a}\bar{b},\, a\}$



The obvious circles in the figure are the decomposition spaces of conjugates of the vertex group $\langle a, c\rangle$.
The point where two circles meet is a cut point corresponding to a conjugate of $\langle a\rangle$.
The tree-of-circles structure mirrors the Bass-Serre tree of the rJSJ.

Decomposition Space of $\{ab\bar{a}\bar{b},\, a,\, b\}$



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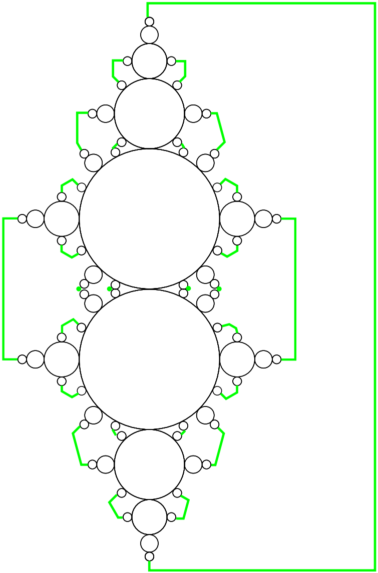


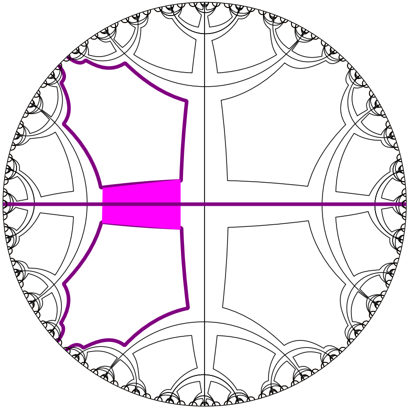
Figure: Decomposition Space

A Rigid Pattern

The pattern generated by $\{ab\bar{a}\bar{b},\, a,\, b\}$ is rigid, and the cube complex X is just the Cayley graph of F_2 with respect to the free generating set $\{a,\, b\}$.

Key Observation:
Edges of the tree are in one to one correspondence with 3-point cut sets of the decomposition space.

Tree vs. Cut Sets in Decomp. Space



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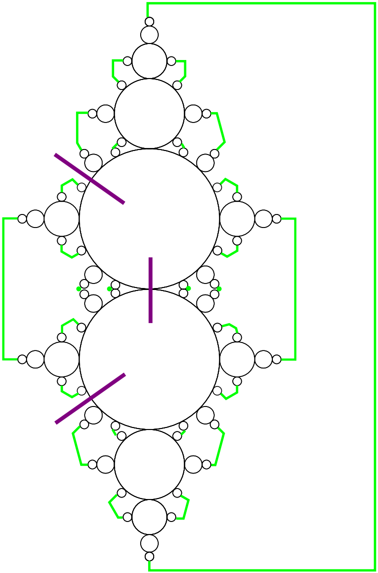

Figure: Decomposition Space

Constructing a Cube Complex

When the decomposition space is connected with no cut points and no cut pairs, small cut sets are well behaved.

Use Sageev construction to build a cube complex encoding the chosen collection of cut sets.

Pattern preserving quasi-isometries induce homeomorphisms of the decomposition space, which preserve the collection of small cut sets. This gives an automorphism of the cube complex.

Whitehead Graph

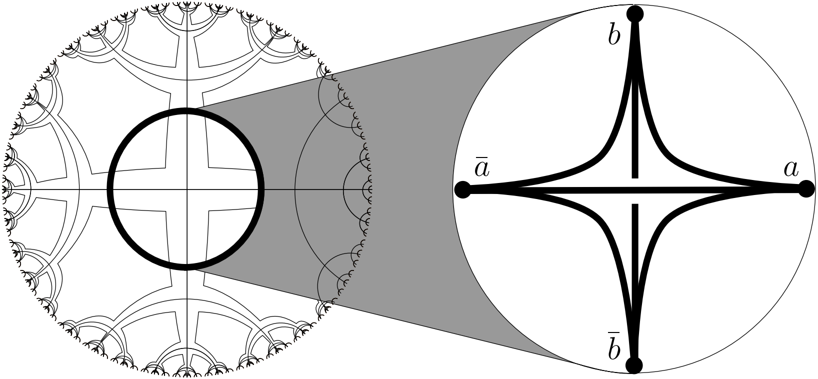

Figure: Whitehead graph from line pattern

Whitehead Graph

The main tool for understanding the topology of the decomposition space is a generalization of the Whitehead graph.

Whitehead used this graph to decide if there is an automorphism of the free group that takes one given multiword to another. (Whitehead's Algorithm, 1936)

A quasi-isometry matching line patterns is a geometric version of an automorphism matching multiwords.