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POISSON STRUCTURES: TOWARDS A CLASSIFICATION

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Abstract. In the present note we give an explicite description of certain class of Poisson structures. The methods lead to a classification of Poisson structures in low dimensions and suggest a possible approach for higher dimensions.

Introduction.

The Poisson structures were first introduced and discussed in the not so well-known paper by S. Lie in 1875 [Lie] , who use the name of function groups. In a systemetic way such structures were investigated by Kirillov [Kir] and Lichnerowicz [Lic]. An interesting example of quadratic Poisson structure was found by E. Sklyanin in 1982 [Skl]. Further progress in this direction has been made by V. Drinfeld [Dr1] while investigating solutions for the Yang-Baxter equation.

Let us mention also the paper [GLZ], where some nonlinear Poisson structures were used for the description of classical (and quantum) integrable systems.

In this paper we would like to start a systematic investigation of Poisson structures, while referring to future work for specific physical applications of our study. Even though we have in mind group manifolds, in this paper we shall mainly work on $\mathbb{R}^n$. Global aspects will be dealt with specific Lie groups in the future.

Let M be a manifold and $\mathcal{F}(M)$ be the space of smooth functions on M.

Definition. *Poisson bracket* $\{\cdot,\cdot\}$ on $\mathcal{F}(M)$ is a Lie bracket being a biderivative of the associative algebra $\mathcal{F}(M)$, i.e. an operation assigning to every pair of functions $F, G \in \mathcal{F}(M)$ a new function $\{F, G\} \in \mathcal{F}(M)$, which is linear in F and G and satisfies the following conditions:

a) the skew-symmetry

$$\{F, G\} = -\{G, F\}, \tag{1}$$

b) the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \tag{2}$$

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c) the Leibnitz rule

$$\{F, GH\} = \{F, G\}H + \{F, H\}G. \tag{3}$$

The identities (1)–(2) are nothing but the axioms of a Lie algebra. In this way, the space $\mathcal{F}(M)$ equipped with the Poisson bracket $\{\cdot, \cdot\}$ becomes an (infinite-dimensional) Lie algebra.

The identities (1) and (3) imply that our bracket is given by a skewsymmetric biderivative, i.e. a bivector field $\Lambda \in \Gamma(\Lambda^2 TM)$, and the Jacobi identity (2) can be rewritten as $[\Lambda, \Lambda]_S = 0$, where $[\cdot, \cdot]_S$ stands for the Schouten bracket (we shall write it without the index in the sequel). Such a bivector field we call *Poisson structure*. Two Poisson structures Λ_1, Λ_2 are called *compatible* if any linear combination of them is again a Poisson structure. In terms of the Schouten bracket it means that $[\Lambda_1, \Lambda_2] = 0$.

Remark. Recall that the Schouten bracket is the unique (up to a constant) extension of the usual Lie bracket of vector fields making the graded commutative algebra of multivector fields into a graded Lie algebra (with n–vector fields being of degree n-1) for which the adjoint action is a graded derivation with respect to the wedge product:

$$[U, V \wedge W] = [U, V] \wedge W + (-1)^{u(v+1)} V \wedge [U, W],$$

u, v being the degrees of U, V.

In particular, for the case of wedge products of two vector fields, we have

$$[X \wedge Y, U \wedge V] = [X, U] \wedge Y \wedge V + X \wedge [Y, U] \wedge V + Y \wedge [X, V] \wedge U + X \wedge U \wedge [Y, V].$$

Let x^j be local coordinates on M and consider the Poisson brackets of the form

$$\{F(x), G(x)\} = \omega^{jk}(x) \partial_j F \partial_k G, \quad \partial_j = \frac{\partial}{\partial x^j}. \tag{4}$$

The Leibnitz rule (3) is automatically satisfied here, while the condition (1) is equivalent to

$$\omega^{jk}(x) = -\omega^{kj}(x) \tag{5}$$

and the condition (2) becomes

$$\omega^{jk} \partial_k \omega^{lm} + \omega^{lk} \partial_k \omega^{mj} + \omega^{mk} \partial_k \omega^{jl} = 0. \tag{6}$$

The corresponding Poisson structure is given by

$$\Lambda = \omega^{jk} \partial_j \wedge \partial_k.$$

Given a function H, we call the vector field $X_H = i_{dH} \Lambda$ the *hamiltonian vector field* with hamiltonian H. From the Jacobi identity (2) easily follows that

$$[X_F, X_H] = X_{\{F,H\}}, \tag{7}$$

so hamiltonian vector fields form a Lie subalgebra in the Lie algebra $\mathcal{X}(M)$ of all smooth vector fields on M. In local coordinates

$$X_H = \omega^{jk}(x) \partial_j H \partial_k. \tag{8}$$

We emphasize that the tensor $\omega^{jk}(x)$ appearing in (4) need not be nondegenerate, in particular, the dimension of M may be odd.

