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Symplectic geometry of the Chern-Simons theory

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Abstract

This article is a review of two original papers [1], [2]. We begin with a description of Kirillov symplectic form and quantum mechanics on coadjoint orbits of a simple Lie group. This theory may be generalized for the case of a Poisson-Lie group. Both these theories are important for understanding of the Chern-Simons model which may be treated as a 3D gauge theory interacting with coadjoint orbits sitting on Wilson lines. Due to topological nature of the Chern-Simons theory one can get rid of the gauge fields in exchange of modification of coadjoint theories. We discover that this modification is exactly the same as we find in the Poisson-Lie case.

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1 Introduction

The nonabelian Chern-Simons theory in 3 dimensions has been solved in [3] using its relation to the 2-dimensional Wess-Zumino-Novikov-Witten model. Recently it has been proved that the Chern-Simons theory on the cylinder (Cartesian product of a Riemann surface and a real axis) may be efficiently reduced to the 2 dimensional topological gauged WZNW model [4], [5]. We learn from these examples that the topological 3-dimensional theory can be related to some solvable two-dimensional theory either conformal or topological.

Here we advocate another approach to 3D-topological theories and demonstrate it on the example of the Chern-Simons model. Namely, instead of dealing with some 2-dimensional model we reduce the problem to a solvable quantum mechanics. The natural question in the Chern-Simons theory is to evaluate a correlation function on some 3D manifold with several Wilson lines inserted. It was suggested in [3] and advocated in [6] that the Chern-Simons theory may be represented as a 3D gauge theory interacting with some quantum mechanical systems living on the Wilson lines. These systems give a physical interpretation of the representation theory of Lie algebras [7]. The models of these family are designed in such a way that their Hilbert spaces coincide with particular irreducible representations of a given Lie algebra. Our aim in this paper is to get rid of the gauge fields in the model and end up with somewhat modified quantum mechanics on the Wilson lines. We restrict ourselves to the geometry of the cylinder and fulfil the described program.

When the gauge field disappears from the system the quantum mechanics on the Wilson lines changes. Fortunately, this particular way to modify the orbit quantum mechanics has been studied previously [8], [1]. It corresponds to the generalization of the notion of the Lie group to Poisson-Lie group when the group manifold carries a nontrivial Poisson bracket. After quantization this idea leads to a definition of quantum groups.

We always stay here on the classical level of consideration as quantum effects in the Chern-Simons model lead only to a finite renormalization of the coupling constant.

The paper is organized as follows. In Section 2 we remind the construction of Kirillov symplectic form and then define quantum mechanical systems appropriate for description of Wilson lines. Section 3 is devoted to machinery of Poisson-Lie groups. There we introduce the modified symplectic structures which will replace naive Kirillov form in the Chern-Simons model. In Section 4 we turn to the main point of the paper and first represent the Chern-Simons theory on a cylinder as an interacting theory of 3D gauge fields and Wilson line quantum mechanics following [6]. As it was pointed out in [3] the problem reduces to analysis of the moduli space of flat connections on a Riemann surface with marked points. We reexamine the symplectic structure of this space and discover that it splits into the direct sum of several terms. Some of these terms may be naturally assigned to the Wilson lines. They coincide with certain symplectic forms related to Poisson-Lie groups and described in Section 3. The other terms have a similar structure and take into account topology of a 3D-manifold. More exactly, each handle of the Riemann surface is roughly speaking equivalent to two marked points.

2 Geometric quantization and Wilson lines

For the purpose of selfconsistency we collect in this section some well-known results concerning Poisson and symplectic structures associated to Lie groups. The most important part of our brief survey is a theory of coadjoint orbits. We concentrate on Kirillov symplectic form and the corresponding action for the dynamical system on the orbit. It appears that a Wilson line observable may be represented as a quantum partition function for such system.

2.1 Kirillov form

Let us fix notations. The main object of our interest is a simple Lie group G. We denote the corresponding Lie algebra by $\mathcal{G}$. The linear space $\mathcal{G}$ is supplied with Lie commutator [,]. If $\{\varepsilon^a\}$ is a basis in $\mathcal{G}$, we can define structure constants f^{ab}_c in the following way:

$$[\varepsilon^a, \varepsilon^b] = \sum_c f^{ab}_c \varepsilon^c \quad . \tag{2.1}$$

The Lie group G has a representation which acts in $\mathcal{G}$. It is called adjoint representation:

$$\varepsilon^g \equiv Ad(g^{-1})\varepsilon \quad . \tag{2.2}$$

For a matrix realization of the group G the adjoint action is represented by conjugation:

$$\varepsilon^g \equiv g^{-1}\varepsilon g \quad . \tag{2.3}$$

The corresponding representation of the algebra $\mathcal{G}$ is realized by the commutator:

$$ad(\varepsilon)\eta = [\varepsilon, \eta] \quad . \tag{2.4}$$

We denote elements of the algebra $\mathcal{G}$ by small Greek letters.

Let us introduce a space $\mathcal{G}^*$ dual to the Lie algebra $\mathcal{G}$. There is a canonical pairing $<,>$ between $\mathcal{G}^*$ and $\mathcal{G}$ and we may construct a basis $\{l_a\}$ in $\mathcal{G}^*$ dual to the basis $\{\varepsilon^a\}$ so that

$$< l_a, \varepsilon^b >= \delta^b_a \quad . \tag{2.5}$$

We use small Latin letters for elements of $\mathcal{G}^*$. Each vector ε from $\mathcal{G}$ defines a linear function on $\mathcal{G}^*$:

$$H_\varepsilon(l) = < l, \varepsilon > \quad . \tag{2.6}$$

In particular, a linear function H^a corresponds to an element ε^a of the basis in $\mathcal{G}$. By duality the group G and its Lie algebra $\mathcal{G}$ act in the space $\mathcal{G}^*$ via the coadjoint representation:

$$< Ad^*(g)l, \varepsilon >=< l, Ad(g^{-1})\varepsilon > \quad , \tag{2.7}$$

$$< ad^*(\varepsilon)l, \eta >= - < l, [\varepsilon, \eta] > \quad . \tag{2.8}$$

The space $\mathcal{G}$ can be considered as a space of left-invariant or right-invariant vector fields on the group G. Let us define the universal right-invariant one-form θ_g on G which takes values in $\mathcal{G}$:

$$\theta_g(\varepsilon) = -\varepsilon \quad . \tag{2.9}$$

We treat ε in the l.h.s. of formula (2.9) as a right-invariant vector field whereas in the r.h.s. as an element of $\mathcal{G}$. Since the one-form θ_g and the vector field ε are right-invariant the result does not depend on the point g of the group. θ_g is known as Maurer-Cartan form.

