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GROWTH TIGHT ACTIONS OF PRODUCT GROUPS

CHRISTOPHER H. CASHEN AND JING TAO

Abstract. A group action on a metric space is called growth tight if the exponential growth rate of the group with respect to the induced pseudo-metric is strictly greater than that of its quotients. A prototypical example is the action of a free group on its Cayley graph with respect to a free generating set. More generally, with Arzhantseva we have shown that group actions with strongly contracting elements are growth tight.

Examples of non-growth tight actions are product groups acting on the L^1 products of Cayley graphs of the factors.

In this paper we consider actions of product groups on product spaces, where each factor group acts with a strongly contracting element on its respective factor space. We show that this action is growth tight with respect to the L^p metric on the product space, for all $1 < p \leq \infty$. In particular, the L^∞ metric on a product of Cayley graphs corresponds to a word metric on the product group. This gives the first examples of groups that are growth tight with respect to an action on one of their Cayley graphs and non-growth tight with respect to an action on another, answering a question of Grigorchuk and de la Harpe.

1. Introduction

The *growth exponent* of a set A with respect to a pseudo-metric d is

$$\delta_{A,d} = \limsup_{r\to\infty} \frac{1}{r} \cdot \log \#\{a \in A \mid d(o,a) \leq r\}$$

where # denotes cardinality and $o \in A$ is some basepoint. The limit is independent of the choice of basepoint.

Let G be a finitely generated group, and let $(\mathcal{X}, d, o)$ be a proper, based, geodesic metric space on which G acts properly discontinuously and cocompactly by isometries.

The metric d induces a left invariant pseudo-metric $\bar{d}$ on any quotient G/N of G by $\bar{d}(gN, g'N) = \min_{n,n'\in N} d(gn.o, g'n'.o)$. When $(\mathcal{X}, d, o)$ is clear we let $\delta_{G/N}$ denote $\delta_{G/N,\bar{d}}$ and let δ_G denote $\delta_{G/\{1\},\bar{d}}$.

Definition 1.1 ([1])**.** $G \curvearrowright \mathcal{X}$ is a *growth tight action* if $\delta_G > \delta_{G/N}$ for every infinite normal subgroup $N \trianglelefteq G$.

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If S is a finite generating set of G, we say G is *growth tight with respect to S* if the action of G via left multiplication on the Cayley graph of G with respect to S is growth tight.

The first examples of such actions were given by Grigorchuk and de la Harpe [9], who showed that a finite rank, non-abelian free group $\mathbb{F}$ is growth tight with respect to any free generating set S. In the same paper, they observe that the product $\mathbb{F}\times\mathbb{F}$ is not growth tight with respect to the generating set $S\times\{1\}\cup\{1\}\times S$, and ask whether there exists a finite generating set with respect to which $\mathbb{F}\times\mathbb{F}$ is growth tight.

We answer this question affirmatively. This is the first example of a group that is growth tight with respect to one generating set and not growth tight with respect to another.

Our main result is for growth tightness of product groups $G_1 \times \cdots \times G_n$. We require that each factor G_i acts cocompactly with a strongly contracting element on a space $\mathcal{X}_i$, see Definition 2.2. Examples include actions of hyperbolic or relatively hyperbolic groups by left multiplication on any of their Cayley graphs, and groups acting cocompactly on proper CAT(0) spaces with rank 1 isometries. With Arzhantseva [1], we have shown that such actions are growth tight.

Theorem 1.2. *For $1 \leq i \leq n$, let G_i be a non-elementary, finitely generated group acting properly discontinuously and cocompactly by isometries on a proper, based, geodesic metric space $(\mathcal{X}_i, d_i, o_i)$ with a strongly contracting element. Let $G = G_1 \times \cdots \times G_n$. Let $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, with $o = (o_1, \ldots, o_n)$ and let d be the L^p metric on $\mathcal{X}$ for some $1 \leq p \leq \infty$. Let $G \curvearrowright \mathcal{X}$ be the coordinate-wise action. Then $G \curvearrowright \mathcal{X}$ is growth tight unless $p = 1$ and $n > 1$.*

Remark. Cocompactness of the factor actions is not strictly necessary. We use it to prove a subadditivity result, Lemma 4.4. There are weaker conditions than cocompactness of the action that can be used to prove such a result. These are discussed in [1, Section 6]. For simplicity, we will stick to cocompact actions in this paper, since this suffices for our main applications.

In the case that $\mathcal{X}_i$ is the Cayley graph of G_i with respect to a finite, symmetric generating set S_i, there is a natural bijection between vertices of $\mathcal{X}$ and elements of G. This bijection is an isometry between vertices of $\mathcal{X}$ with the L^1 metric and elements of G with the word metric corresponding to the generating set:

$$S^1 = \bigcup_{1 \leq i \leq n} \{(s_1, \ldots, s_n) \mid s_j = 1 \text{ for } j \neq i \text{ and } s_i \in S_i\}$$

The same bijection is also an isometry between vertices of $\mathcal{X}$ with the L^∞ metric and elements of G with the word metric corresponding to the generating set:

$$S^\infty = \left\{(s_1, \ldots, s_n) \mid s_i \in S_i \cup \{1\}\right\}$$

Corollary 1.3. *For $1 \leq i \leq n$, let G_i be a non-elementary group with a finite, symmetric generating set S_i. Let $\mathcal{X}_i$ be the Cayley graph of G_i with respect to S_i, and suppose that the action of G_i on $\mathcal{X}_i$ by left multiplication has a strongly contracting element. When $n \geq 2$, the product $G = G_1 \times \cdots \times G_n$ admits a finite generating set S^1 for which the action on the corresponding Cayley graph is* *not growth tight and another finite generating set S^∞ for which the action on the corresponding Cayley graph is growth tight.*

Non-elementary, finitely generated, relatively hyperbolic groups, and finite rank free groups in particular, act with a strongly contracting element on any one of their Cayley graphs, so:

Corollary 1.4. *If $\mathbb{F}$ is a finite rank free group and S is a finite, symmetric free generating set of $\mathbb{F}$ then $\mathbb{F}\times\mathbb{F}$ is growth tight with respect to the generating set $(S\cup\{1\})\times(S\cup\{1\})$.*

Another common way to think of $\mathbb{F}\times\mathbb{F}$ is as the Right Angled Artin Group with defining graph the join of two sets of vertices of cardinality equal to the rank of $\mathbb{F}$. The universal cover of the corresponding Salvetti complex is the product of Cayley graphs of $\mathbb{F}$ with respect to free generating sets. There are two natural metrics to consider on the vertex set of the universal cover of the Salvetti complex: the induced length metric from the piecewise Euclidean structure, which is the restriction of the L^2 metric on the product, and the induced length metric in the 1–skeleton, which is the restriction of the L^1 metric on the product.

Corollary 1.5. *The action of $\mathbb{F}\times\mathbb{F}$ on the universal cover of its Salvetti complex is growth tight with respect to the piecewise Euclidean metric but not growth tight with respect to the 1–skeleton metric.*

We sketch a direct proof of Corollary 1.4. The proof of Theorem 1.2 follows the same outline.