Note that orbits of hamiltonian vector fields form a generalized foliation, i.e. the orbits are integral manifolds of the corresponding distribution which is invariant with respect to local flows of vector fields with values in the distribution.

Let us discuss typical examples of Poisson brackets.

Example 1. The "classical" Poisson bracket on a symplectic manifold (e.g. on the cotangent bundle of a manifold) may be written in canonical coordinates $(\mathbf{q}, \mathbf{p})$ in the form

$$\{F, G\} = \sum_j \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}.$$

Example 2. If tensor $\omega^{jk}(x)$ does not depend on x, by a linear change of variables this case can be reduced to the bracket as above, but with $(\mathbf{q}, \mathbf{p})$ being only a part of coordinates. We get symplectic Poisson bracket if these are all coordinates, what is possible only in even dimension.

Example 3. The next important case is that of linear coefficients $\omega^{jk}(x)$ (we assume here that M is a linear space):

$$\omega^{jk}(x) = C_l^{jk} x^l. \tag{9}$$

Since

$$\{x^j, x^k\} = C_l^{jk} x^l, \tag{10}$$

C_l^{jk} are structure constant of a Lie algebra. Conversely, having a Lie algebra $(g, [\cdot,\cdot])$ with a basis x^l, we can regard this basis as coordinate (in fact linear) system for the dual space g^* and consider the unique Poisson bracket $\{\cdot,\cdot\}$ on g^* for which $\{x^i, x^j\} = [x^i, x^j]$. The Jacobi identity for the bracket in g implies that this bracket on $\mathcal{F}(g^*)$ given by

$$\{F(x), G(x)\} = C_l^{jk} x^l \partial_j F \partial_k G \tag{11}$$

is really a Poisson bracket. It is usually called the *Kostant–Kirillov–Souriau bracket* and can be also written formally as

$$\{F(x), G(x)\} = (x, [\frac{\partial F}{\partial x}, \frac{\partial G}{\partial x}]). \tag{12}$$

It is easy to see that the orbits of this Poisson structure–the leaves of the corresponding generalized foliation–are exactly orbits of the coadjoint representation. Notice also that the Poisson bracket of two polynomial functions on g^* is again a polynomial function, so that the space $\mathcal{P}(g^*)$ of all polynomials on g^* is a Lie subalgebra.

Example 4. Having associated our bracket with a tensor field, it is clear that we can perform a coordinate transformation and generate "new" brackets if the transformation is not canonical.

Let M be the dual space of the Lie algebra of the group $SO(3)$, $M = \{x : x = (x_1, x_2, x_3)\}$. The Poisson structure of Kostant–Kirillov-Souriau is given in this case by

$$\{x_j, x_k\} = \varepsilon_{jkl} x_l, \quad j, k, l = 1, 2, 3, \tag{13}$$

where ε_{jkl} is a totally skew-symmetric tensor, $\varepsilon_{123} = 1$. The Poisson structure is degenerate, and on the orbits $\mathcal{O}_r = \{x : |x|^2 = x_1^2 + x_2^2 + x_3^2 = r^2\}$ of the coadjoint representation it can be expressed in spherical coordinates as

$$\{F(\theta, \varphi), G(\theta, \varphi)\} = \frac{1}{r \sin\theta}(\frac{\partial F}{\partial \theta}\frac{\partial G}{\partial \varphi} - \frac{\partial F}{\partial \varphi}\frac{\partial G}{\partial \theta}). \tag{14}$$

Here $x_1 = r\sin\theta\cos\varphi, x_2 = r\sin\theta\sin\varphi, x_3 = r\cos\theta$. The dynamics in M is given by the equation

$$\dot{x} = [x, \frac{\partial H}{\partial x}], \tag{15}$$

where the bracket is the vector product. We notice that for a quadratic Hamiltonian

$$H = \frac{1}{2}\sum a_j x_j^2 \tag{16}$$

these equations turn into Euler's equations describing the motion of a rigid body about a fixed point.

Example 5. Consider a Poisson structure $\Lambda = \omega^{jk}\partial_j \wedge \partial_k$ with constant coefficients. Using the coordinate transformation $\xi_i = e^{x_i}$ we get the bracket $\{\xi_j, \xi_k\} = \omega^{jk}\xi_j\xi_k$ corresponding to the Poisson structure

$$\Lambda = \omega^{jk}\xi_j\xi_k \frac{\partial}{\partial\xi_j} \wedge \frac{\partial}{\partial\xi_k}.$$

From Poisson structures to differential forms.