Similarly, the universal left-invariant one-form μ_g can be introduced:

$$\mu_g(\varepsilon) = \varepsilon \quad , \quad \mu_g = Ad(g^{-1})\theta_g \quad , \tag{2.10}$$

where ε is a left-invariant vector field, Ad acts on values of θ_g.

In the case of matrix group G the invariant forms θ_g and μ_g look like follows:

$$\theta_g = \delta g \, g^{-1} \quad , \tag{2.11}$$

$$\mu_g = g^{-1}\delta g \quad . \tag{2.12}$$

For any group G there exist two covariant differential operators ∇_L and ∇_R taking values in the space $\mathcal{G}^*$. These are left and right derivatives:

$$< \nabla_L f, \varepsilon > (g) = -\frac{\delta}{\delta t} f(exp(t\varepsilon)g) \quad , \tag{2.13}$$

$$< \nabla_R f, \varepsilon > (g) = \frac{\delta}{\delta t} f(g \, exp(t\varepsilon)) \quad , \tag{2.14}$$

where exp is the exponential map from a Lie algebra to a Lie group. The simple relation for left and right derivatives of the same function f holds:

$$\nabla_R f = -Ad^*(g^{-1})\nabla_L f \quad . \tag{2.15}$$

The space $\mathcal{G}^*$ carries a natural Poisson structure invariant with respect to the coadjoint action of G on $\mathcal{G}^*$. Let us remark that the differential of any function on $\mathcal{G}^*$ is an element of the dual space , i.e. of the Lie algebra $\mathcal{G}$. It gives us a possibility to define the following Kirillov-Kostant Poisson bracket:

$$\{f, h\}(l) = < l, [\delta f(l), \delta h(l)] > \quad . \tag{2.16}$$

In particular, for linear functions H_ε the r.h.s. of (2.16) simplifies:

$$\{H_\varepsilon, H_\eta\} = H_{[\varepsilon,\eta]} \quad , \tag{2.17}$$

$$\{H^a, H^b\} = \sum_c f_c^{ab} H^c \quad . \tag{2.18}$$

The last formula simulates the commutation relations (2.1).

In general situation the space $\mathcal{G}^*$ supplied with Poisson bracket (2.16) is not a symplectic manifold. The Kirillov-Kostant bracket is degenerate. For example, in the simplest case of $\mathcal{G} = su(2)$ the space $\mathcal{G}^*$ is 3-dimensional. The matrix of Poisson bracket is antisymmetric and degenerates as any antisymmetric matrix in an odd-dimensional space.

The relation between symplectic and Poisson theories is the following. Any Poisson manifold with degenerate Poisson bracket splits into a set of symplectic leaves. A symplectic leaf is defined so that its tangent space at any point consists of the values of all hamiltonian vector fields at this point:

$$v_h(f) = \{h, f\} \quad . \tag{2.19}$$

Each symplectic leaf inherits the Poisson bracket from the manifold. However, being restricted onto the symplectic leaf the Poisson bracket becomes nondegenerate and we can define the symplectic two-form Ω so that:

$$\Omega(v_f, v_h) = \{f, h\} \quad . \tag{2.20}$$

The relation (2.20) defines Ω completely because any tangent vector to the symplectic leaf may be represented as a value of some hamiltonian vector field.

If we choose dual bases $\{e_a\}$ and $\{e^a\}$ in tangent and cotangent spaces to the symplectic leaf we can rewrite the bracket and the symplectic form as follows:

$$\{f, h\} = -\sum_{ab} P^{ab} < \delta f, e_a >< \delta h, e_b > \quad , \tag{2.21}$$

$$\Omega = \sum_{ab} \Omega_{ab}\, e^a \otimes e^b = \frac{1}{2} \sum_{ab} \Omega_{ab}\, e^a \wedge e^b \quad . \tag{2.22}$$

Using definition (2.20) of the form Ω and formulae (2.21), (2.22) one can check that the matrix Ω_{ab} is inverse to the matrix P^{ab}:

$$\sum_c \Omega_{ac} P^{cb} = \delta_a^b \quad . \tag{2.23}$$

For the particular case of the space $\mathcal{G}^*$ with Poisson structure (2.16), there exists a nice description of the symplectic leaves. They coincide with the orbits of coadjoint action (2.7) of the group G. Starting from any point l_0, we can construct an orbit

$$\mathcal{O}_{l_0} = \{l = Ad^*(g) l_0 \quad , \quad g \in G\} \quad . \tag{2.24}$$

Any point of $\mathcal{G}^*$ belongs to some coadjoint orbit. The orbit $\mathcal{O}_{l_0}$ can be regarded as a quotient space of the group G over its subgroup S_{l_0}:

$$\mathcal{O}_{l_0} \approx G/S_{l_0} \quad , \tag{2.25}$$

where S_{l_0} is defined as follows:

$$S_{l_0} = \{g \in G \quad , \quad Ad^*(g) l_0 = l_0\} \quad . \tag{2.26}$$

In the case of $G = SU(2)$ the coadjoint action is represented by rotations in the 3-dimensional space $\mathcal{G}^*$. The orbits are spheres and there is one exceptional zero radius orbit which is just the origin. The group S_{l_0} is isomorphic to $U(1)$ and corresponds to rotations around the axis parallel to l_0. For the exceptional orbit $S_{l_0} = G$ and the quotient space G/G is a point.