Sketch proof of Corollary 1.4. Let $\mathcal{X}$ be the Cayley graph of $\mathbb{F}$ with respect to S. Let $G=\mathbb{F}\times\mathbb{F}$ be generated by $(S\cup\{1\})\times(S\cup\{1\})$, which induces the L^∞ metric on $\mathcal{X}\times\mathcal{X}$. We have $\delta_G=2\delta_{\mathbb{F}}>0$.

Let N be a non-trivial normal subgroup of G. If N has trivial projection to, say, the first factor, then $G/N=\mathbb{F}\times(\mathbb{F}/\pi_2(N))$. Since $\mathbb{F}$ is growth tight with respect to every word metric, $\delta_{\mathbb{F}/\pi_2(N)}<\delta_{\mathbb{F}}$, so $\delta_{G/N}=\delta_{\mathbb{F}}+\delta_{\mathbb{F}/\pi_2(N)}<2\delta_{\mathbb{F}}=\delta_G$.

If N has non-trivial projection to both factors, then there is an element $(h_1,h_2)\in N$ with both coordinates non-trivial. For each $(a_1,a_2)N\in(\mathbb{F}\times\mathbb{F})/N=G/N$, choose an element $(a_1',a_2')\in(a_1,a_2)N$ such that

$$d\big((a_1',a_2'),(1,1)\big)=d\big((a_1,a_2)N,(1,1)\big).$$

Let $A=\{(a_1',a_2')\mid(a_1,a_2)N\in G/N\}$. We call A a *minimal section of the quotient map*. We have $\delta_{A,d}=\delta_{G/N,\bar{d}}$.

Given a non-trivial, reduced word f, let $W(f)$ be the subset of elements of $\mathbb{F}$ whose expression as a reduced word in S contains f as a subword. Denote by $\bar{a}$ the inverse of a word a in $\mathbb{F}$. If $(a_1',a_2')\in W(h_1)\times W(h_2)$ then there exist b_i and c_i such that $a_i'=b_ih_ic_i$ for $i=1$, 2, and

$$(a_1',a_2')=(b_1h_1c_1,b_2h_2c_2)=(b_1c_1,b_2c_2)\cdot(\overline{c_1}h_1c_1,\overline{c_2}h_2c_2)$$

So $(b_1c_1,b_2c_2)N=(a_1',a_2')N$, but this contradicts the fact that $(a_1',a_2')\in A$, since $|(b_1c_1,b_2c_2)|_\infty<|(a_1',a_2')|_\infty$. Therefore, $A\subset(\mathbb{F}-W(h_1))\times\mathbb{F}\cup\mathbb{F}\times(\mathbb{F}-W(h_2))$. However, for any non-trivial f the growth exponent of $\mathbb{F}-W(f)$ is strictly less than that of $\mathbb{F}$, so the growth exponent of A is strictly less than that of $\mathbb{F}\times\mathbb{F}$. $\square$

The fact that the growth exponent of $F - W(f)$ is strictly less than that of F has analogues in formal language theory. A language $\mathcal{L}$ over a finite alphabet is known as 'growth-sensitive' or 'entropy-sensitive' if for every finite set of words in $\mathcal{L}$, called the *forbidden words*, the sub-language of words that do not contain one of the forbidden words as a subword has strictly smaller growth exponent than $\mathcal{L}$. It has been a topic of recent interest to decide what kinds of languages are growth-sensitive [5, 6, 10].

Our approach to growth tightness is to prove a coarse-geometric version of growth sensitivity, where the forbidden word is a power of a strongly contracting element.

The first coarse-geometric version of growth sensitivity was used by Arzhantseva and Lysenok [2] to prove growth tightness for hyperbolic groups. With Arzhantseva, [1] we gave a more general construction that applied to group actions with strongly contracting elements. The idea is that the action of a strongly contracting element closely resembles the action of an infinite order element of a hyperbolic group on a Cayley graph.

In [1] we proved a coarse-geometric version of the statement that the growth exponent of the set of reduced words in $\mathbb{F}$ that do not contain f or $\overline{f}$ as subwords is strictly less than the growth exponent of $\mathbb{F}$. For products this is not enough, since, for example, if $(f, f) \in N \trianglelefteq \mathbb{F} \times \mathbb{F}$ we cannot make the element $(f, \overline{f})$ shorter by applying powers of (f, f). We really want to forbid only positive occurrences of f in each coordinate, so we need to strengthen our coarse-geometric statement to take orientation into account.

After preliminaries in Section 2, we show in Section 3 that an infinite normal subgroup of G that has infinite projection to each factor contains an element $\underline{h}$ for which each coordinate is strongly contracting for the action of the factor group on the factor space.

In Section 4 we prove the main technical lemma, Lemma 4.7, which is our oriented growth sensitivity result.

In Section 5 we complete the proof of Theorem 1.2.

2. Preliminaries

For any group G, we use $\bar{g}$ to denote the multiplicative inverse of $g \in G$.

A group is *elementary* if it is finite or has an infinite cyclic subgroup of finite index.

A *quasi-map* $\pi\colon \mathcal{X} \to \mathcal{Y}$ between metric spaces assigns to each point $x \in \mathcal{X}$ a subset $\pi(x) \subset \mathcal{Y}$ of uniformly bounded diameter.

2.1. Strongly Contracting Elements.

We define strongly contracting elements following Sisto[1] [11]. See also [1] for additional reference.

Definition 2.1. Let $(\mathcal{X}, d)$ be a proper geodesic metric space, and let $\mathcal{A} \subset \mathcal{X}$ be a subset. Given a constant $C > 0$, a map $\pi_{\mathcal{A}}\colon \mathcal{X} \to \mathcal{A}$ is called a *C–strongly contracting projection* if $\pi_{\mathcal{A}}$ satisfies the following properties:

[1]Sisto considers '$\mathcal{PS}$–contracting projections'. We use 'strong' to indicate the special case that $\mathcal{PS}$ is the collection of all geodesic segments in $\mathcal{X}$.

- For every $a \in \mathcal{A}$, $d\big(a, \pi_{\mathcal{A}}(a)\big) \leq C$.
- For every $x, y \in \mathcal{X}$, if $d\big(\pi_{\mathcal{A}}(x), \pi_{\mathcal{A}}(y)\big) > C$, then for every geodesic segment $\mathcal{P}$ with endpoints x and y, we have $d(\pi_{\mathcal{A}}(x), \mathcal{P}) \leq C$ and $d(\pi_{\mathcal{A}}(y), \mathcal{P}) \leq C$.

We say the map $\pi_{\mathcal{A}}$ is a *strongly contracting projection* if it is C–strongly contracting for some $C > 0$.

Fix a base point $o \in \mathcal{X}$. Let G be a finitely-generated group that admits a proper, cocompact, and isometric action on $\mathcal{X}$.

Definition 2.2. An element $h \in G$ is *C–strongly contracting* if $i \mapsto h^i.o$ is a quasi-geodesic and if there exists $C > 0$ such that, for every geodesic segment $\mathcal{P}$ with endpoints on $\langle h \rangle.o$, there exists a C–strongly contracting projection $\pi_{\mathcal{P}} \colon \mathcal{X} \to \mathcal{P}$.