As differential calculus with differential forms is more familiar, we associate $(n-k)$–forms with k–vector fields on an n–dimensional orientable manifold M putting

$$\Psi_\Lambda = i_\Lambda \Omega,$$

where Ω stands for a volume form. The mapping Ψ yields an isomorphism between k–vector fields and $(n-k)$–forms. Since for $\Lambda = X_1 \wedge ... \wedge X_k$

$$(i_\Lambda\Omega)(Y_1, ..., Y_{n-k}) = \Omega(X_1, ..., X_k, Y_1, ..., Y_{n-k}),$$

for a bivector field $\Lambda = c^{ij}\partial_i \wedge \partial_j$ and $\Omega = dx^1 \wedge ... \wedge dx^n$ we have

$$\Psi_\Lambda = 2\sum_{i<j}(-1)^{i+j}c^{ij}dx^1 \wedge ... \overset{i}{V} ... \overset{j}{V} ... \wedge dx^n,$$

where "$\overset{i}{V}$" stands for the omission. Note that Ψ depends on the choice of Ω.

For vector fields X, Y we have

$$i_{[X,Y]} = i_X i_Y d - di_{X\wedge Y} + i_X di_Y - i_Y di_X$$

and one can prove that for bivector fields $\Lambda_1,\ \Lambda_2$ we have similarly

$$i_{[\Lambda_1,\Lambda_2]} = -i_{\Lambda_1}i_{\Lambda 2}d - di_{\Lambda_2\wedge\Lambda_1} + i_{\Lambda_1}di_{\Lambda_2} + i_{\Lambda_2}di_{\Lambda_1},$$

that easily implies the following:

Theorem 1. *A bivector field* Λ *is a Poisson structure if and only if*

$$2i_\Lambda d\Psi_\Lambda = d\Psi_{\Lambda\wedge\Lambda}. \tag{17}$$

Two Poisson structures $\Lambda_1,\ \Lambda_2$ *are compatible if and only if*

$$d\Psi_{\Lambda_1\wedge\Lambda_2} = i_{\Lambda_1}d\Psi_{\Lambda_2} + i_{\Lambda_2}d\Psi_{\Lambda_1}.$$

The isomorphism defined by Ψ suggests to compose it with operators available on forms, say $d,\ L_X,\ i_X$, (cf. [LX] ,[Kos]) so that we can state properties of the Schouten bracket in terms of differential forms. We have, as it is well known,

1) $$L_X\Omega = div(X)\Omega = di_X\Omega$$

2) $$d(i_{X\wedge Y}\Omega) = i_{[Y,X]}\Omega + i_X di_Y\Omega - i_Y di_X\Omega,$$

so defining $D = \Psi^{-1}\circ d\circ\Psi$, we get

1′) $$D(X) = div(X)$$

2′) $$D(X\wedge Y) = [Y,X] + div(Y)X - div(X)Y$$

for vector fields X, Y and

$$D(\Lambda_1\wedge\Lambda_2) = [\Lambda_1,\Lambda_2] + D(\Lambda_1)\wedge\Lambda_2 - \Lambda_1\wedge D(\Lambda_2)$$

for bivector fields Λ_1, Λ_2. Hence we can rewrite (17) in the form

$$D(\Lambda\wedge\Lambda) = 2\Lambda\wedge D(\Lambda). \tag{18}$$

We shall call a Poisson structure Λ *closed* if $D(\Lambda)=0$ (this implies $D(\Lambda\wedge\Lambda)=0$). This is clearly equivalent to the fact that the form Ψ_Λ (and hence $\Psi_{\Lambda\wedge\Lambda}$) is closed. It should be mentioned that this definition is volume dependend, i.e. if Ω is replaced by $f\Omega$ the Poisson structure may be not closed any more.

Differentiating the identity

$$df\wedge i_{\Lambda\wedge\Lambda}\Omega = -2i_{X_f\wedge\Lambda}\Omega$$

we get

$$df\wedge di_{\Lambda\wedge\Lambda}\Omega = 2di_{X_f\wedge\Lambda}\Omega.$$

Using now (17) we get a new version of it in terms of hamiltonians.

Theorem 2. *Given a volume form Ω a bivector field Λ is a Poisson structure if and only if for any smooth function f, we have*

$$df \wedge i_\Lambda d i_\Lambda \Omega = d i_{X_f \wedge \Lambda} \Omega.$$

Starting with a given volume Ω we can try to decompose Ψ_Λ into a closed part plus a remaining term. What is interesting in this decomposition is the fact that this remainder is usually associated with rank 2 Poisson structure. We first outline the main idea of the construction and then we try to say in which assumptions the results can be stated in a more global setting. From $\Psi = i_\Lambda \Omega$ we can derive the (n-1)–form $d\Psi$ (or even an (n-2)–form if the last is exact) and define the vector field X_Λ by $i_{X_\Lambda}\Omega = d\Psi$. This vector field satisfies

1) $$div(X_\Lambda) = 0$$

2) $$L_{X_\Lambda}\Lambda = 0$$

(cf. [LX]).