Let us denote by p_{l_0} the projection from G to $\mathcal{O}_{l_0}$:

$$p_{l_0} : \quad g \longrightarrow l_g = Ad^*(g) l_0 \quad . \tag{2.27}$$

We may investigate the symplectic form Ω on the orbit directly. However, for technical reasons it is more convenient to consider its pull-back $\Omega^G_{l_0} = p^*_{l_0}\Omega$ defined on the group G itself. We reformulate the famous Kirillov's result in the following form. Let $\mathcal{O}_{l_0}$ be a coadjoint orbit of the group G and p_{l_0} be the projection (2.27). The Poisson structure (2.16) defines a symplectic form Ω on $\mathcal{O}_{l_0}$.

Theorem 1 *The pull-back of Ω along the projection p_{l_0} is the following:*

$$\Omega^G_{l_0} = \frac{1}{2} < \delta l_g \stackrel{\wedge}{,} \theta_g > \quad . \tag{2.28}$$

We do not prove formula (2.28) but the proof of its Poisson-Lie counterpart in subsection 3.4 will fill this gap. Let us make only few remarks. First of all, the form $\Omega^G_{l_0}$ actually is a pull-back of some two-form on the orbit $\mathcal{O}_{l_0}$. Then, $\Omega^G_{l_0}$ is a closed form:

$$\delta \Omega^G_{l_0} = 0 \quad . \tag{2.29}$$

This is a direct consequence of the Jacobi identity for the Poisson bracket (2.16). The form $\Omega^G_{l_0}$ is exact, while the original form Ω belongs to a nontrivial cohomology class. The left-invariant one-form

$$\alpha =< l_g, \theta_g >=< l_0, \mu_g > \tag{2.30}$$

satisfies the equation

$$\delta\alpha = \Omega^G_{l_0} \quad . \tag{2.31}$$

In physical applications the form α defines an action for a hamiltonian system on the orbit:

$$S = \int \alpha \quad . \tag{2.32}$$

This action plays a crucial role in the representation of a Wilson line via functional integral. (see section 2.2).

The rest of this subsection is devoted to the cotangent bundle T^*G of the group G. Actually, the bundle T^*G is trivial. The group G acts on itself by means of right and left multiplications. Both these actions may be used to trivialize T^*G. So we have two parametrizations of

$$T^*G = G \times \mathcal{G}^* \tag{2.33}$$

by pairs (g, l) and (g, m) where l and m are elements of $\mathcal{G}^*$. In the left parametrization G acts on T^*G as follows:

$$\mathrm{L} \qquad h : \ (g, m) \longrightarrow (hg, m) \quad , \tag{2.34}$$

$$\mathrm{R} \qquad h:\ (g,m) \longrightarrow (gh^{-1}, Ad^*(h)m) \quad . \tag{2.35}$$

In the right parametrization left and right multiplications change roles:

$$\mathrm{L} \qquad h:\ (g,l) \longrightarrow (hg, Ad^*(h)l) \quad , \tag{2.36}$$

$$\mathrm{R} \qquad h:\ (g,l) \longrightarrow (gh^{-1}, l) \quad . \tag{2.37}$$

The two coordinates l and m are related:

$$l = Ad^*(g)m \quad . \tag{2.38}$$

The cotangent bundle T^*G carries the canonical symplectic structure Ω^{T^*G} [9]. Using coordinates (g,l,m), we write a formula for Ω^{T^*G} without proof:

$$\Omega^{T^*G} = \frac{1}{2}(< \delta m \,\overset{\wedge}{,}\, \mu_g > + < \delta l \,\overset{\wedge}{,}\, \theta_g >) \quad . \tag{2.39}$$

The symplectic structure on T^*G is a sort of universal one. We can recover the Kirillov two-form (2.28) for any orbit starting from (2.39). More exactly, let us impose in (2.39) the condition:

$$m = m_0 = const \quad . \tag{2.40}$$

It means that instead of T^*G we consider a reduced symplectic manifold with the symplectic structure (for justification see subsection 3.3):

$$\Omega_r = \frac{1}{2} < \delta l, \theta_g > \quad , \tag{2.41}$$

where l is subject to constraint

$$l = Ad^*(g)m_0 \quad . \tag{2.42}$$

Formulae (2.41), (2.42) reproduce formulae (2.27), (2.28) and we can conclude that the reduction leads to the orbit O_{m_0} of the point m_0 in $\mathcal{G}^*$.

2.2 Functional integral for a coadjoint orbit

Our main motivation to consider geometric quantization and Kirillov symplectic form is the application of this theory to the Chern-Simons model. More exactly, we rewrite the expression for a Wilson line observable as a certain functional integral over a coadjoint orbit of the group G. It is convenient to restrict ourselves to the case of G being a simple Lie group as it is the main example which we are interested in in the framework of the Chern-Simons theory.

First, let us remind that the quantization of Kirillov-Kostant bracket (2.16) reproduces the Lie algebraic commutator (2.1). So, we expect that after quantization the Lie algebra $\mathcal{G}$ acts in the Hilbert space of the corresponding quantum system. If we start with an orbit we expect that the corresponding representation is irreducible. This guess is based on the observation that before quantization the group action can move any given point on the orbit to any other point. Procedure of geometric quantization [10] provides a mathematical proof of this conjecture. However, in this paper we use a physical language and treat the quantization procedure in the framework of path integral formulation.