An element $h \in G$ *strongly contracting* if there exists a $C > 0$ such that h is C–strongly contracting.

The property of strongly contracting is independent of the base point o. Since the action is by isometries, a conjugate of a strongly contracting element is strongly contracting.

Let $h \in G$ be a strongly contracting element. Let $E(h) < G$ be the subgroup such that $g \in E(h)$ if and only if the Hausdorff distance between $\langle h \rangle.o$ and $g\langle h \rangle.o$ is bounded. Then $E(h)$ is *hyperbolically embedded* in the sense of Dahmani-Guirardel-Osin [7], and $E(h)$ is the unique maximal virtually cyclic subgroup containing h [7, Lemma 6.5]. Thus, $E(h)$ is the subgroup that often called the *elementarizer* or *elementary closure* of $\langle h \rangle$.

Definition 2.3. Given a strongly contracting element $h \in G$ and a point $o \in \mathcal{X}$, the set $\mathcal{H} = E(h).o$ is called a *quasi-axis* in $\mathcal{X}$ for h.

Lemma 2.4 ([1, Lemma 2.20])**.** *If $h \in G$ is strongly contracting, then there exists a strongly contracting projection quasi-map $\pi_{\mathcal{H}} \colon G \to \mathcal{H}$ such that $\pi_{\mathcal{H}}$ is $E(h)$–equivariant.*

Definition 2.5. If $h \in G$ is strongly contracting and $g \notin E(h)$ define $\pi_{g\mathcal{H}} \colon \mathcal{X} \to g\mathcal{H}$ by $\pi_{g\mathcal{H}}(x) = g.\pi_{\mathcal{H}}(\bar{g}.x)$.

Combining Lemma 2.4 and Definition 2.5, we may assume that the strongly contracting projection quasi-maps $\pi_{g\mathcal{H}}$ to translates of $\mathcal{H}$ are G–equivariant.

Lemma 2.6. *If $h \in G$ is C–strongly contracting there exist non-negative constants λ, ϵ, and μ such that $i \to h^i.o$ is a (λ, ϵ)–quasi-geodesic and for $0 \leq \alpha \leq \beta$ every geodesic from o to $h^\beta.o$ passes within distance μ of $h^\alpha.o$.*

Proof. There exist λ and ϵ such that $i \to h^i.o$ is a (λ, ϵ)–quasi-geodesic by definition of contracting element. Let γ be a geodesic segment from o to $h^\beta.o$. Then γ is Morse, by [11, Lemma 2.9]. Thus, there is a μ depending only on C, λ, and ϵ such that every (λ, ϵ)–quasi-geodesic segment with endpoints on γ is contained in the μ–neighborhood of γ. But $i \mapsto h^i.o$ for $i \in [0, \beta]$ is such a (λ, ϵ)–quasi-geodesic, so there is a point of γ at distance at most μ from $h^\alpha.o$. $\square$

2.2. **Actions on Quasi-trees.** Let h be a contracting element for $G \curvearrowright \mathcal{X}$ as in the previous section, and let $\mathcal{H}$ be the quasi-axis of h.

In Lemma 4.7 we will consider a *free product subset*

$$Z^* * h^m = \bigcup_{i=1}^{\infty} \{z_1 h^m \cdots z_i h^m \mid z_j \in Z - \{1\}\}$$

for a certain subset $Z \subset G$ and a sufficiently large m. We wish to know that the orbit map from G into $\mathcal{X}$ is an embedding on this free product set.

This statement recalls the following well known result:

Proposition 2.7 (Baumslag's Lemma [3])**.** *If $z_1, \ldots, z_k$ and h are elements of a free group such that h does not commute with any of the z_i, then $z_1 h^{m_1} \cdots z_k h^{m_k} \neq 1$ if all the $|m_i|$ are sufficiently large.*

A convenient way to prove such an embedding result is to work in a tree, so that the global result, that $z_1 h^{m_1} \cdots z_k h^{m_k} \neq 1$, can be certified by a local 'no-backtracking' condition. In our situation, we do not have an action on a tree to work with, but a construction of Bestvina, Bromberg, and Fujiwara [4] produces an action of G on a quasi-tree, a space quasi-isometric to a simplicial tree, from the action of G on the G–translates of $\mathcal{H}$. In [1] we use this quasi-tree construction and a no-backtracking argument to prove that the orbit map is an embedding of a certain free product subset. The proof of Lemma 4.7 consists of choosing an appropriate free product set to which we can apply the argument from [1]. The details of the construction of the quasi-tree and the proof of the free product subset embedding are somewhat technical, so we will not repeat them here (see [4, Section 3] and [1, Section 2.4] for more details). However, we will make use of some of Bestvina, Bromberg, and Fujiwara's 'projection axioms', which hold for quasi-axes of contracting elements by work of Sisto [11], as recounted below.

Let $\mathbb{Y}$ be the collection of all distinct G–translates of $\mathcal{H}$. For each $Y \in \mathbb{Y}$, let π_Y be the projection map from the above. Set

$$d^\pi_Y(x, y) = \operatorname{diam}\{\pi_Y(x), \pi_Y(y)\}.$$

Lemma 2.8 ([1, Section 2.4], cf [11, Theorem 5.6])**.** *There exists $\xi \geqslant 0$ such that for all distinct $X, Y, Z \in \mathbb{Y}$:*

- (P0) $\operatorname{diam} \pi_Y(X) \leqslant \xi$
- (P1) *At most one of $d^\pi_X(Y, Z)$, $d^\pi_Y(X, Z)$ and $d^\pi_Z(X, Y)$ is strictly greater than ξ.*
- (P2) $|\{V \in \mathbb{Y} \mid d^\pi_V(X, Y) > \xi\}| < \infty$

3. Elements that are Strongly Contracting in each Coordinate

Let G be a finitely generated, non-elementary group acting properly discontinuously and cocompactly by isometries on a based proper geodesic metric space $(\mathcal{X}, d, o)$ such that there exists an element $h \in G$ that is strongly contracting for $G \curvearrowright \mathcal{X}$. Let $\mathcal{H} = E(h).o$. Let C be the contraction constant for $\pi_{\mathcal{H}}$ from Lemma 2.4, and let ξ be the constant of Lemma 2.8. For any $x \in \mathcal{X}$ and any $r > 0$), denote by $B_r(x)$ the open ball of radius r about x.

Lemma 3.1. *Let p be a point of $\mathcal{H}$. Let g be an element of G. There exists a constant D such that either some non-trivial power of g is contained in $\langle h\rangle$ or for all $n > 0$ we have $d^{\pi}_{\mathcal{H}}(g^n.p, p) \leq 2d(p, g.p) + D$.*

Proof. Since $\langle h\rangle$ is a finite index subgroup of $E(h)$, if some non-trivial power of g is contained in $E(h)$ then some non-trivial power of g is contained in $\langle h\rangle$, and we are done. Thus, we may assume that no non-trivial power of g is contained in $E(h)$. This implies that if $m \neq n$ then $g^m\mathcal{H} \neq g^n\mathcal{H}$.

