MOMENTS AND DISTRIBUTION OF THE LOCAL TIME OF A TWO-DIMENSIONAL RANDOM WALK

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Abstract. Let $\ell(n, x)$ be the local time of a random walk on $\mathbb{Z}^2$. We prove a strong law of large numbers for the quantity $L_n(\alpha) = \sum_{x \in \mathbb{Z}^2} \ell(n, x)^\alpha$ for all $\alpha \geq 0$. We use this result to describe the distribution of the local time of a typical point in the range of the random walk.

1. Introduction

Let $X_i, i \in \mathbb{N}$, be a sequence of i.i.d. random vectors on some probability space $(\Omega, \mathbb{P})$, which have values in $\mathbb{Z}^2$, mean 0, and a finite non-singular covariance matrix $\Sigma$. We write

$$S_0 := 0, \quad S_n := \sum_{i=1}^{n} X_i, \quad n \geq 1,$$

for a $\mathbb{Z}^2$-valued random walk. Let $\ell(n, x)$ be its local time,

$$\ell(n, x) := \sum_{i=0}^{n} \mathbb{1}\{S_i = x\}, \quad x \in \mathbb{Z}^2.$$

We will always assume that the characteristic function of $X_i$, $\chi(k) := \mathbb{E}\exp\left(i\langle k, X_1 \rangle\right), \quad k \in J := [-\pi, \pi)^2,$

satisfies $\chi(k) = 1 \iff k = 0$. Here $\langle \cdot, \cdot \rangle$ stands for the standard scalar product in $\mathbb{R}^2$.

In this paper we prove the following strong law of large numbers for random variables

$$L_n(\alpha) := \sum_{x \in \mathbb{Z}^2} \ell(n, x)^\alpha, \quad \alpha \geq 0, n \in \mathbb{N}.$$

**Theorem 1.** For all $\alpha \geq 0$, $\mathbb{P}$-a.s.,

$$\lim_{n \to \infty} \frac{L_n(\alpha)}{n(\log n)^{\alpha-1}} = \frac{\Gamma(\alpha + 1)}{(2\pi \sqrt{\det \Sigma})^{\alpha-1}}.$$

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Remark. This result is trivial for $\alpha = 1$ and well known for $\alpha = 0$. In the second case, $L_n(0) = \sum_x 1\{\ell(n, x) \geq 1\} =: R(n)$ is the size of the range of the random walk. For the simple random walk it was proved in [DE51] that the range satisfies

$$\lim_{n \to \infty} \frac{\log n}{n} R(n) = \pi, \quad \mathbb{P}\text{-a.s.} \quad (6)$$

For a non-simple walk with a covariance matrix $\Sigma$ the right hand side of (6) must be multiplied by $2\sqrt{\det \Sigma}$.

There are at least two reasons why the quantity $L_n(\alpha)$ is worth to study. First, if $\alpha$ is an integer, then $L_n(\alpha)$ is related to the number of $\alpha$-fold self-intersections of the random walk (see also (11) below). This is of much importance, mainly with $\alpha = 2$ or $\alpha = 0$, for the so-called self-interacting random walk, see e.g. [BS95]. In this paper, however, we do not require $\alpha$ being integer. $L_n(\alpha)$ can be then considered as a possible candidate for a definition of the number of $\alpha$-fold self-intersections for all real positive $\alpha$.

The second related subject, which was the original motivation for studying $L_n(\alpha)$, is so-called random walk in random scenery and with it closely connected problem of aging in trap models. We describe this problem briefly. Let $\tau_x, x \in \mathbb{Z}^2$, be a collection of i.i.d. random variables independent of $X_i$. Define

$$Z_n := \sum_{i=0}^n \tau_{S_i}. \quad (7)$$

This process (called usually random walk in random scenery) was first time considered for one-dimensional random walks in [KS79]. Two-dimensional walks were studied in [Bol89], where the random scenery $\tau_x$ was required to have mean zero and a finite variance $\sigma^2$. It was proved there that the process $Z_{\lfloor nt \rfloor}/\sqrt{n \log n}$ converges to the standard Brownian motion with a variance depending on $\sigma$ and $\Sigma$.