To begin with we need an action which describes our physical system. As we live on the orbit, our nearest concern is to introduce some efficient coordinates. Actually, it has been done in the previous subsection where we parametrized a point on the orbit by the group element (2.27):

$$T = v^{-1}Dv. \tag{2.43}$$

Here we introduced special notations for the case of the simple group G. We denote a point on the orbit represented by matrix from $\mathcal{g}$ by T. The fixed point D is a diagonal matrix which defines the orbit. The group G acts by conjugations:

$$T^g = g^{-1}Tg. \tag{2.44}$$

We remind T gives a momentum mapping corresponding to this action. In terms of D and $v \in G$ Kirillov form (2.28) looks as:

$$\varpi = TrD(\delta v v^{-1})^2. \tag{2.45}$$

So, the action for such a system may be written as

$$S_D(v) = \int TrD(\delta v v^{-1}) - \int H dt. \tag{2.46}$$

Here Hamiltonian H is an arbitrary function on the orbit. For our purposes it is convenient to choose it to be a linear function:

$$H = iTr(AT), \tag{2.47}$$

where $A = A(t)$ is a time-dependent source. The main problem of this theory is to evaluate the partition function:

$$Z_D(A) = \int \mathcal{D}v e^{iS_D(v)}. \tag{2.48}$$

We shall consider this integral with periodic boundary conditions. Strictly speaking, it is not well-defined because of the gauge symmetry with respect to the left action of the diagonal subgroup of G: $u \to hu$. However, this symmetry may be taken into account by the standard renormalization of the integration measure. As for any functional integral, we can rewrite the partition function using an ordered exponent of the Hamiltonian:

$$Z_D(A) = Tr_{\mathcal{H}} Pexp(\int \sum_a A_a(t)\hat{T}^a dt). \tag{2.49}$$

Here $\hat{T}^a$ is an operator corresponding to T^a after quantization, $\mathcal{H}$ is a Hilbert space of the resulting theory. As we discussed, this Hilbert space is expected to be an irreducible representation of the Lie algebra $\mathcal{g}$. The problem is how to find out which representation we get starting from the action $S_D(u)$. The answer looks like follows. Let us represent the highest weight $w(D)$ of the corresponding representation as a diagonal matrix. Then

$$w(D) = D - \rho, \tag{2.50}$$

for ρ being a half sum of positive roots of $\mathcal{G}$.

Let us conclude that we obtained a nice representation for a Wilson line observable in the Chern-Simons theory. Namely, such an observable may be always represented as a partition function in the auxiliary theory on the certain coadjoint orbit:

$$W_{w(D)}(\Gamma) = Z_D(A(t)), \tag{2.51}$$

where $A(t)$ is the restriction of the gauge field A on the curve Γ.

For further information on the orbit functional integral we send the reader to original papers [11],[7],[12]. The representation (2.51) has been applied to the Chern-Simons theory in [6].

3 Symplectic structures associated to Poisson-Lie groups

In this Section we develop machinery of Poisson-Lie groups and find out how Kirillov form modifies when we introduce a nontrivial Poisson bracket on a group manifold. We follow the approach of [1].

3.1 Heisenberg double of Lie bialgebra.

One of the ways to introduce deformation leading to Poisson-Lie groups is to consider the bialgebra structure on $\mathcal{G}$. Following [13], we consider a pair $(\mathcal{G}, \mathcal{G}^*)$, where we treat $\mathcal{G}^*$ as another Lie algebra with the commutator $[,]^*$. For a given commutator $[,]$ in $\mathcal{G}$ we can not choose an arbitrary commutator $[,]^*$ in $\mathcal{G}^*$. The axioms of bialgebra can be reformulated as follows. The linear space

$$\mathcal{D} = \mathcal{G} + \mathcal{G}^* \tag{3.1}$$

with the commutator $[,]_{\mathcal{D}}$:

$$[\varepsilon, \eta]_{\mathcal{D}} = [\varepsilon, \eta] \quad , \tag{3.2}$$

$$[x, y]_{\mathcal{D}} = [x, y]^* \quad , \tag{3.3}$$

$$[\varepsilon, x]_{\mathcal{D}} = ad^*(\varepsilon)x - ad^*(x)\varepsilon \quad . \tag{3.4}$$

must be a Lie algebra. In the last formula (3.4) $ad^*(\varepsilon)$ is the usual ad^*-operator for the Lie algebra $\mathcal{G}$ acting on $\mathcal{G}^*$. The symbol $ad^*(x)$ corresponds to the coadjoint action of the Lie algebra $\mathcal{G}^*$ on its dual space $\mathcal{G}$.

The only thing we have to check is the Jacobi identity for the commutator $[,]_{\mathcal{D}}$. If it is satisfied, we call the pair $(\mathcal{G},\mathcal{G}^*)$ Lie bialgebra. Algebra $\mathcal{D}$ is called Drinfeld double. It has the nondegenerate scalar product $<,>_{\mathcal{D}}$:

$$< (\varepsilon, x), (\eta, y) >_{\mathcal{D}} = < y, \varepsilon > + < x, \eta > \quad , \tag{3.5}$$

where in the r.h.s. $<,>$ is the canonical pairing of $\mathcal{G}$ and $\mathcal{G}^*$. It is easy to see that

$$< \mathcal{G}, \mathcal{G} >_{\mathcal{D}} = 0 \quad , \quad < \mathcal{G}^*, \mathcal{G}^* >_{\mathcal{D}} = 0 \quad . \tag{3.6}$$

In other words, $\mathcal{G}$ and $\mathcal{G}^*$ are isotropic subspaces in $\mathcal{D}$ with respect to the form $<,>_{\mathcal{D}}$. We call the form $<,>_{\mathcal{D}}$ on the algebra $\mathcal{D}$ standard product in $\mathcal{D}$.