Let z be a point on a geodesic from p to $g.p$ in $B_C(\pi_{\mathcal{H}}(g.p))$. Let ξ be the constant of Lemma 2.8. Axiom (P0) of Lemma 2.8 says $\operatorname{diam} \pi_{\mathcal{H}}(g\mathcal{H}) \leq \xi$.

$$\begin{aligned} d(p, g.p) &= d(p, z) + d(z, g.p) \\ &\geq d\big(p, \pi_{\mathcal{H}}(g.p)\big) - C + d(z, g.p) \\ &\geq d^{\pi}_{\mathcal{H}}(p, g\mathcal{H}) - C - \xi + d(z, g.p) \\ &\geq d^{\pi}_{\mathcal{H}}(p, g\mathcal{H}) - C - \xi \end{aligned}$$

By a similar argument, for every $k \neq 0, \pm 1$,

$$d(p, g.p) \geq d^{\pi}_{g^k\mathcal{H}}(\mathcal{H}, g^{\pm 1}\mathcal{H}) - 2C - 2\xi$$

Using the above we obtain that, for any $n > 1$,

$$\begin{aligned} d^{\pi}_{\bar{g}\mathcal{H}}(\mathcal{H}, g^{n-1}\mathcal{H}) = d^{\pi}_{\mathcal{H}}(g\mathcal{H}, g^n\mathcal{H}) &\geq d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) - d^{\pi}_{\mathcal{H}}(g\mathcal{H}, p) \\ &\geq d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) - d(g.p, p) - C - \xi \end{aligned}$$

Suppose that $d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) - d(g.p, p) - C - \xi > \xi$. The previous inequality says $d^{\pi}_{\bar{g}\mathcal{H}}(\mathcal{H}, g^{n-1}\mathcal{H}) > \xi$, so (P1) of Lemma 2.8 implies $\xi \geq d^{\pi}_{\mathcal{H}}(\bar{g}\mathcal{H}, g^{n-1}\mathcal{H})$, hence:

$$\begin{aligned} \xi \geq d^{\pi}_{\mathcal{H}}(\bar{g}\mathcal{H}, g^{n-1}\mathcal{H}) &\geq d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) - d^{\pi}_{\mathcal{H}}(p, \bar{g}\mathcal{H}) - d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, g^{n-1}\mathcal{H}) \\ &\geq d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) - d^{\pi}_{\mathcal{H}}(p, \bar{g}\mathcal{H}) - d^{\pi}_{\bar{g}^n\mathcal{H}}(\mathcal{H}, \bar{g}\mathcal{H}) \\ &\geq d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) - 2d(g.p, p) - 3C - 3\xi \end{aligned}$$

Thus, $d^{\pi}_{\mathcal{H}}(g^n\mathcal{H}, p) \leq 2d(g.p, p) + D$ for $D = 3C + 4\xi$. □

Lemma 3.2. *For every $g \in G$ there exists an $l > 0$ and an $n' \geq 0$ such that for all $m > 0$ and all $n \geq n'$, except possibly one, the elements $g^{lm}h^n$ and $h^n g^{lm}$ are strongly contracting.*

Proof. Suppose there exists a minimal $a > 0$ and b such that $g^a = h^b$. If $b > 0$ let $l = a$ and let $n' = 0$, so that $g^{lm}h^n = h^{bm+n}$ is a positive power of h. If $b = 0$ let $l = a$ and $n' = 1$ so that $g^{lm}h^n = h^n$ is a positive power of h. If $b < 0$ let $l = a$, $n' = 0$, and $n \geq n'$ such that $n \neq -mb$. Then $g^{lm}h^n$ is a non-zero power of h.

If no non-trivial power of g is contained in $\langle h\rangle$, let $l = 1$. By Lemma 3.1, there exists a D' such that for every $p \in \mathcal{H}$ and every $m > 0$ we have $d^{\pi}_{\mathcal{H}}(g^m.p, p) \leq 2d(p, g.p) + D'$. Let D be the maximum of D' and the constant D from [11, Lemma 5.2]. Let $p \in \mathcal{H}$ be a point such that $d(p, g.p)$ is minimal. Let n' be large enough so that $d(p, h^{n'}.p) \geq 4d(p, g.p) + 3D$. Then for $n \geq n'$ we have $d(p, h^n.p) \geq d(\pi_{\mathcal{H}}(g^m.p), p) + d(p, \pi_{\mathcal{H}}(\bar{g}^m.p)) + D$. This implies $g^{lm}h^n$ is strongly contracting by [11, Lemma 5.2]. $h^n g^{lm}$ is also strongly contracting as it is conjugate to $g^{lm}h^n$. □

For $i = 1, \dots, n$, let G_i be a non-elementary group acting properly discontinuously and cocompactly by isometries on a proper, based, geodesic metric space $(\mathcal{X}_i, d_i, o_i)$. Assume, for each i, that $G_i \curvearrowright \mathcal{X}_i$ has a strongly contracting element. Let $G = G_1 \times \cdots \times G_n$. Let $\chi_i \colon G \to G_i$ be projection to the i–th coordinate.

Lemma 3.3. *Let N be an infinite normal subgroup of G such that $\chi_i(N)$ is infinite for all i. There exists an element $\underline{h} = (h_1, \dots, h_n) \in N$ such that h_i is a strongly contracting element for $G_i \curvearrowright X_i$.*

Proof. $\chi_i(N)$ is an infinite normal subgroup of G_i, so it contains a strongly contracting element by [1, Proposition 3.1]. For each i, let $\underline{g}_i = (a_{i,1}, \dots, a_{i,n}) \in N$ such that $a_{i,i}$ is a strongly contracting element for $G_i \curvearrowright \mathcal{X}_i$.

We will show by induction that there is a product of the $\underline{g}_i$ that gives the desired element $\underline{h}$. The element $\underline{g}_1$ has a strongly contracting element in its first coordinate. Suppose that there is a product $\underline{f} = (f_1, \dots, f_n)$ of $\underline{g}_1, \dots, \underline{g}_i$ such that the first i coordinates are strongly contracting elements in their coordinate spaces.

For $1 \leq j \leq i$ there exists an l_j and an n'_j as in Lemma 3.2 such that for all m and all $n \geq n'_j$, except possibly one, we have $a_{i+1,j}^{l_j m} f_j^n$ is strongly contracting. Similarly, there are l_{i+1} and n'_{i+1} such that $a_{i+1,i+1}^n f_{i+1}^{l_{i+1} m}$ is strongly contracting for all $m > 0$ and $n \geq n'_{i+1}$.

Let l be the least common multiple of $l_1, \dots, l_i$. Let m be large enough so that $ml \geq n'_{i+1}$. Let $\lambda_k = l_{i+1}(k + \max_{j=1,\dots,i} n'_j)$, where $k \geq 0$ varies.