Remark. If we start with $d\Psi = 0$ and change the volume, we get

$$i_{X_\Lambda} f\Omega = df \wedge \Psi,$$

i.e. $X_\Lambda = -X_f/f$.

Having constructed $d\Psi$ we can look for a 1–form θ such that

1) $$\Omega = \theta \wedge d\Psi$$

2) $$\theta(X_\Lambda) = 1$$

3) $$d\theta = 0.$$

In these conditions, as can be easily seen,

$$X_\theta \wedge X_\Lambda - \Lambda$$

is closed. We shall notice that θ is defined only up to a closed 1–form which is a constant of the motion of X_Λ and such that $\theta \wedge \Psi = 0$. In the proof of this claim the following properties play a crucial role:

A. X_θ and X_Λ define an involutive distribution;
B. $L_{X_\theta}\Lambda = 0$.

The first one says that $X_\theta \wedge X_\Lambda$ is a Poisson structure and the second one implies compatibility with Λ.

Theorem 3. *If Λ is a Poisson structure then there is X defined on the support of X_Λ such that $\Lambda - X \wedge X_\Lambda$ is a closed Poisson structure.*

Lets observe that in some cases we are able to define X_θ globally or to replace it with a global one satisfying properties A and B. For instance having a polynomially quadratic Poisson structure on $\mathbb{R}^n$ we can choose X to be $\frac{1}{n}x^i\partial_i$ (cf. [LX]).

It is obvious that if a bivector field Λ is a Poisson structure with the leaves of dimension at most two, then $\Lambda \wedge \Lambda = 0$ and we have $i_\Lambda d\Psi_\Lambda = 0$. Looking for the converse, we get the following.

Theorem 4. *If $f, f_1, ..., f_{n-2}$ are smooth functions on an open and dense subset of an n–dimensional manifold M with a given volume form Ω such that $\Psi = f df_1 \wedge ... \wedge df_{n-2}$ is a smooth $(n-2)$–form on M corresponding to a bivector field Λ then Λ is a Poisson structure with orbits of dimension at most two for which $f_1, ..., f_{n-2}$ are Casimir functions and with The Poisson bracket defined by*

$$\{g, h\} = f dg \wedge dh \wedge df_1 \wedge ... \wedge df_{n-2}/\Omega.$$

Conversely, if Λ is a Poisson structure with orbits of dimension at most two, then there are smooth functions $f, f_1, ...f_{n-2}$ defined on an open and dense subset of M such that $\Psi_\Lambda = f df_1 \wedge ... \wedge f_{n-2}$.

Proof. Let $\tilde{\Omega}$ be the corresponding contravariant volume. Since $\Lambda = i_\Psi \tilde{\Omega}$, it is easy to see that $i_{df_j}\Lambda = 0$. Hence $i_\Lambda \Psi = 0$ and $i_\Lambda d\Psi = 0$, so (17) holds.

Conversely, if Λ has orbits of dimension at most two then in a neighbourhood of a nonsingular point there are coordinates y_j such that the orbits of Λ give us the foliation defined by $y_j = const,\ j = 1, ..., n-2$. Since the form Ψ_Λ vanishes on $\partial_{n-1},\ \partial_n$, it is of the form $f dy_1 \wedge ... \wedge dy_{n-2}$.

Remark. This theorem holds true if we replace the exact 1–forms df_i with closed 1–forms.

If we are on the vector space $\mathbb{R}^n$, we can use the dilation vector field $\Delta = x_i\partial_i$ to define homogeneous Poisson structures. We say that a tensor field τ is homogeneous of degree k if $L_\Delta \tau = k\tau$, where L denotes the Lie derivative. We shall say that a Poisson structure Λ on $\mathbb{R}^n$, is *polynomially homogeneous of degree* k if $L_\Delta \Lambda = (k-2)\Lambda$. In particular, the polynomially quadratic Poisson structures ($L_\Delta \Lambda = 0$) have the form

$$\Lambda = R^{rs}_{ij} x_r x_s \partial_i \wedge \partial_j$$

and using our association with forms, we easily see that the corresponding $(n-2)$–form Ψ_Λ is of degree n.

We are now ready to tackle the problem of the description of Poisson brackets in $\mathbb{R}^3$ and $\mathbb{R}^4$.

The three dimensional case.