We shall need two operators P and P^* acting in $\mathcal{D}$. P is defined as a projector onto the subspace $\mathcal{G}$:

$$P(x+\varepsilon) = \varepsilon \quad . \tag{3.7}$$

The operator P^* is its conjugate with respect to form (3.5). It appears to be a projector onto the subspace $\mathcal{G}^*$:

$$P^*(x+\varepsilon) = x \quad . \tag{3.8}$$

The standard product in $\mathcal{D}$ enables us to define the canonical isomorphism J : $\mathcal{D}^* \longrightarrow \mathcal{D}$ by means of the formula

$$< J(a^*), b >_{\mathcal{D}} = < a^*, b > \quad , \tag{3.9}$$

where a^* is an element of $\mathcal{D}^*$ and b belongs to $\mathcal{D}$. In the r.h.s. we use the canonical pairing of $\mathcal{D}$ and $\mathcal{D}^*$. The standard product can be defined on the space $\mathcal{D}^*$:

$$< a^*, b^* >_{\mathcal{D}^*} = < J(a^*), J(b^*) >_{\mathcal{D}} \quad , \tag{3.10}$$

where a^* and b^* belong to $\mathcal{D}^*$. The scalar product $<,>_{\mathcal{D}}$ is invariant with respect to the commutator in $\mathcal{D}$:

$$< [a,b], c >_{\mathcal{D}} + < b, [a,c] >_{\mathcal{D}} = 0 \quad . \tag{3.11}$$

It is easy to check that the operator J converts ad^* into ad:

$$J ad^*(a) J^{-1} = ad(a) \quad . \tag{3.12}$$

Using the standard scalar product in $\mathcal{D}$, one can construct elements r and r^* in $\mathcal{D}\otimes\mathcal{D}$ which correspond to the operators P and P^*:

$$< a \otimes b, r >_{\mathcal{D}\otimes\mathcal{D}} = < a, Pb >_{\mathcal{D}} \quad , \tag{3.13}$$

$$< a \otimes b, r^* >_{\mathcal{D}\otimes\mathcal{D}} = - < a, P^* b >_{\mathcal{D}} \quad . \tag{3.14}$$

In terms of dual bases $\{\varepsilon^a\}$ and $\{l_a\}$ in $\mathcal{G}$ and $\mathcal{G}^*$

$$r = \sum_a \varepsilon^a \otimes l_a \quad , \quad r^* = -\sum_a l_a \otimes \varepsilon^a \quad . \tag{3.15}$$

The Lie algebra $\mathcal{D}$ may be used to construct the Lie group D. We suppose that D exists (for example, for finite dimensional algebras it is granted by the Lie theorem) and we choose it to be connected. Originally the double is defined as a connected and simply connected group. However, we may use any connected group D corresponding to Lie algebra $\mathcal{D}$. Property (3.12) can be generalized for Ad and Ad^*:

$$J Ad^*(d) J^{-1} = Ad(d) \quad , \tag{3.16}$$

where d is an element of D.

Let us denote by G and G^* the subgroups in D corresponding to subalgebras $\mathcal{G}$ and $\mathcal{G}^*$ in $\mathcal{D}$. In the vicinity D_0 of the unit element of D the following two decompositions are applicable:

$$d = gg^* = h^*h \quad , \tag{3.17}$$

where d is an element of D, coordinates g, h belong to the subgroup G, coordinates g^*, h^* belong to the subgroup G^*. In general, the subset D_0 does not cover the whole group D. However, it is open and dense. In the further consideration we restrict ourselves to the cell D_0 in D and send the reader to [1] for complete description.

Now we turn to the description of the Poisson brackets on the manifold D. Double D admits two natural Poisson structures. First of them was proposed by Drinfeld [13]. For two functions f and h on D the Drinfeld bracket is equal to

$$\{f, h\} =< \nabla_L f \otimes \nabla_L h, r > - < \nabla_R f \otimes \nabla_R h, r > \quad , \tag{3.18}$$

where $<,>$ is the canonical pairing between $\mathcal{D}\otimes\mathcal{D}$ and $\mathcal{D}^*\otimes\mathcal{D}^*$. Poisson bracket (3.18) defines a structure of a Poisson-Lie group on D. However, the most important for us is the second Poisson structure on D suggested by Semenov-Tian-Shansky [14]:

$$\{f, h\} = -(< \nabla_L f \otimes \nabla_L h, r > + < \nabla_R f \otimes \nabla_R h, r^* >) \quad . \tag{3.19}$$

The manifold D equipped with bracket (3.19) is called Heisenberg double or D_+. It is a natural analogue of T^*G in the Poisson-Lie case. When $\mathcal{G}^*$ is abelian, $G^* = \mathcal{G}^*$ and $D_+ = T^*G$. If the double D is a matrix group, we can rewrite the basic formula (3.19) in the following form:

$$\{d^1, d^2\} = -(rd^1d^2 + d^1d^2r^*) \quad , \tag{3.20}$$

where $d^1 = d \otimes I$, $d^2 = I \otimes d$.

For concrete calculations let us choose the left identification of the tangent space to D with $\mathcal{D}$. We can rewrite the Poisson bracket (3.19) in terms of left derivatives ∇_L:

$$\begin{aligned}\{f, h\}(d) = -(< \nabla_L f \otimes \nabla_L h, r > + < Ad^*(d^{-1})\nabla_L f \otimes Ad^*(d^{-1})\nabla_L h, r^* >) = \\ = - < \nabla_L f \otimes \nabla_L h, r + Ad(d) \otimes Ad(d)\, r^* > \quad .\end{aligned} \tag{3.21}$$

Here we use relation (2.15) between left and right derivatives on a group.

Given a hamiltonian h one can produce the hamiltonian vector field v_h so that the formula

$$< \delta f, v_h >= \{h, f\} \tag{3.22}$$

holds for any function f. Using (3.21), (3.22) we can reconstruct the field v_h:

$$v_h =< \nabla_L h, r + Ad(d) \otimes Ad(d)\, r^* >_2 \quad . \tag{3.23}$$

Having identified $\mathcal{D}$ and $\mathcal{D}^*$ by means of the operator J, we can rewrite the r.h.s. of (3.23) as follows:

$$v_h\big|_d = \mathcal{P}\delta h = (P - Ad(d)P^* Ad(d^{-1}))J(\nabla_L h(d)) \quad , \tag{3.24}$$

where $\mathcal{P}$ acts in $\mathcal{D}$:

$$\mathcal{P} = P - Ad(d)P^* Ad(d^{-1}) \quad . \tag{3.25}$$

It is called Poisson operator.