In [BCM06] we needed to control the behaviour of $Z_n$ for a scenery $\tau_x$ in the domain of attraction of a non-negative, $\alpha$-stable, $\alpha \in (0, 1)$, law. The interest in this kind of scenery originated in the study of aging in so called Bouchaud’s trap model. This model was proposed by [Bou92] in physics literature to explain basic mechanisms that can be responsible for peculiar dynamical properties (like aging) of complex disordered systems. The $\alpha$-stable sceneries with small $\alpha$ correspond to the low-temperature regime in these systems that is particularly interesting. In the simplest case, Bouchaud’s trap model is a Markov process $\mathcal{X}(t)$ on $\mathbb{Z}^2$ (or some other graph) which is defined as a random time change of the random walk, $\mathcal{X}(t) := S_{Z^{-1}(t)}$ (here $Z^{-1}$ denotes the
right-continuous inverse of \( Z_n \). To show aging behaviour in this model
entails, e.g., to prove that the probability of the event \( \mathcal{X}(\theta t) = \mathcal{X}(t), \quad \theta > 0 \), converges to some non-trivial value as \( t \to \infty \). Since \( \mathcal{X}(t) \) is a
time change of the random walk, the first step in proving such a claim
should be logically the behaviour of the time-change process \( Z_n \).

What is the connection of \( Z_n \) with \( L_n(\alpha) \)? Consider for simplicity
\( \tau_x \) to be \( \alpha \)-stable with \( \mathbb{E} \exp(-\lambda \tau_x) = \exp(-c\lambda^\alpha) \). Then the Laplace
transformation of \( Z_n \) can be rewritten as

\[
\mathbb{E}_{\tau,X} e^{-\lambda Z_n} = \mathbb{E}_X \exp \left( -c \lambda^\alpha \sum_x \ell(n, x)^\alpha \right) = \mathbb{E}_X e^{-c\lambda^\alpha L_n(\alpha)}. \tag{8}
\]

Here the first expectation is over both \( \tau_x \) and \( X_i \). When we started
to investigate aging on \( Z^2 \), we did not find any useful result about
\( L_n(\alpha) \) in the literature. Therefore in [BCM06] we used methods which
do not rely on formula (8) to show that for \( \alpha \)-stable \( \tau_x \), the process
\( Z_{[nt]}/\sqrt{n(\log n)^{\alpha-1}} \) converges to an \( \alpha \)-stable subordinator for a.e. random
environment. Going back, this result together with (8) allows to
deduce a weak law of large numbers for \( L_n(\alpha), \alpha \in (0,1) \). It is however
not possible without a major effort to use the techniques of [BCM06]
to show a strong law. This consequently induces complications when
one tries to extend the convergence to \( \alpha > 1 \). That is why different
methods are used here.

To close the introduction it should be remarked that even knowing
the behaviour of \( L_n(\alpha) \), the proof of aging would be not completely
straightforward. The methods used in [BCM06] describe more precisely
the process \( \mathcal{X}(t) \) and not only the time change \( Z_n \).

The proof of Theorem 1 for \( \alpha \in \mathbb{N} \) is relatively standard, as will be
seen later. The main question is how to extend it to all \( \alpha \geq 0 \). This
extension is made possible by the following theorem that describes the
distribution of the local time of a “typical” point in the range of the
random walk.

**Theorem 2.** Given \( X := \{X_1, X_2, \ldots \} \) let \( Y_n \) be a point chosen uni-
formly in the range of the random walk up to time \( n \), that is

\[
\mathbb{P}[Y_n = x|X] = R(n)^{-1}\mathbb{I}\{\ell(n, x) \geq 1\}. \tag{9}
\]

Then for \( \mathbb{P} \)-a.e. \( X \), the normalised random variable \( \ell(n, Y_n) \) is asymp-
totically exponentially distributed, namely

\[
\mathbb{P} \left[ 2\pi \sqrt{\det \Sigma} \frac{\ell(n, Y_n)}{\log n} \geq u \bigg| X \right] \overset{n \to \infty}{\rightarrow} e^{-u}. \tag{10}
\]

**Remark.** This result is, to a certain extent, related to the fact that the
distribution of the normalised local time of the origin, \((\log n)^{-1}\ell(n, 0), \)
converges to the exponential distribution with mean $\pi$, which was proved for the simple random walk in [ET60]. A possible interpretation of Theorem 2 is then: “The origin becomes asymptotically typical.”