Consider $\underline{g}_{i+1}^{ml} \underline{f}^{\lambda_k}$. For $1 \leq j \leq i$, the j-th coordinate is strongly contracting for all except possibly one value of k, since ml is a multiple of l_j and $\lambda_k \geq n'_j$. Similarly, the $(i+1)$–st coordinate is strongly contracting for all except possibly one value of k since λ_k is a multiple of l_{i+1} and $ml \geq n'_{i+1}$. By choosing a k that is not among the at most $i+1$ forbidden values, we have that the first $i+1$ coordinates of $\underline{g}_{i+1}^{ml} \underline{f}^{\lambda_k}$ are strongly contracting in their coordinate space. $\square$

We will say an element $g \in G_i$ has a K–long h_i–projection if there exists an $f \in G_i$ such that $d^{\pi}_{f\mathcal{H}_i}(o_i, g.o_i) \geq K$.

Lemma 3.4. *Given $\underline{h}$ as in Lemma 3.3, there exists an element $\underline{h'} = (h'_1, \dots, h'_n) \in N$ such that h'_i is strongly contracting for each $G_i \curvearrowright X_i$ and there exists a K such that powers of h'_i have no K–long h_i–projections and powers of h_i have no K–long h'_i–projections.*

Proof. For each i, the group G_i is non-elementary, so there exists a $g_i \in G_i - E(h_i)$. Let $\underline{g} = (g_1, \dots, g_n)$. [1, Proposition 3.1] shows that $\underline{h'} = \underline{g}\underline{h}^m \bar{\underline{g}}\bar{\underline{h}}^m \in N$ is strongly contracting in each coordinate for any sufficiently large m, so K can be taken to be $\max_i d^{\pi}_{g_i\mathcal{H}_i}(\mathcal{H}_i, g_i h_i^m \bar{g}_i \mathcal{H}_i) + 2\xi_i$, where ξ_i is chosen by Lemma 2.8. $\square$

4. Elements without Long, Positive Projections

In the following, let G be any finitely generated, non-elementary group (not necessarily a product) acting properly discontinuously and cocompactly by isometries on a based proper geodesic metric space $(\mathcal{X}, d, o)$. Suppose there exists a strongly contracting element $h \in G$ for $G \curvearrowright \mathcal{X}$. Let $\mathcal{H} = E(h).o$ and let C be the contraction constant for $\pi_{\mathcal{H}}$.

Let $D = \operatorname{diam}(G\backslash\mathcal{X})$ and let $D' = \operatorname{diam}\big(\langle h\rangle \backslash\mathcal{H}\big)$.

Definition 4.1. For x_0 and x_1 in $\mathcal{X}$, the ordered pair (x_0, x_1) has a *K–long, positive h–projection* if there exists a $k \in G$ such that $d^{\pi}_{k\mathcal{H}}(x_0, x_1) \geq K$ and $d(k.o, \pi_{k\mathcal{H}}(x_0)) \leq D'$ and there exists $\alpha > 0$ such that $d(kh^{\alpha}.o, \pi_{k\mathcal{H}}(x_1)) \leq D'$.

It is immediate that the property of having a K–long, positive h–projection is invariant under the G–action. We also remark that the 'positive' restriction is vacuous if $K > 2D'$ and there exists an element of G that flips the ends of $\mathcal{H}$.

Definition 4.2. Let $\hat{G}(K)$ be the elements $g \in G$ such that there exist points $x \in B_D(o)$ and $y \in B_D(g.o)$ and a geodesic γ from x to y such that no subsegment of γ has a K–long, positive h–projection.

For any $g \in G$, set $|g| = d(o, g.o)$.

Lemma 4.3. *For all sufficiently large K and for every $g \in G - \hat{G}(K)$ there exists a $k \in G$ and an interval $[\alpha', \alpha''] \subset \mathbb{Z}^+$ such that $|kh^{-\alpha}\bar{k}g| < |g|$ for all $\alpha' \leq \alpha \leq \alpha''$. The lower bound α' depends only on h and the upper bound α'' depends linearly on K.*

Proof. Let γ be a geodesic from $\gamma(0) = o$ to $\gamma(T) = g.o$. Since $g \notin \hat{G}(K)$, there are times t_0 and t_1 in $[0, T]$ such that $(\gamma(t_0), \gamma(t_1))$ has a K–long, positive h–projection. Let $k \in G$ such that $d^{\pi}_{k\mathcal{H}}(\gamma(t_0), \gamma(t_1)) \geq K$ and $d(\pi_{k\mathcal{H}}(\gamma(t_0)), k.o)$, and let $\beta > 0$ be such that $d(kh^{\beta}, \pi_{k\mathcal{H}}(\gamma(t_1))) \leq D'$.

Let λ, ϵ, and μ be the constants of Lemma 2.6 for h. Let ξ be the constant of Lemma 2.8 Since $i \mapsto h^i.o$ is (λ, ϵ)–quasi-geodesic and $d(1, h^{\beta}.o) \geq K - 2D' - 2\xi$ we have $\beta \geq \frac{1}{\lambda}(K - 2D' - 2\xi) - \epsilon$.

Set $\alpha'' = \beta$ and $\alpha' = \lambda(4(C + D' + \xi) + \epsilon + 2\mu + 1))$. We assume that K is large enough so that $\alpha'' \geq \alpha'$. For all $\alpha' \leq \alpha \leq \alpha''$ we have:

$$d(k.o, kh^{\beta}.o) \geq d(k.o, kh^{\alpha}.o) + d(kh^{\alpha}.o, kh^{\beta}.o) - 2\mu$$

Rearranging, and using the quasi-geodesic condition for $k\mathcal{H}$:

$$\begin{aligned} d(kh^{\alpha}.o, kh^{\beta}.o) &\leq d(k.o, kh^{\beta}.o) - d(k.o, kh^{\alpha}.o) + 2\mu \\ &\leq d(k.o, kh^{\beta}.o) - (\alpha/\lambda - \epsilon) + 2\mu \\ &< d(k.o, kh^{\beta}.o) - 4(C + D' + \xi) \end{aligned}$$

Now we use the fact that γ passes $C + D' + \xi$ close to $k.o$ and $kh^{\beta}.o$:

$$\begin{aligned} |g| &= d\big(\gamma(0), \gamma(t_0)\big) + d\big(\gamma(t_0), \gamma(t_1)\big) + d\big(\gamma(t_1), \gamma(T)\big) \\ &\geq d\big(\gamma(0), \gamma(t_0)\big) + d\big(\gamma(t_0), k.o\big) + d(k.o, kh^{\beta}.o) \\ &\qquad + d\big(kh^{\beta}.o, \gamma(t_1)\big) + d\big(\gamma(t_1), \gamma(T)\big) - 4(C + D' + \xi) \end{aligned}$$

So:

$$\begin{aligned}|kh^{-\alpha}\overline{k}g| &\leq d(\gamma(0),\gamma(t_0)) + d(\gamma(t_0),k.o) + d(k.o,kh^{-\alpha}\overline{k}kh^{\beta}.o)\\ &\qquad + d(kh^{-\alpha}\overline{k}kh^{\beta}.o,kh^{-\alpha}\overline{k}\gamma(t_1)) + d(kh^{-\alpha}\overline{k}\gamma(t_1),kh^{-\alpha}\overline{k}\gamma(T))\\ &= d(\gamma(0),\gamma(t_0)) + d(\gamma(t_0),k.o) + d(kh^{\alpha}.o,kh^{\beta}.o)\\ &\qquad + d(kh^{\beta}.o,\gamma(t_1)) + d(\gamma(t_1),\gamma(T))\\ &\leq |g| + 4(C+D'+\xi) - d(k.o,kh^{\beta}.o) + d(kh^{\alpha}.o,kh^{\beta}.o)\\ &< |g|\end{aligned}$$

□

Lemma 4.4. *Fix K and let $P(r) = \#(B_r(o)\cap\hat{G}(K).o)$. The function $\log P(r)$ is subadditive in r, up to bounded error.*

Proof. Let $g.o \in B_r(o)\cap\hat{G}(K).o$. Let x, y, and γ be as in Definition 4.2. Let $m+n=r$. If $d(x,y)>m$ let z be the point on γ at distance m from x. Otherwise, let $z=y$. There exists an $f\in G$ such that $d(z,f.o)\leq D$.