The problem which appears immediately in the theory of D_+ is the possible degeneracy of Poisson structure (3.19) in some points of D. Stratification of D_+ into the set of symplectic leaves is described in [1]. Here we need only a simple fact about this stratification:

Lemma 1 *The subset*

$$D_0 = GG^* \cap G^* G \tag{3.26}$$

is a symplectic leave in D with respect to the Poisson bracket (3.19).

It means that the bracket (3.19) is actually nondegenerate on D_0.

3.2 Symplectic structure of the Heisenberg double

The subject of this subsection is to find an efficient description of the symplectic form Θ on D_0. Let us introduce two sets of coordinates on D_0:

$$d = gg^* = h^* h. \tag{3.27}$$

In terms of (g, g^*) and (h^*, h) we can write down the answer for Θ.

Theorem 2 *The symplectic form Θ on D_0 can be represented as follows:*

$$\Theta = \frac{1}{2}(< \theta_{h^*} \overset{\wedge}{,} \theta_g > + < \mu_{g^*} \overset{\wedge}{,} \mu_h >) \quad . \tag{3.28}$$

In the formula (3.28) $\theta_g, \theta_{h^*}, \mu_h, \mu_{g^*}$ are Maurer-Cartan forms on G and G^*. The pairing $<,>$ is applied to their values, which can be treated as elements of $\mathcal{G}$ and $\mathcal{G}^*$ embedded to $\mathcal{D} = \mathcal{G} + \mathcal{G}^*$. So we can use $<,>_{\mathcal{D}}$ as well as $<,>$.

Proof of Theorem 2

The strategy of the proof is quite straightforward. We consider Poisson bracket (3.19) on the symplectic leaf D_0. If we use dual bases $\{e_a\}$ and $\{e^a\}$ ($a = 1, \ldots, n = dimD$) of right-invariant vector fields and one-forms on D, the formula (3.19) acquires the following form:

$$\begin{aligned}\{f, h\}(d) &= - < \nabla_L f \otimes \nabla_L h, r + Ad(d) \otimes Ad(d)\, r^* >= \\ &= - \sum_{a,b=1}^{n} < \nabla_L f, e_a >< \nabla_L h, e_b >< e^a, \mathcal{P} J e^b > \quad .\end{aligned} \tag{3.29}$$

The last multiplier in (3.29) is Poisson matrix corresponding to the bracket (3.19):

$$\mathcal{P}^{ab} = < e^a, \mathcal{P} J e^b > \quad . \tag{3.30}$$

Here $\mathcal{P}$ is the same as in (3.25). It is ensured by *Lemma 1* that the matrix $\mathcal{P}^{ab}$ is nondegenerate. The symplectic form Θ can be represented as follows (see subsection 2.1):

$$\Theta = \sum_{a,b=1}^{n} \Theta_{ab} e^a \otimes e^b \quad , \tag{3.31}$$

where the matrix Θ satisfies the following condition:

$$\sum_{c=1}^{n} \Theta_{ac} \mathcal{P}^{cb} = \delta_a^b \quad . \tag{3.32}$$

So what we need is inverse matrix $\mathcal{P}^{-1}$ for $\mathcal{P}^{ab}$. To this end let us introduce two operators $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\mathcal{P}_1 = P + Ad(d) P^* \quad , \tag{3.33}$$

$$\mathcal{P}_2 = P^* - Ad(d) P \quad . \tag{3.34}$$

$\mathcal{P}$ may be decomposed in two ways, using $\mathcal{P}_1$ and $\mathcal{P}_2$:

$$\mathcal{P} = \mathcal{P}_1 \mathcal{P}_2^* = -\mathcal{P}_2 \mathcal{P}_1^* \quad . \tag{3.35}$$

The definition of $\mathcal{P}_1$ and $\mathcal{P}_2$ permit us to write down the answer for Θ_{ab}:

$$\Theta_{ab} = < e_a, \Theta e_b >_{\mathcal{D}} \quad , \quad \Theta = P \mathcal{P}_1^{-1} - P^* \mathcal{P}_2^{-1} \quad . \tag{3.36}$$

We must check condition (3.32):

$$\begin{aligned} \delta_a^b = \sum_{c=1}^{n} \Theta_{ac} \mathcal{P}^{cb} = \\ = \sum_{c=1}^{n} < e_a, \Theta e_c >< e^c, \mathcal{P} J(e^b) >= \\ = < e_a, \Theta \mathcal{P} J(e^b) >_{\mathcal{D}} \quad . \end{aligned} \tag{3.37}$$

The product $\Theta\mathcal{P}$ can be easily calculated using (3.35), (3.36):

$$\begin{aligned} \Theta\mathcal{P} = P \mathcal{P}_1^{-1} \mathcal{P}_1 \mathcal{P}_2^* + P^* \mathcal{P}_2^{-1} \mathcal{P}_2 \mathcal{P}_1^* = \\ = P(P - P^* Ad(d^{-1})) + P^*(P^* + P Ad(d^{-1})) = \\ = P + P^* = I \quad . \end{aligned} \tag{3.38}$$

So, the answer is

$$< e_a, \Theta \mathcal{P} J(e^b) >_{\mathcal{D}} = < e^b, e_a > = \delta_a^b \tag{3.39}$$

as it is required by (3.32).