The following strategy will be used in the proofs. We first prove Theorem 1 for $\alpha \in \mathbb{N}$. This will allow us to show Theorem 2 and then extend Theorem 1 to $\alpha \geq 0$.

2. PROOFS OF THE THEOREMS

We first prove Theorem 1 for $\alpha \in \mathbb{N}$. We compute the expected value, $\mathbb{E}L_n(\alpha)$, and bound from above the variance, $\text{Var} L_n(\alpha)$, using relatively standard techniques (see e.g. [Bol89] which we follow closely). We then use these estimates to prove a strong law of large numbers along sufficiently fast increasing sequences, and finally we fill the gaps in these sequences.

Expected value. For $\alpha \in \mathbb{N}$ the random variable $L_n(\alpha)$ can be written as

$$L_n(\alpha) = \sum_{x \in \mathbb{Z}^2} \left( \sum_{i=0}^{n} \mathbb{1}\{S_i = x\} \right)^\alpha = \sum_{k_1, \ldots, k_\alpha = 0}^{n} \mathbb{1}\{S_{k_1} = \cdots = S_{k_\alpha}\}. \quad (11)$$

Therefore,

$$\mathbb{E}L_n(\alpha) = \sum_{k_1, \ldots, k_\alpha = 0}^{n} \mathbb{P}[S_{k_1} = \cdots = S_{k_\alpha}] = \sum_{\beta=1}^{\alpha} C(\alpha, \beta) \sum_{0 \leq k_1 < \cdots < k_\beta \leq n} \mathbb{P}[S_{k_1} = \cdots = S_{k_\beta}], \quad (12)$$

where $C(\alpha, \beta)$ are combinatorial factors depending only on $\alpha$ and on $\beta$, which is the number of different values in sequence $k_1, \ldots, k_\alpha$. In particular $C(\alpha, \alpha) = \alpha! = \Gamma(\alpha + 1)$. Values of all others $C(\alpha, \beta)$ are irrelevant, as we will see. Using the Markov property we get

$$a_\beta(n) := \sum_{0 \leq k_1 < \cdots < k_\beta \leq n} \mathbb{P}[S_{k_1} = \cdots = S_{k_\beta}] = \sum_{m \in M_n} \prod_{i=1}^{\beta-1} \mathbb{P}[S_{m_i} = 0], \quad (13)$$

where

$$M_n = \{m = (m_0, \ldots, m_\beta) \in \mathbb{N}_0^{\beta+1}, m_1, \ldots, m_{\beta-1} \geq 1, \sum m_i = n\}. \quad (14)$$

We set $\rho_\beta(\lambda) = \sum_{n=0}^{\infty} \lambda^n a_\beta(n)$ and use the fact that