We claim that f contributes to $P(m+2D)$ and $\overline{f}g$ contributes to $P(n+2D)$. This is because $d(o,f.o)\leq m+2D$, and the subsegment of γ from x to z is a geodesic for f satisfying Definition 4.2. Similarly, $d(o,\overline{f}g.o)=d(f.o,g.o)\leq n+2D$, and the subsegment of $\overline{f}.\gamma$ from $\overline{f}.z$ to $\overline{f}.y$ is a geodesic for $\overline{f}g$ satisfying Definition 4.2.

This shows that for any $m+n=r$ we have $P(r)\leq P(m+2D)\cdot P(n+2D)$. Applying this relation for $(m-2D)+4D=m+2D$ and $(n-2D)+4D=n+2D$ yields:

$$P(r)\leq (P(6D))^2\cdot P(m)\cdot P(n)$$

Thus:

$$\log P(m+n)\leq \log P(m)+\log P(n)+2\log P(6D).$$

□

There is a result known as Fekete's Lemma that says if (a_i) is a subadditive sequence then $\lim_{i\to\infty}\frac{a_i}{i}$ exists and is equal to $\inf_i\frac{a_i}{i}$. We will need the following generalization for almost subadditive sequences:

Lemma 4.5. *Let (a_i) be an unbounded, increasing sequence of positive numbers. Suppose there exists b such that $a_{m+n}\leq a_m+a_n+b$ for all m and n. Then $L=\lim_{i\to\infty}\frac{a_i}{i}$ exists and $a_i\geq Li-b$ for all i.*

Proof. Let $L^+=\limsup_i\frac{a_i}{i}$. Let $L^-=\liminf_i\frac{a_i}{i}$. Suppose that $L^+>L^-$. Let $\epsilon=\frac{L^+-L^-}{3}$. Since the sequence is increasing and unbounded, there exists an I such that for all $i>I$ we have $\frac{a_i+b}{a_i}<\sqrt{\frac{L^+-\epsilon}{L^-+\epsilon}}$. Fix an $i>I$ such that $\frac{a_i}{i}<L^-+\epsilon$. Choose a j such that $\frac{a_j}{j}>L^+-\epsilon$ and $\frac{q+1}{q}<\sqrt{\frac{L^+-\epsilon}{L^-+\epsilon}}$, where $qi\leq j<(q+1)i$.

$$L^+-\epsilon<\frac{a_j}{j}\leq\frac{a_j}{qi}<\frac{(q+1)(a_i+b)}{qi}<\frac{L^+-\epsilon}{L^-+\epsilon}\cdot\frac{a_i}{i}\leq\frac{L^+-\epsilon}{L^-+\epsilon}(L^-+\epsilon)=L^+-\epsilon$$

This is a contradiction, so $L=L^+=L^-$.

If for some i we have $a_i<Li-b$ then

$$L=\lim_{j\to\infty}\frac{a_{ij}}{ij}\leq\lim_{j\to\infty}\frac{j(a_i+b)}{ij}=\frac{a_i+b}{i}<L,$$

which is a contradiction. □

4.1. **Divergence.** For any subset $A \subset G$, define:

$$\Theta_A(s) = \sum_{r=0}^{\infty} \#\big(B_r(o) \cap A.o\big)e^{-rs}$$

The growth exponent δ_A is the *critical exponent* of Θ_A, that is, Θ_A diverges for all $s < \delta_A$ and converges for all $s > \delta_A$. We say A is *divergent* if Θ_A diverges at δ_A.

Lemma 4.6. *$\hat{G}(K)$ is divergent.*

Proof. Let $P(r) = \#(B_r(o) \cap \hat{G}(K).o)$. By Lemma 4.4 and Lemma 4.5, $\log P(r) \geq r\delta_{\hat{G}(K)} - 2\log P(6D)$ for all r. Thus:

$$\Theta_{\hat{G}(K)}(\delta_{\hat{G}(K)}) = \sum_{r=0}^{\infty} P(r)\exp(-r\delta_{\hat{G}(K)}) \geq \sum_{r=0}^{\infty} \frac{1}{P(6D)^2} = \infty. \qquad \square$$

Lemma 4.7. *For sufficiently large K, the growth exponent of $\hat{G}(K)$ is strictly smaller than the growth exponent of G.*

Proof. Let $h' \in G$ and D be the element and constant, respectively, of Lemma 3.4 (in this case the product has only one factor). Let $K > D$.

Define a map ϕ on $\hat{G}(K)$ as follows.

$$\phi(g) = \begin{cases} h'g\bar{h}' & \text{if } d^{\pi}_{\mathcal{H}}(o, g.o) \geq K \text{ and } d^{\pi}_{g\mathcal{H}}(o, g.o) \geq K \\ h'g & \text{if } d^{\pi}_{\mathcal{H}}(o, g.o) \geq K \\ g\bar{h}' & \text{if } d^{\pi}_{g\mathcal{H}}(o, g.o) \geq K \\ g & \text{otherwise} \end{cases}$$

Then for all $g \in \hat{G}(K)$ we have $d^{\pi}_{\mathcal{H}}\big(o, \phi(g).o\big) < K$ and $d^{\pi}_{\phi(g)\mathcal{H}}\big(o, \phi(g).o\big) < K$.

Let $\hat{G}'(K)$ be the image of ϕ. Then ϕ is a bijection between $\hat{G}(K)$ and $\hat{G}'(K)$, and for all $g \in \hat{G}(K)$ we have $|g| = |\phi(g)| \pm 2|h'|$. It follows that $\delta_{\hat{G}(K)} = \delta_{\hat{G}'(K)}$ and $\hat{G}'(K)$ is divergent.

Let Z be a maximal $2K$–separated subset of $\hat{G}'(K)$. Then $\delta_Z = \delta_{\hat{G}'(K)}$ and Z is divergent. For z and z' in Z, if $z\mathcal{H} = z'\mathcal{H}$ then since $d^{\pi}_{z\mathcal{H}}(o, z.o) < K$ and $d^{\pi}_{z'\mathcal{H}}(o, z'.o) < K$ we have $d(z.o, z'.o) < 2K$, so $z = z'$.