We start with $\mathbb{R}^3$, standard coordinates (x_1, x_2, x_3) and standard volume form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. For a bivector field $\Lambda = c^{ij}\partial_i \wedge \partial_j$ we have $f := \Psi_\Lambda = f^k dx_k$, where $f^k = \varepsilon_{ijr}\delta^{rk}c^{ij}$. Since in this dimension clearly $\Lambda \wedge \Lambda = 0$, due to Theorem 1, the Jacobi identity for the Poisson bracket has the form $i_\Lambda d(f^k dx_k) = 0$. This in turn is equivalent to $df \wedge f = 0$, i.e. f admits an integrating factor, so we can write $f = u d\varphi$ for some smooth functions u, φ defined at least on an open and dense subset of the support of Λ. One can explain it also in a different way. It is easy to see that f vanishes on the tangent spaces to (two dimensional) orbits of Λ. In neighbourhoods of nonsingular points of Λ we can choose the coordinates (y_1, y_2, y_3) such the the orbits are described by $y_3 = const$. Hence the form dy_3 has the same kernel as f, so they differ by a factor. This way we get the following.

Theorem 5. *Every Poisson structure Λ on $\mathbb{R}^3$ corresponds to the one form $f = i_\Lambda \Omega$ which can be written as $f = u d\varphi$ for smooth functions u, φ defined on an open* *and dense subset of the support of* Λ. *The most general Poisson bracket in* $\mathbb{R}^3$ *can be therefore written in the form*

$$\{x_i, x_j\} = \varepsilon_{ijk} u \frac{\partial \varphi}{\partial x_k}.$$

Remark. It is worth noting that we really have to admit not everywere defined (singular) functions. For example the bracket in $\mathbb{R}^3$ defined by $\{x_1, x_3\} = x_1$, $\{x_2, x_3\} = x_2$, $\{x_1, x_2\} = 0$, corresponds to the one–form $f = x_2 dx_1 - x_1 dx_2$. The integrating factor is here $\frac{1}{x_1^2+x_2^2}$, so we can write

$$f = (x_1^2 + x_2^2) d(arctg \frac{x_1}{x_2}).$$

We list now some relevant examples.

I. Assume that $u = 1$.

1. The function φ is linear in x_1, x_2, x_3. Then after orthogonal transformation we may consider $\varphi = a x_3$ and we obtain

$$\{x_1, x_2\} = a, \quad \{x_3, x_1\} = \{x_3, x_2\} = 0.$$

We may take $x_3 = c$ and we have the standard Poisson brackets.

2. The function φ is quadratic in x_1, x_2, x_3. Then by means of orthogonal transformation we obtain

$$\varphi = \frac{1}{2}(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2).$$

We have

$$\{x_1, x_2\} = a_3 x_3, \quad \{x_2, x_3\} = a_1 x_1, \quad \{x_3, x_1\} = a_2 x_2.$$

Let us consider some special cases

(i)

$$a_1 = a_2 = a_3 = 1;$$

we get the K–K–S bracket corresponding to the Lie algebra $so(3)$.

(ii)

$$a_1 = a_2 = -a_3 = 1;$$

we get the K–K–S bracket corresponding to the Lie algebra $so(2,1)$.

(iii)

$$a_3 = 0; \quad \{x_1, x_2\} = 0, \quad \{x_2, x_3\} = x_1, \quad \{x_3, x_1\} = x_2;$$

we get the K–K–S bracket corresponding to the Lie algebra $e(2)$.

(iv)

$$a_1 = a_2 = 0,$$

we get the K–K–S bracket corresponding to the Heisenberg-Weyl algebra $\mathcal{W}_1$

$$\{x_1, x_2\} = x_3, \quad \{x_2, x_3\} = \{x_3, x_1\} = 0.$$

2'. When $\varphi = x_1 x_2 + x_2 x_3 + x_1 x_3$, the quadratic form associated with φ can be reduced to the canonical form $\varphi = -2y_1^2 + y_2^2 + y_3^2$ and we get again $so(2,1)$.

3. The function φ is cubic; Here we have 10 monomials, i.e. in general we have a 10–parameter family of brackets. We give only two examples.

(i) $\varphi = x_1x_2x_3$. We get the bracket

$$\{x_1, x_2\} = x_1x_2, \quad \{x_2, x_3\} = x_2x_3, \quad \{x_3, x_1\} = x_3x_1.$$

(ii) $\varphi = \frac{1}{3}(x_1^3 + x_2^3 + x_3^3)$. We get the bracket

$$\{x_1, x_2\} = x_3^2, \quad \{x_2, x_3\} = x_1^2, \quad \{x_3, x_1\} = x_2^2.$$

By using a polynomial function of degree k we get the bracket which is polynomial of degree $k-1$.

II.

1. Put now $f = ud\varphi$, where $\varphi = \psi(x_1) + \psi(x_2) + \psi(x_3)$, $u = \chi(x_1)\chi(x_2)\chi(x_3)$, where $\chi(x) = \frac{1}{\psi'(x).}$ We get

$$\{x_1, x_2\} = \chi(x_1)\chi(x_2), \quad \{x_2, x_3\} = \chi(x_2)\chi(x_3), \quad \{x_3, x_1\} = \chi(x_3)\chi(x_1).$$

For example, if $\psi(x) = \frac{1}{1-m}x^{1-m}$, we find $ud\varphi = (x_2x_3)^m dx_1 + (x_3x_1)^m dx_2 + (x_1x_2)^m dx_3$.