$$\mathbb{P}(S_j = x) = (2\pi)^{-2} \int_{\mathbb{R}} \chi(k)^j \exp(-i\langle k, x \rangle) \, dk. \quad (15)$$
An easy computation yields
\[
\rho_\beta(\lambda) = (1 - \lambda)^{-2}\left(\int_J \frac{dk}{(2\pi)^2} \frac{\lambda \chi(k)}{1 - \lambda \chi(k)}\right)^{\beta - 1}.
\] (16)
As in [Bol89], for two positive functions \(f_\delta(\lambda)\) and \(g_\delta(\lambda)\), \(\delta > 0\), \(\lambda \in (0, 1)\), which diverge for \(\lambda \to 1\) we write
\[
f_\delta(\lambda) \sim \delta \to 0 g_\delta(\lambda)
\] (17) if
\[
\lim_{\delta \to 0} \liminf_{\lambda \to 1} \frac{f_\delta(\lambda)}{g_\delta(\lambda)} = \lim_{\delta \to 0} \limsup_{\lambda \to 1} \frac{f_\delta(\lambda)}{g_\delta(\lambda)} = 1.
\] (18)
Let \(U_\delta \subset J\), \(k \in U_\delta \iff \langle k, \Sigma k \rangle \leq \delta\). It is easy to see that
\[
\left|\int_{J \setminus U_\delta} \frac{dk}{(2\pi)^2} \frac{\lambda \chi(k)}{1 - \lambda \chi(k)}\right| \leq \text{const.} \delta^{-1} \text{ for all } \lambda \leq 1.
\] (19)
To treat the integral over \(U_\delta\), we observe first that the characteristic function of \(X_i\), \(\chi(k)\), has the following expansion around 0:
\[
\chi(k) = 1 - \frac{1}{2} \langle k, \Sigma k \rangle + R(k), \quad \text{where } |R(k)| = o(|k|^2) \text{ for } k \to 0.
\] (20)
Using this expansion it can be shown that
\[
\int_{U_\delta} \frac{dk}{(2\pi)^2} \frac{\lambda \chi(k)}{1 - \lambda \chi(k)} \sim (2\pi \sqrt{\det \Sigma})^{-1} \log \frac{1}{1 - \lambda}.
\] (21)
Inserting this back into (16) it follows from the Tauberian theorem for sequences (see [Fel71], Theorem XIII 5.5), and the fact that \(a_\beta(n)\) are monotone that
\[
a_\beta(n) = n \left(\frac{\log n}{2\pi \sqrt{\det \Sigma}}\right)^{\beta - 1} (1 + o(1)), \quad \text{as } n \to \infty.
\] (22)
In particular \(a_\alpha(n) \gg a_\beta(n)\) for all \(\beta < \alpha\). Therefore, using also (12), for all \(\alpha \in \mathbb{N}\)
\[
\mathbb{E}L_n(\alpha) = \frac{\Gamma(\alpha + 1)}{(2\pi \sqrt{\det \Sigma})^{\alpha - 1}} n(\log n)^{\alpha - 1}(1 + o(1)), \quad \text{as } n \to \infty.
\] (23)
**Variance.** The computation of the variance is similar but relatively complicated. We will show that
\[
\text{Var } L_n(\alpha) = O\left(n^2(\log n)^{2\alpha - 4}\right).
\] (24)
We first rewrite \( \text{Var} L_n(\alpha) \) in spirit of (11),

\[
\text{Var} L_n(\alpha) = \sum_{k_1, \ldots, k_\alpha, l_1, \ldots, l_\alpha} \mathbb{P}[S_{k_1} = \cdots = S_{k_\alpha}, S_{l_1} = \cdots = S_{l_\alpha}]
- \mathbb{P}[S_{k_1} = \cdots = S_{k_\alpha}] \mathbb{P}[S_{l_1} = \cdots = S_{l_\alpha}]
= \sum_{\alpha} C(\alpha, \beta, \gamma) \sum_{0 \leq k_1 < \cdots < k_\beta \leq n} \mathbb{P}[S_{k_1} = \cdots = S_{k_\beta}, S_{l_1} = \cdots = S_{l_\gamma}]
- \mathbb{P}[S_{k_1} = \cdots = S_{k_\beta}] \mathbb{P}[S_{l_1} = \cdots = S_{l_\gamma}]
=: \sum_{\alpha} C(\alpha, \beta, \gamma) a_{\beta, \gamma}(n). 
\]

(25)

Here again the precise values of the combinatorial factors \( C(\alpha, \beta, \gamma) \) are irrelevant.

We want to compute \( a_{\beta, \gamma}(n) \) using the same methods as for the expectation. To this end we need several definitions. Given two ordered sequences \( k_1, \ldots, k_\beta \) and \( l_1, \ldots, l_\gamma \) we define a sequence of pairs

\[
(j_i, \kappa_i), \quad i \in \{1, \ldots, \beta + \gamma\},
\]

which satisfies \( j_i \in \{0, \ldots, n\}, \kappa_i \in \{0, 1\}, j_i \leq j_{i+1} \) for all \( i \leq \beta + \gamma - 1 \) and

\[
\{j_i : \kappa_i = 0\} = \{k_1, \ldots, k_\beta\}, \quad \{j_i : \kappa_i = 1\} = \{l_1, \ldots, l_\gamma\}. 
\]