Consider the free product set

$$Z^* * h^m = \bigcup_{i=1}^{\infty} \big\{z_1 h^m \cdots z_i h^m \mid z_j \in Z - \{1\}\big\}.$$

By the same arguments as [1, Proposition 4.1], for all sufficiently large m, the orbit map is an injection of $Z^* * h^m$ into $\mathcal{X}$. This fact, together with divergence of Z, implies that $\delta_Z < \delta_G$, by [8, Criterion 2.4]. □

5. Proof of the Main Theorem

Let $(\mathcal{X}_1, d_1, o_1), \ldots, (\mathcal{X}_n, d_n, o_n)$ a finite collection of proper geodesic metric spaces. Let $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, and let $o = (o_1, \ldots, o_n)$. Let $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ be any points in $\mathcal{X}$. For any $1 \leq p < \infty$, the L^p metric on $\mathcal{X}$ is defined to be:

$$d^p(\underline{x}, \underline{y}) = \left(\sum_{i=1}^{n} (d_i(x_i, y_i))^p \right)^{1/p}$$

The L^∞ metric on $\mathcal{X}$ is:

$$d^\infty(\underline{x}, \underline{y}) = \max_i d_i(x_i, y_i)$$

Proposition 5.1. *For $i = 1, \ldots, n$, let G_i be a non-elementary, finitely generated group acting properly discontinuously and cocompactly by isometries on a proper geodesic metric space $\mathcal{X}_i$. Let $G = G_1 \times \cdots \times G_n$. For each i, let A_i be a subset of G_i such that $\log P_i(r)$ is subadditive in r, up to bounded error, for $P_i(r) = \#(B_r(o_i) \cap A_i.o_i)$. Let $\delta_i = \delta_{A_i.o_i}$ be the growth exponent of A_i. For $1 \leq p \leq \infty$, the growth exponent δ_A of $A = \prod_i^n A_i$ with respect to the L^p metric on $\mathcal{X}$ is the L^q–norm of $(\delta_1, \ldots, \delta_n)$, where $1/p + 1/q = 1$, and $\frac{1}{\infty}$ is understood to be 0.*

Proof. For each $g \in G_i$ let $|g|_i = d_i(o_i, g.o_i)$. For $\underline{g} = (g_1, \ldots, g_n) \in G$, let $|\underline{g}|_p = d^p(o, \underline{g}.o)$. Let B_r^p be the closed r–ball with respect to the L^p metric.

Let $P(r) = \# B_r^p(o) \cap A.o$.

Let $\mathbb{R}^n$ be equipped with the L^p norm $|| \cdot ||_p$, and let S_r^p be the vectors of norm r. Let $\phi \colon \mathbb{R}^n \to \mathbb{R}$ be the linear function $\phi(x_1, \ldots, x_n) = \sum_{i=1}^n \delta_i x_i$. For every $r > 0$ the duality of L^q and L^p implies:

$$||(\delta_1, \ldots, \delta_n)||_q = ||\phi||_p = \sup_{(x_1, \ldots, x_n) \in S_r^p} \frac{|\phi(x_1, \ldots, x_n)|}{r}$$

Since $\delta_i \geq 0$ for all i, the supremum can be restricted to the positive sector of S_r^p. Furthermore, letting

$$Z_r^p = \left\{ (r_1, \ldots, r_n) \mid ||(r_1, \ldots, r_n)||_p \leq r, \, r_i \in \mathbb{N} \right\},$$

we have:

$$||\phi||_p = \lim_{r \to \infty} \max_{(r_1, \ldots, r_n) \in Z_r^p} \frac{\phi(r_1, \ldots, r_n)}{r}$$

Given two positive valued functions $f(r)$ and $g(r)$, we write $f(r) \sim g(r)$ if $\lim_{r \to \infty} \frac{\log f(r)}{\log g(r)} = 1$.

Lemma 4.4 and Lemma 4.5 imply $P_i(r) \sim e^{\delta_i r}$ for each $i = 1, \ldots, n$, so:

$$\begin{aligned} ||\phi||_p &= \lim_{r \to \infty} \max_{(r_1, \ldots, r_n) \in Z_r^p} \frac{\phi(r_1, \ldots, r_n)}{r} \\ &= \lim_{r \to \infty} \max_{(r_1, \ldots, r_n) \in Z_r^p} \frac{\log \prod_{i=1}^n P_i(r_i)}{r} \end{aligned}$$

For any fixed r there is $(z_{r,1}, \ldots, z_{r,n}) \in Z_r^p$ such that:

$$\prod_{i=1}^{n} P_i(z_{r,i}) = \max_{(r_1, \ldots, r_n) \in Z_r^p} \prod_{i=1}^{n} P_i(r_i)$$

We also note that:

$$\prod_{i=1}^{n} P_i(z_{r,i}) \leq P(r) \leq \sum_{(r_1,\ldots,r_n)\in Z_r^p} \prod_{i=1}^{n} P_i(r_i) \leq \#Z_r^p \cdot \prod_{i=1}^{n} P_i(z_{r,i})$$

Since $\#Z_r^p \leq r^n$, this means $P(r) \sim \prod_i^n P_i(z_{r,i})$.

Therefore:

$$\begin{aligned}\delta_A = \limsup_{r\to\infty} \frac{\log P(r)}{r} &= \lim_{r\to\infty} \frac{\log \prod_{i=1}^n P_i(z_{r,i})}{r} \\ &= \lim_{r\to\infty} \max_{(r_1,\ldots,r_n)\in Z_r^p} \frac{\log \prod_{i=1}^n P_i(r_i)}{r} \\ &= ||\phi||_p = ||(\delta_1,\ldots,\delta_n)||_q\end{aligned}$$

□

Proof of Theorem 1.2. The existence of a strongly contracting element implies that each factor group has strictly positive growth exponent, and the main theorem of [1] says that $G_i \curvearrowright \mathcal{X}_i$ is growth tight, so we are done if $n = 1$.

Assume $n > 1$ and let $1 \leq q \leq \infty$ be such that $1/p + 1/q = 1$. If $p = 1$, then by Proposition 5.1 the growth exponent of G is the maximum of the growth exponents of the G_i. Thus, we may kill the slowest growing factor without changing the growth exponent, and the action of G on $\mathcal{X}$ with the L^1 metric is not growth tight.

Now assume $p > 1$. Let $\chi_i \colon G \to G_i$ be projection to the i–th coordinate. Let N be an infinite normal subgroup of G.

First we assume that $\chi_i(N)$ is infinite for all i.

By Lemma 3.3, there exists an element $\underline{h} = (h_1,\ldots,h_n) \in N$ such that h_i is a strongly contracting element for $G_i \curvearrowright X_i$ for each i.

Let A be a *minimal section* of the quotient map $G \to G/N$. That is, A consists of a representative for each coset gN and $d(o, \underline{a}.o) = d(N.o, \underline{a}N.o)$ for all $\underline{a} \in A$, where d is the L^p metric on $\mathcal{X}$.

Proposition 5.2. *For all sufficiently large K and for all $\underline{a} = (a_1,\ldots,a_n) \in A$ there exists an index $1 \leq i \leq n$ such that $a_i \in \hat{G}_i(K)$.*

Proof. For each i, let $\hat{G}_i(K)$ be as in Definition 4.2 for each G_i. Assume K is greater than the constants K from Lemma 4.7 and Lemma 4.3 applied to each G_i.