2. Here

$$\varphi = \psi(x_1)\psi(x_2)\psi(x_3)(\frac{1}{\psi(x_1)} + \frac{1}{\psi(x_2)} + \frac{1}{\psi(x_3)})$$

and u as above. Now

$$ud\varphi = (\frac{\psi(x_2)}{\psi'(x_2)}\frac{1}{\psi'(x_3)} + \frac{\psi(x_3)}{\psi'(x_3)}\frac{1}{\psi'(x_2)})dx_1 + cyclic.$$

For instance, with $\psi(x) = \frac{1}{1-m}x^{1-m}$, we get $ud\varphi = c(x_2x_3^m + x_3x_2^m)dx_1 + cyclic.$

3. For $ud\varphi = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$ the Poisson bracket is

$$\{x_1, x_2\} = 0, \quad \{x_2, x_3\} = f_1, \quad \{x_3, x_1\} = f_2.$$

4. $ud\varphi = F(x_1, x_2, x_3)dx_3$ gives

$$\{x_1, x_2\} = F, \quad \{x_2, x_3\} = 0, \quad \{x_3, x_1\} = 0.$$

Remark. For computational purposes it is convenient to notice that in solving for Λ the equation $i_\Lambda \Omega = \psi$ we may start from a volume in the contravariant form, let say $\tilde{\Omega} = \partial_1 \wedge ... \wedge \partial_n$ and consider the contraction $i_\psi \tilde{\Omega} = \Lambda$. In our three dimensional examples this procedure gives $i_{ud\varphi}\tilde{\Omega} = u\frac{\partial\varphi}{\partial x_1}\partial_2 \wedge \partial_3 + u\frac{\partial\varphi}{\partial x_2}\partial_3 \wedge \partial_1 + u\frac{\partial\varphi}{\partial x_3}\partial_1 \wedge \partial_2$. This remark allows to conclude immediately that φ is a Casimir of our bivector field as in Theorem 4.

We notice that by using our formula on compatible Poisson structures we find $d\Psi_{\Lambda\wedge\Lambda} = 0$ because of dimensionality arguments and the remaining term

$$i_{\Lambda_1}d(f_3df_4) + i_{\Lambda 2}d(f_1df_2)$$

reduces to

$$d(f_1 df_2 + f_3 df_4) \wedge (f_3 df_4 + f_1 df_2) = 0,$$

i.e.

$$(f_1 df_3 - f_3 df_1) \wedge df_2 \wedge df_4 = 0.$$

For the Poisson structures compatible with the K–K–S structure on $so(3)^*$ we find $f_2 = (x_1^2 + x_2^2 + x_3^2)/2,\ f_1 = 0$. Therefore we get

$$df_3 \wedge df_4 \wedge d(x_1^2 + x_2^2 + x_3^2) = 0.$$

This condition is equivalent to the closure of $f_3 df_4$ on every three dimensional sphere. Thus the general solution is

$$f_3 df_4 = g df_2 + dh$$

with arbitrary g and h.

As a way to illustrate our association of bivector fields with one–forms we reproduce in a quick way a classification of all three dimensional Lie algebras. We start with coordinates (x_1, x_2, x_3) for $\mathbb{R}^3$ and write $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. The linear bivector field Λ gives us the following 1–form:

$$\Psi = i_\Lambda \Omega = A_{ij} x^i dx^j.$$

We have to reduce Ψ to normal form using linear transformations. We first notice that

$$A_{ij} x^i dx^j = \frac{1}{2}(A_{ij} - A_{ji}) x^i dx^j + d(\frac{1}{4}(A_{ij} + A_{ji}) x^i x^j).$$

By using linear transformation we can set

$$\frac{1}{2}(A_{ij} - A_{ji}) x^i dx^j = n(ydz - zdy).$$

Now, using a linear transformation that preserves $dy \wedge dz$, we can reduce the quadratic form to $ax^2 + by^2 + cz^2$. We obtain the normal form for Ψ to be

$$\Psi = n(ydz - zdy) + d(ax^2 + by^2 + cz^2).$$

Finally we impose the Jacobi condition $\Psi \wedge d\Psi = 0$ to get $na = 0$. Therefore all three dimensional Lie algebras are parametrized by (a, b, c, n) with $na = 0$.

The four dimensional case.