(27)

To rule out possible ties we require: if \( j_i = j_{i+1} \), then \( \kappa_i < \kappa_{i+1} \). We then set \( m_0 = j_1, m_{\beta + \gamma} = n - j_{\beta + \gamma} \), and

\[
\epsilon_i = \kappa_{i+1} - \kappa_i, \quad m_i = j_{i+1} - j_i, \quad \text{for } i = 1, \ldots, \beta + \gamma - 1. 
\]

(28)

Let \( E(\beta, \gamma) \subset \{-1, 0, 1\}^{\beta + \gamma - 1} \) be the set of all possible sequences \( \epsilon = \{\epsilon_i, i = 1, \ldots, \beta + \gamma - 1\} \) that can be produced using this construction. This set is obviously finite. Let further \( M_{\beta, \gamma}(\epsilon, n) \) be the set of all \( m = (m_0, \ldots, m_{\beta + \gamma}) \) such that \( m_i \in \mathbb{N}_0, \sum m_i = n \), and \( m \) is compatible with \( \epsilon \). To be compatible with \( \epsilon \) imposes \( m_i \geq 1 \) for some \( \epsilon \)-dependent \( i \)'s. Since we are looking for an upper bound we will generally ignore these restrictions.

We can now compute \( a_{\beta, \gamma}(n) \). Observe first that if there is only one \( \epsilon_i \neq 0 \), then \( k_\beta \leq l_1 \) or \( l_\gamma \leq k_1 \), and by Markov property the positive and negative term of \( a_{\beta, \gamma}(n) \) in definition (25) exactly cancel each other. Therefore we can consider only \( \epsilon \in E'(\beta, \gamma) := \{\epsilon : \sum |\epsilon_i| \geq 2\} \). For these \( \epsilon \) we first completely ignore the negative term. Therefore, again
by Markov property,

$$a_{\beta,\gamma}(n) \leq \sum_{\varepsilon \in E'(\beta,\gamma)} \sum_{i \in \mathbb{Z}^2} \sum_{m \in M_{\beta,\gamma}(\varepsilon,n)} \prod_{i=1}^{\beta+\gamma-1} P[S_{m_i} = \varepsilon_i z] =: \sum_{\varepsilon \in E'} a(\varepsilon,n).$$

Taking \( \rho(\lambda) = \sum_{n=0}^{\infty} a(\varepsilon,n)\lambda^n \) and setting \( M_{\beta,\gamma}(\varepsilon,n) = \bigcup_{n} M_{\beta,\gamma}(\varepsilon,n) \)
we get

$$\rho(\lambda) = \sum_{\varepsilon \in \mathbb{Z}^2} \sum_{m \in M_{\beta,\gamma}(\varepsilon)} \lambda^{m_0 + m_{\beta+\gamma}} \prod_{i=1}^{\beta+\gamma-1} \int_{j} \frac{dk_j}{(2\pi)^2} (\lambda \chi(k_j))^{m_j} e^{-i(k_j,\varepsilon_j)}.$$

Since \( \varepsilon \in E' \), there are at least two \( j \)'s such that \( \varepsilon_j \neq 0 \). Suppose, for simplicity, that \( \varepsilon_1 \) is one of them. Using the substitution \( k'_i = \sum_{i=1}^{\beta+\gamma} \varepsilon_i k_i, k'_j = k_j \) for \( j \geq 2 \) and applying the Fourier inversion we get

$$\rho(\lambda) \leq \text{const}.(1 - \lambda)^{-2} \int_{B_{\delta \sqrt{\lambda}}} \prod_{i=2}^{\beta+\gamma-1} \frac{dk_i}{1 - \lambda \chi(k_i)} \frac{1}{1 + \lambda \chi(f(k))}, \quad (31)$$