We can rewrite formula (3.36) in more invariant way:

$$\Theta = < \theta_d \overset{\otimes}{,} \Theta \theta_d >_{\mathcal{D}} \quad , \tag{3.40}$$

where θ_d is the Maurer-Cartan on D. Expression (3.36) for the operator Θ still includes inverse operators $\mathcal{P}_{1,2}^{-1}$ implying that some equations must be solved. To this end we represent the Maurer-Cartan form θ_d in two different ways:

$$\theta_d = \theta_g + Ad(d)\mu_{g^*} \quad , \tag{3.41}$$

$$\theta_d = \theta_{h^*} + Ad(d)\mu_h \quad . \tag{3.42}$$

Representations (3.41), (3.42) allow us to calculate $\mathcal{P}_{1,2}^{-1}p^*\theta_d$ explicitly:

$$\mathcal{P}_1^{-1}\theta_d = \theta_g + \mu_{g^*} \quad , \tag{3.43}$$

$$\mathcal{P}_2^{-1}\theta_d = \theta_{h^*} - \mu_h \quad . \tag{3.44}$$

Putting together (3.36), (3.40), (3.43) and (3.44), we obtain the following formula for the symplectic form:

$$\begin{aligned}\Theta =&< (\theta_g + Ad(d)\mu_{g^*}) \stackrel{\otimes}{,} \theta_g >_{\mathcal{D}} - < (\theta_{h^*} + Ad(d)\mu_h) \stackrel{\otimes}{,} \theta_{h^*} >_{\mathcal{D}}= \\ =&< Ad(d)\mu_{g^*} \stackrel{\otimes}{,} \theta_g >_{\mathcal{D}} - < Ad(d)\mu_h \stackrel{\otimes}{,} \theta_{h^*} >_{\mathcal{D}} \quad .\end{aligned} \tag{3.45}$$

Actually, the form (3.45) is antisymmetric. To make it evident, let us consider the identity

$$\begin{aligned}&< p^*\theta_d \stackrel{\otimes}{,} p^*\theta_d >_{\mathcal{D}}= \\ =&< Ad(d)\mu_{g^*} \stackrel{\otimes}{,} \theta_g >_{\mathcal{D}} + < \theta_g \stackrel{\otimes}{,} Ad(d)\mu_{g^*} >_{\mathcal{D}}= \\ =&< Ad(d)\mu_h \stackrel{\otimes}{,} \theta_{h^*} >_{\mathcal{D}} + < \theta_{h^*} \stackrel{\otimes}{,} Ad(d)\mu_h >_{\mathcal{D}} \quad .\end{aligned} \tag{3.46}$$

Or, equivalently,

$$\begin{aligned}&< Ad(d)\mu_{g^*} \stackrel{\otimes}{,} \theta_g >_{\mathcal{D}} - < Ad(d)\mu_h \stackrel{\otimes}{,} \theta_{h^*} >_{\mathcal{D}}= \\ =& - < \theta_g \stackrel{\otimes}{,} Ad(d)\mu_{g^*} >_{\mathcal{D}} + < \theta_{h^*} \stackrel{\otimes}{,} Ad(d)\mu_h >_{\mathcal{D}} \quad .\end{aligned} \tag{3.47}$$

Applying 3.47 to make (3.45) manifestly antisymmetric, one gets:

$$\Theta = \frac{1}{2}(< Ad(d)\mu_{g^*} \stackrel{\wedge}{,} \theta_g >_{\mathcal{D}} + < \theta_{h^*} \stackrel{\wedge}{,} Ad(d)\mu_h >_{\mathcal{D}}) \quad . \tag{3.48}$$

Using representation of d in terms of (g, g^*) and (h^*, h), it is easy to check that formula (3.48) coincides with

$$\Theta = -\frac{1}{2}(< \mu_g \stackrel{\wedge}{,} \theta_{g^*} >_{\mathcal{D}} + < \theta_h \stackrel{\wedge}{,} \mu_{h^*} >_{\mathcal{D}}) \quad . \tag{3.49}$$

To obtain formula (3.28) one can use (3.41), (3.42):

$$\theta_d = \theta_g + Ad(d)\mu_{g^*} = \theta_{h^*} + Ad(d)\mu_h \quad . \tag{3.50}$$

Or, equivalently,

$$\theta_g - Ad(d)\mu_h = \theta_{h^*} - Ad(d)\mu_{g^*} \quad . \tag{3.51}$$

Due to antisymmetry we have

$$< (\theta_g - Ad(d)\mu_h) \stackrel{\wedge}{,} (\theta_{h^*} - Ad(d)\mu_{g^*}) >_{\mathcal{D}} = 0 \quad . \tag{3.52}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}(< \theta_{h^*} \stackrel{\wedge}{,} \theta_g >_{\mathcal{D}} + < \mu_{g^*} \stackrel{\wedge}{,} \mu_h >_{\mathcal{D}}) = \\ = &\frac{1}{2}(< Ad(d)\mu_{g^*} \stackrel{\wedge}{,} \theta_g >_{\mathcal{D}} + < \theta_{h^*} \stackrel{\wedge}{,} Ad(d)\mu_h >_{\mathcal{D}}) = \Theta \quad , \end{aligned} \tag{3.53}$$

which coincides with (3.28).

One can easily check that the r.h.s. of formula (3.28) does represent the pull-back of some two-form on D_0.

It is known from general Poisson theory that

$$\delta\Theta = 0 \quad , \tag{3.54}$$

but it is interesting to check that form (3.28) is closed by direct calculations. Rewriting equation (3.51) we get:

$$\theta_g - \theta_{h^*} = Ad(d)\mu_h - Ad(d)\mu_{g^*} \quad . \tag{3.55}$$

Taking the cube of the last equation we get:

$$\begin{aligned} &< \theta_g \stackrel{\wedge}{,} \theta_g \wedge \theta_g >_{\mathcal{D}} - < \theta_{h^*} \stackrel{\wedge}{,} \theta_{h^*} \wedge \theta_{h^*} >_{\mathcal{D}} + \\ &+3 < \theta_g \stackrel{\wedge}{,} \theta_{h^*} \wedge \theta_{h^*} >_{\mathcal{D}} -3 < \theta_g \wedge \theta_g \stackrel{\wedge}{,} \theta_{h^*} >_{\mathcal{D}} = \\ &=< \mu_h \stackrel{\wedge}{,} \mu_h \wedge \mu_h >_{\mathcal{D}} - < \mu_{g^*} \stackrel{\wedge}{,} \mu_{g^*} \wedge \mu_{g^*} >_{\mathcal{D}} + \\ &+3 < \mu_h \stackrel{\wedge}{,} \mu_{g^*} \wedge \mu_{g^*} >_{\mathcal{D}} -3 < \mu_h \wedge \mu_h \stackrel{\wedge}{,} \mu_{g^*} >_{\mathcal{D}} \quad . \end{aligned} \tag{3.56}$$