In $\mathbb{R}^4$ we expect to have bivector fields that on some open submanifold are the inverse of a symplectic structure or are of rank two so that, at least locally, they admit two Casimir functions. The most general bivector field Λ will be associated with a two form, let us call it $F = F_{mn} dx^m \wedge dx^n,\ m, n = 0, ..., 3$ and, by Theorem 1, the equation for Λ to be a Poisson structure reads

$$2 i_\Lambda dF = d i_\Lambda F.$$

We can also write $F = (E_i dx^i)\wedge dx^0 - B_{ij}dx^i\wedge dx^j$. If we introduce $H^k = \varepsilon^{ijk}B_{ij}$, we can also rewrite F in terms of $\mathbf{H}$. Of course $F\wedge F = 2\mathbf{EH}dx^0\wedge ... \wedge x^3$. Therefore our bivector field will be degenerated at the points where $\mathbf{EH} = 0$. Assume this equation, since otherwise we are in the symplectic case. If our 2–form F satisfies the Maxwell equations for electromagnetism, we have $dF = 0$ and our associated bivector field will automatically satisfy the Jacobi identity due to Theorem 1 . By using a vector potential to write F and by noting that F is defined up to a "gauge", i.e. a closed 1–form, we see that in this case our bracket is associated with three functions. If F is non–degenerate on an open submanifold of $\mathbb{R}^4$, this describes the situation completely, i.e. $i_\Lambda dF = 0$ implies $dF = 0$ if $F\wedge F \neq 0$. If there is degeneracy, i.e. $F\wedge F = 0$ but $F \neq 0$ on some open submanifold, we can write locally $F = f_1 df_2 \wedge df_3$. By using the contraction with $\tilde{\Omega}$ we get $\Lambda = f_1 i_{df_2} i_{df_3}\tilde{\Omega}$ which shows that $\Lambda(df_2) = 0$, $\Lambda(df_3) = 0$, i.e. both f_2 and f_3 are Casimirs, therefore $i_\Lambda dF = i_\Lambda(df_1\wedge df_2\wedge df_3) = 0$. Thus if $F\wedge F = 0$ the most general Poisson bracket is locally characterized by three functions. It should be noticed that in our argument the conclusion remains true if df_2 and df_3 are both replaced by closed 1–forms which are not exact, actually it is not even required that the monomial form we use is globally true, the condition $i_\Lambda dF = 0$ can be tested on each neighbourhood. For instance $F = \varepsilon_{ijk}x_i dx_j \wedge dx_k$ satisfies our requirement on open submanifolds and $i_\Lambda dF = 0$ globally.

Theorem 6. *If a Poisson structure Λ in $\mathbb{R}^4$ is degenerate (i.e. has orbits of dimension at most two) then it corresponds to a two form $F = f_1 df_2 \wedge df_3$ for densely defined smooth functions f_1, f_2, f_3.*

Remark that in general df_2 and df_3 can be replaced by closed 1–forms.

We close these general considerations by saying that when F reduces to purely "electric" or "magnetic" or "radiation" field, we will have a degenerate Poisson bracket.

Example 7. Due to Theorem 4, the form $F = d(a_i x_i^2)\wedge d(b_j x_j^2)$ where $a_i, b_i,\ i = 0,1,2,3$, corresponds to a Poisson structure Λ on $\mathbb{R}^4$. Since clearly $F = 2(a_i b_j - b_i a_j)x_i x_j dx_i \wedge dx_j$, so $\Lambda = \varepsilon_{ijkl}(a_i b_j - b_i a_j)x_i x_j \partial_k \wedge \partial_l$ and we get the Poisson bracket

$$\{x_k, x_l\} = \varepsilon_{klij}(a_i b_j - b_i a_j)x_i x_j.$$

If we put $b_0 = 0,\ b_1 = b_2 = b_3 = 1,\ a_0 = 1$, we get the Sklyanin bracket [Skl]

$$\{x_k, x_l\} = \varepsilon_{jkl}x_0 x_j, \qquad \{x_k, x_0\} = \varepsilon_{jkl}(a_j - a_l)x_j x_l,$$

where $j, k, l = 1, 2, 3$.

Example 8. Put $F = (E_i dx^i)\wedge dx^0 - \varepsilon_{ijk}H_i dx^j \wedge dx^k$. Then $\Lambda = \partial_0 \wedge (H_i\partial_i) + \varepsilon_{ijk}E_k\partial_i \wedge \partial_j$, i.e. $\{x^0, x^i\} = H_i,\ \{x^i, x^j\} = \varepsilon_{ijk}E_k$. Let us look for quadratic brackets. From $F = dA$ and the quadratic requirement $A = A_i dx^i$ must have $A_i$of degree three, therefore $d(x_0(x_i^2 dx_i) + (x_1^3 + x_2^3 + x_3^3)d(x_1 + x_2 + x_3)) = F$ will provide us with an instance of quadratic bracket in four dimension. It is possible of course to manifacture many more examples. For instance by taking $c_1 = x_1 x_0 - x_2 x_3,\ \ c_2 = x_0^2 + ... + x_3^2$ we get a quadratic Poisson bracket by setting $\Lambda = i_{dc_1} i_{dc_2}\tilde{\Omega}$. It should be noticed that if we consider the Amper two–form $G = (H_i dx^i)\wedge dx^0 + D_{ij}dx^i \wedge dx^j$ with $G\wedge G \neq 0$ the associated bivector field will satisfy the Jacobi identity except at the world lines of the sources. For the electromagnetic field of a moving charge the Jacobi identity fails only along the world line of the charge. This situation generalizes to four dimension the situation for magnetic monopoles in $\mathbb{R}^3$.