where \( f(k) = \varepsilon_1 \sum_{i=2}^{\beta+\gamma-1} \varepsilon_i k_i \). Let \( \delta > 0 \) and let \( U_{\delta} = \left\{ |k_i, \Sigma k_i| \leq \delta, \forall i = 2, \ldots, \beta + \gamma - 1 \right\} \). The integral over \( U_{\delta} \) can be rewritten using again the expansion (20) and several easy substitutions as

$$\int_{U_{\delta}} \prod_{i=2}^{\beta+\gamma-1} \frac{dk_i}{1 - \lambda \chi(k_i)} \frac{1}{1 + \lambda \chi(f(k))} \approx \text{const}.(1 - \lambda)^{-1} \int_{B_{\delta \sqrt{\lambda}}} \prod_{i=2}^{\beta+\gamma-1} \frac{dk_i}{1 + k_i^2} \frac{1}{1 + (f(k))^2},$$

where \( B_{r} \) is the ball in \( \mathbb{R}^2 \) with radius \( r \) centered at the origin. Integrating over all \( k_i \) that are not contained in \( f(k) \), that means over all \( k_i \) such that \( \varepsilon_i = 0 \), say there is \( \omega_e \) of them, we get a factor \( (\log 1/(1 - \lambda))^{\omega_e} \). The integral over the remaining \( k_i \)’s stays bounded as \( \lambda \to 1 \). Therefore, the last expression is

$$\rho(\lambda) \sim \text{const}.(1 - \lambda)^{-1}(\log 1/(1 - \lambda))^{\omega_e}, \quad (33)$$

It can be seen easily that the integral over the set \( J^{\beta+\gamma-2} \setminus U_{\delta} \) diverges at most as fast as the integral over \( U_{\delta} \). The equations (31) and (33) yield

$$\rho(\lambda) \sim \text{const}.(1 - \lambda)^{-3}(\log 1/(1 - \lambda))^{\omega_e}. \quad (34)$$

The Tauberian theorem then implies that \( a(\varepsilon,n) = O(n^2(\log n)^{\omega_e}) \).
If $\omega_\varepsilon \leq 2\alpha - 4$, this bound would be strong enough to imply (24). This is however not always the case. There is one exception: $\beta = \gamma = \alpha$ and $\varepsilon_i \neq 0$ only for two values of $i$, call them $u$, $v$. In this case $\omega_\varepsilon = 2\alpha - 3$. So that we cannot ignore the negative term in (25), and the computation must be refined. For simplicity we assume that $u < v$ and $\varepsilon_u = 1$, then $\varepsilon_v = -1$. Using again the Markov property we get for the contribution of this $\varepsilon$

$$
\sum_{m \in M_{\alpha,\alpha}(\varepsilon, n)} \sum_{z \in \mathbb{Z}^2} \mathbb{P}[S_{m_u} = z] \mathbb{P}[S_{m_v} = -z] \prod_{i=1}^{2\alpha - 1} \mathbb{P}[S_{m_i} = 0] \quad (35)
- \mathbb{P}[s_{m_u + \ldots + m_v} = 0] \prod_{i=1}^{2\alpha - 1} \mathbb{P}[S_{m_i} = 0] =: b_{u,v}(n).
$$

Setting $\rho_{u,v}(\lambda) = \sum_{n=0}^{\infty} \lambda^n b_{u,v}(n)$, after a standard computation we get

$$
\rho_{u,v}(\lambda) = \text{const.} (1 - \lambda)^{-2} \left( \log \frac{1}{1 - \lambda} \right)^{u - 2 + 2\alpha - v}
\left\{ \int \frac{1}{1 - \lambda \chi(k_u)} \prod_{i=u}^{v-1} \frac{dk_i}{1 - \lambda \chi(k_i)}
- \int \frac{dk_u}{(1 - \lambda \chi(k_u))^2} \prod_{i=u+1}^{v-1} \frac{dk_i}{1 - \lambda \chi(k_i) \chi(k_u)} \right\}.
$$

(36)