Suppose $\underline{a}$ is such that for all i we have $a_i \in G_i - \hat{G}_i(K)$. For each i, let $k_i \in G_i$ and $[\alpha_i', \alpha_i'']$ be the k and interval, respectively, from Lemma 4.3 applied to a_i. The α_i' depend only on their respective h_i, while the α_i'' depend linearly on K. By choosing K large enough, we may choose α such that $\max_i \alpha_i' \leq \alpha \leq \min_i \alpha_i''$, so that $\alpha \in [\alpha_i', \alpha_i'']$ for all i. Let $\underline{k} = (k_1,\ldots,k_n)$. The i–th coordinate of $\underline{k}\bar{\underline{h}}^{\alpha}\bar{\underline{k}}\underline{a}$ is $k_i\bar{h_i}^{\alpha}\bar{k_i}a_i$, which is shorter than a_i by Lemma 4.3. But this means that $\underline{k}\bar{\underline{h}}^{\alpha}\bar{\underline{k}}\underline{a}$ is shorter than $\underline{a}$. This contradicts the fact that $\underline{a}$ belongs to a minimal section, since $\underline{k}\bar{\underline{h}}^{\alpha}\bar{\underline{k}}\underline{a} = \underline{a}(\bar{\underline{a}}\underline{k}\bar{\underline{h}}^{\alpha}\bar{\underline{k}}\underline{a}) \in \underline{a}N$. □

Continuing the proof of Theorem 1.2, by Proposition 5.2,

$$A \subset \bigcup_{i=1}^{n} G_1 \times \cdots \times \hat{G}_i \times \cdots \times G_n,$$

where $\hat{G}_i = \hat{G}_i(K)$ for some sufficiently large K. By Proposition 5.1, the growth exponent of $G_1 \times \cdots \times \hat{G}_i \times \cdots \times G_n$ is $||(\delta_1, \dots, \hat{\delta}_i, \dots, \delta_n)||_q$, where δ_i is the growth exponent of G_i and $\hat{\delta}_i$ is the growth exponent of $\hat{G}_i$. Thus, the growth exponent of A is $\max_i ||(\delta_1, \dots, \hat{\delta}_i, \dots, \delta_n)||_q$. By Lemma 4.7, $\hat{\delta}_i < \delta_i$ for each i, so, since $q < \infty$:

$$\delta_{G/N} = \delta_A = \max_i ||(\delta_1, \dots, \hat{\delta}_i, \dots, \delta_n)||_q < ||(\delta_1, \dots, \delta_n)||_q = \delta_G$$

It remains to consider the case that some $\chi_i(N)$ is finite. By reordering, if necessary, we may assume $\chi_i(N)$ is finite for $i \leq m$ and infinite for $i > m$. Since N is infinite, $m < n$. Let $G^1 = G_1 \times \cdots \times G_m$ with $\chi^1 = \chi_1 \times \cdots \times \chi_m \colon G \to G^1$. Let $G^\infty = G_{m+1} \times \cdots \times G_n$ with $\chi^\infty = \chi_{m+1} \times \cdots \times \chi_n \colon G \to G^\infty$.

Now $\ker(\chi^1) \cap N$ is a finite index subgroup of N that is normal in G, so G/N is a quotient of $G/(\ker(\chi^1) \cap N)$ by a finite group, and they have the same growth rates. Replacing N with $\ker(\chi^1) \cap N$, we can assume that $\chi_i(N)$ is trivial for $1 \leq i \leq m$ and infinite for $m < i \leq n$. The theorem applied to G^∞ shows that $\delta_{G^\infty/\chi^\infty(N)} < \delta_{G^\infty}$, so, since $q < \infty$:

$$\delta_{G/N} = ||(\delta_{G^1}, \delta_{G^\infty/\chi^\infty(N)})||_q < ||(\delta_{G^1}, \delta_{G^\infty})||_q = \delta_G \qquad \square$$

In the case that the normal subgroup has infinite projection to each factor, our proof uses the existence of a contracting element in each factor in an essential way. One wonders if the theorem is still true without this hypothesis:

Question. *If, for $1 \leq i \leq n$, G_i is a non-elementary, finitely generated group acting properly discontinuously and cocompactly by isometries on a proper geodesic metric space $\mathcal{X}_i$, and if, for all i, $G_i \curvearrowright \mathcal{X}_i$ is growth tight, is it still true that the product group is growth tight with respect to the action on the product space with the L^p metric for some/all $p > 1$?*

References

[1] Goulnara N. Arzhantseva, Christopher H. Cashen, and Jing Tao, *Growth tight actions*, Pacific J. Math., in press, 2013, arXiv:1401.0499v3.

[2] Goulnara N. Arzhantseva and Igor G. Lysenok, *Growth tightness for word hyperbolic groups*, Math. Z. **241** (2002), no. 3, 597–611, DOI:10.1007/s00209-002-0434-6. MR 1938706 (2003h:20077)

[3] Gilbert Baumslag, *On generalised free products*, Math. Z. **78** (1962), 423–438, DOI:10.1007/BF01195185. MR 0140562 (25 #3980)

[4] Mladen Bestvina, Kenneth Bromberg, and Koji Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publ. Math. Inst. Hautes Études Sci., in press, 2010, arXiv:1006.1939v5.

[5] Tullio Ceccherini-Silberstein and Wolfgang Woess, *Growth and ergodicity of context-free languages*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4597–4625, DOI:10.1090/S0002-9947-02-03048-9. MR 1926891 (2003g:68067)

[6] Tullio Ceccherini-Silberstein and Wolfgang Woess, *Growth-sensitivity of context-free languages*, Theoret. Comput. Sci. **307** (2003), no. 1, 103–116, Words, DOI:10.1016/S0304-3975(03)00095-1. MR 2022843 (2005h:68066)

[7] François Dahmani, Vincent Guirardel, and Denis Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, preprint, 2011, arXiv:1111.7048v5.

[8] Françoise Dal'Bo, Marc Peigné, Jean-Claude Picaud, and Andrea Sambusetti, *On the growth of quotients of Kleinian groups*, Ergodic Theory Dynam. Systems **31** (2011), no. 3, 835–851, DOI:10.1017/S0143385710000131. MR 2794950 (2012f:37069)

[9] Rostislav Grigorchuk and Pierre de la Harpe, *On problems related to growth, entropy, and spectrum in group theory*, J. Dynam. Control Systems **3** (1997), no. 1, 51–89, DOI:10.1007/BF02471762. MR 1436550 (98d:20039)

[10] Wilfried Huss, Ecaterina Sava, and Wolfgang Woess, *Entropy sensitivity of languages defined by infinite automata, via Markov chains with forbidden transitions*, Theoret. Comput. Sci. **411** (2010), no. 44-46, 3917–3922, DOI:10.1016/j.tcs.2010.07.020. MR 2768800 (2011k:68082)

[11] Alessandro Sisto, *Contracting elements and random walks*, preprint, 2011, arXiv:1112.2666v2.

E-mail address: christopher.cashen@univie.ac.at

Fakultät für Mathematik, Universität Wien

E-mail address: jing@math.ou.edu

Department of Mathematics, University of Oklahoma