Example 9. Consider $\mathbb{R}^4$ as the group $G = \mathbb{R}_+ \times SU(2)$ of 2×2 matrices introducing coordinates

$$\begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

With the element $X \wedge Y$ in the wedge product of the Lie algebra of G and where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is associated a Lie–Poisson structure on G defined by

$$\Lambda(x) = (L_x)_*(X \wedge Y) - (R_x)_*(X \wedge Y),$$

where L_x, R_x denote the left and right translations. The corresponding Poisson bracket in $\mathbb{R}^4$ is quadratic and has the form

$$\{x_2, x_1\} = x_3^2 + x_4^2; \qquad \{x_1, x_4\} = x_2x_4; \qquad \{x_1, x_3\} = x_2x_3;$$

$$\{x_2, x_4\} = -x_1x_4; \qquad \{x_2, x_3\} = -x_1x_3; \qquad \{x_3, x_4\} = 0.$$

This bracket is invariant with respect to the left and right action of the group of matrices

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

(which is the Cartan subgroup). Here one of Casimir functions is $C = \frac{1}{2}(x_1^2 + ... + x_4^2)$, so the structure reduces to spheres, in particular to the $SU(2)$–group (cf. [Gr1]). We find the form Ψ_Λ to be

$$\Psi_\Lambda = (x_3^2 + x_4^2)\frac{x_3dx_4 - x_4dx_3}{x_3^2 + x_4^2} \wedge dC$$

(cf. Theorem 6).

Example 10. Similarly as above, starting with the element

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the wedge product of the Lie algebra $gl(2, \mathbb{R})$, we get a Lie–Poisson structure on $\mathbb{R}^4 \simeq GL(2, \mathbb{R})$. The bracket is

$$\{x_2, x_1\} = x_2x_1; \quad \{x_1, x_4\} = -2x_2x_3; \quad \{x_1, x_3\} = -x_1x_3;$$

$$\{x_2, x_4\} = -x_2x_4; \quad \{x_4, x_3\} = x_4x_3; \quad \{x_3, x_2\} = 0.$$

This bracket is invariant with respect to the left and right action of the group of matrices

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

(which is the Cartan subgroup).

One of Casimir functions is here the determinant $C = x_1x_4 - x_2x_3$ and the structure reduces to $SL(2,\mathbb{R})$. Here we can write Ψ_Λ in the form

$$\Psi_\Lambda = (x_2^2 + x_3^2)\frac{x_3dx_2 - x_2dx_3}{x_2^2 + x_3^2} \wedge dC.$$

We close this section by mentioning how the Lie–Poisson condition can be stated in terms of Ψ. We consider Λ to be a Poisson structure on a Lie group. The following multiplicative condition characterizes a Lie–Poisson structure

$$\Lambda(gh) = (L_g)_*\Lambda(h) - (R_h)_*\Lambda(g).$$

By using a volume Ω which is invariant under the left and right multiplication, we find the following transformation property for $\Psi = i_\Lambda\Omega$:

$$\Psi(gh) = (L_{g^{-1}})^*\Psi(h) - (R_{h^{-1}})^*\Psi(g)$$

when Λ is assumed to define a Lie–Poisson structure.

Conclusions.

There are sound reasons for the current interest in quantum groups. Let us quote some important applications: solvable two–dimensional systems, rational conformal field theory, two–dimensional gravity, three–dimensional Chern–Simons theory, and other interesting topics. It has also been shown [Dr2] that the classical limit of a quantum group is provided by a Lie–Poisson group. It seems therefore of interest to study Poisson structures on Lie groups which are candidates to be Lie–Poisson structures. Our paper is an attempt to start an effective systematic study of Poisson structures, even though we have not yet tackled the Lie–Poisson aspect. Our approach seems to be rather powerful for Poisson brackets of rank two. For this particular situation we expect (cf. [Gr2]) that a star–quantization is possible providing a star–product on the algebra of functions. Additional work is required to systematically investigate higher rank Poisson structures, we hope it will be easier by using the presented approach. Our procedure allows us to deal with compatible Poisson structures in a constructive way and therefore to deal with completely integrable systems that arise in such a framework, however these aspects shall be taken up in future work.

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