Here, the logarithmic factor on the first line comes from those terms in (35) where $i < u$ or $i > v$. Narrowing the domain of integration to a $\delta$-neighbourhood of the origin (which gives as always a leading divergence), using again (20) and some obvious substitutions, we get that the difference in the braces is of the order of

$$
(1 - \lambda)^{-1} \int_{B_{1/\sqrt{\pi}}} \frac{1}{1 + k_u^2} \prod_{j=u}^{v-1} \frac{dk_j}{1 + k_j^2} \left[ 1 - \prod_{i=u+1}^{v-1} \frac{1 + k_i^2}{1 + k_i^2 + k_u^2} \right].
$$

(37)

The difference in the brackets can be telescoped as $1 - abc = (1 - a) + a(1 - b) + ab(1 - c)$, giving a sum of several integrals. All of them can be shown to be at most $O((\log 1/(1 - \lambda))^{v-u-2})$. That is the power smaller by one than if the difference in the brackets was replaced by one. This is exactly what we needed. The usual reasoning then gives that $b_{u,v}(n) = O(n^2 (\log n)^{2\alpha - 4})$ and since there is only finitely many $u$’s and $v$’s the proof of (24) is finished.
Strong law of large numbers for $\alpha \in \mathbb{N}$. The result for $\alpha = 1$ is trivial, therefore we consider $\alpha \geq 2$. Let $n_k = \exp k^\theta$, $1/2 < \theta < 1$. Then by Chebyshev inequality

$$\sum_{k=0}^{\infty} \mathbb{P}\left[\left(L_{n_k}(\alpha) - \mathbb{E}L_{n_k}(\alpha)\right) \geq \varepsilon \mathbb{E}L_{n_k}(\alpha)\right] \leq C(\varepsilon) \sum_{k=0}^{\infty} (\log n_k)^2 < \infty.$$  

(38)

Therefore $L_{n_k}(\alpha)/\mathbb{E}L_{n_k}(\alpha) \to 1$ a.s. as $k \to \infty$. Let now $n_k \leq n < n_{k+1}$. Then

$$L_{n_k}(\alpha) - \mathbb{E}L_{n_{k+1}}(\alpha) \leq L_n(\alpha) - \mathbb{E}L_n(\alpha) \leq L_{n_{k+1}}(\alpha) - \mathbb{E}L_{n_k}(\alpha).$$  

(39)

The absolute value of the two extremal terms is a.s. for all $n$ large enough bounded by

$$\varepsilon L_{n_{k+1}}(\alpha) + \mathbb{E}L_{n_{k+1}}(\alpha) - \mathbb{E}L_{n_k}(\alpha) \leq 3\varepsilon \mathbb{E}L_n(\alpha).$$  

(40)

This finishes the proof of Theorem 1 for $\alpha \in \mathbb{N}$.

Proof of Theorem 2. We want to show that the distribution of

$$Z_n := 2\pi \sqrt{\det \Sigma} \frac{\ell(n, Y_n)}{\log n}$$  

(41)

converges a.s to the exponential distribution. We compute integer moments of $Z_n$.

$$\mathbb{E}[Z_n^\alpha | X] = (2\pi \sqrt{\det \Sigma})^\alpha R(n)^{-1} \sum_{x \in \mathbb{Z}^d} \ell(n, x)^\alpha \frac{(\log n)^\alpha}{(\log n)^\alpha}$$

$$= \frac{(2\pi \sqrt{\det \Sigma})^{\alpha-1} \sum_{x \in \mathbb{Z}^d} \ell(n, x)^\alpha}{n(\log n)^{\alpha-1}} \frac{2\pi n(\log n)^{-1} \sqrt{\det \Sigma}}{R(n)}.$$  

(42)

By Theorem 1 and (6) the last expression converges a.s. to $\Gamma(\alpha + 1)$. Since the $\alpha$-th moment of the exponential distribution with mean one is $\Gamma(1 + \alpha)$, and this distribution is determined by its integer moments, Theorem 2 is proved.

Proof of Theorem 1 for $\alpha \geq 0$. This proof is now trivial. It is sufficient to read (42) from right to left and use the fact that by Theorem 2 and by the convergence of integer moments for all integers larger than $\alpha$,

$$\lim_{n \to \infty} \mathbb{E}[Z_n^\alpha | X] = \Gamma(\alpha + 1)$$  

(43)

a.s. for all $\alpha \geq 0$.  

□
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