As $\theta_g \wedge \theta_g = \frac{1}{2}[\theta_g \stackrel{\wedge}{,} \theta_g]$ and $\mu_h \wedge \mu_h = \frac{1}{2}[\mu_h \stackrel{\wedge}{,} \mu_h]$ take values in $\mathcal{G}$, $\theta_{h^*} \wedge \theta_{h^*} = \frac{1}{2}[\theta_{h^*} \stackrel{\wedge}{,} \theta_{h^*}]$ and $\mu_{g^*} \wedge \mu_{g^*} = \frac{1}{2}[\mu_{g^*} \stackrel{\wedge}{,} \mu_{g^*}]$ take values in $\mathcal{G}^*$ we may use the pairing $<,>_{\mathcal{D}}$ for them. Moreover, as both $\mathcal{G}$ and $\mathcal{G}^*$ are isotropic subspaces in $\mathcal{D}$, we rewrite (3.56) as follows:

$$\begin{aligned} &< \theta_g \stackrel{\wedge}{,} \theta_{h^*} \wedge \theta_{h^*} >_{\mathcal{D}} - < \theta_g \wedge \theta_g \stackrel{\wedge}{,} \theta_{h^*} >_{\mathcal{D}} - \\ &- < \mu_h \stackrel{\wedge}{,} \mu_{g^*} \wedge \mu_{g^*} >_{\mathcal{D}} + < \mu_h \wedge \mu_h \stackrel{\wedge}{,} \mu_{g^*} >_{\mathcal{D}} = 0 \quad . \end{aligned} \tag{3.57}$$

We remind that $\delta\theta_g = \theta_g \wedge \theta_g$ and $\delta\mu_g = -\mu_g \wedge \mu_g$. Thus,

$$\begin{aligned} \delta\Theta = &- < \delta\theta_g \stackrel{\wedge}{,} \theta_{h^*} >_{\mathcal{D}} + < \theta_g \stackrel{\wedge}{,} \delta\theta_{h^*} >_{\mathcal{D}} - \\ &- < \delta\mu_h \stackrel{\wedge}{,} \mu_{g^*} >_{\mathcal{D}} + < \mu_h \stackrel{\wedge}{,} \delta\mu_{g^*} >_{\mathcal{D}} = 0 \quad . \end{aligned} \tag{3.58}$$

Now it is interesting to consider the classical limit of our theory to recover the standard answer for T^*G. There is no deformation parameter in bracket (3.19) but it may be introduced by hand:

$$\{f, h\}_\gamma = \gamma\{f, h\} \quad . \tag{3.59}$$

For the new bracket (3.59) we have the symplectic form:

$$\Theta^\gamma = \frac{1}{\gamma}\Theta \quad . \tag{3.60}$$

To recover coordinates on T^*G one have to parametrize a vicinity of the unit element in the group G^* by means of the exponential map:

$$g^* = \exp(\gamma m) \quad , \tag{3.61}$$

$$h^* = \exp(\gamma l) \quad , \tag{3.62}$$

where m and l belong to $\mathcal{G}^*$. Coordinates m and l are adjusted in such a way that they have finite values after the limit procedure. When γ tends to zero, the formula

$$d = gg^* = h^*h \tag{3.63}$$

leads to the following relations:

$$g = h \quad , \quad l = Ad^*(g)m \quad . \tag{3.64}$$

Expanding the form Θ^γ into the series in γ we keep only the constant term (singularity γ^{-1} disappears from the answer because the corresponding two-form is identically equal to zero). The answer is the following:

$$\Theta^\gamma = \frac{1}{2}(< \delta m \stackrel{\wedge}{,} \mu_g > + < \delta l \stackrel{\wedge}{,} \theta_g >) \tag{3.65}$$

and it recovers classical answer (see subsection 2.1). Deriving formula (3.65), we use the expansions for the Maurer-Cartan forms on G^*:

$$\theta_{g^*} = \gamma\delta m + O(\gamma^2) \quad , \tag{3.66}$$

$$\mu_{h^*} = \gamma\delta l + O(\gamma^2) \quad . \tag{3.67}$$

We have considered general properties of the symplectic structure on the main cell D_0 of the Heisenberg double D_+. Our next aim is the Poisson-Lie analogue of the theory of coadjoint orbits. The necessary technical tools will be introduced in the next subsection.

3.3 Dual pairs

One of powerful tools in Hamiltonian mechanics is the language of dual pairs. Let X be a symplectic space. Obviously, it carries a nondegenerate Poisson structures.

Definition 1 *A pair of Poisson mappings*

$$\begin{aligned} \mu : X \to Y, \\ \nu : X \to Z \end{aligned} \tag{3.68}$$

is called a dual pair if

$$\{\{f, h\} = 0, \forall f = \tilde{f} \circ \mu, \tilde{f} : Y \to C\} \Leftrightarrow \{\exists \tilde{h} : Z \to C, h = \tilde{h} \circ \nu\}. \tag{3.69}$$

In other words, any function lifted from Y is in involution with any function lifted from Z and moreover, if some function commute with any function lifted from Y it means that it is lifted from Z.

The standard source of dual pairs is Hamiltonian reduction. If we have a Hamiltonian action of a group G on a symplectic manifold X, the following pair of projections is dual:

$$\begin{aligned} \mu &: X \rightarrow \mathcal{G}^*, \\ \nu &: X \rightarrow X/G. \end{aligned} \tag{3.70}$$

Here the mapping μ is the momentum mapping from the manifold X to the space dual to the Lie algebra $\mathcal{G}$.

Dual pairs provide the method to classify symplectic leaves in the Poisson spaces Y and Z. For any point $y \in Y$ the subspace $\nu(\mu^{-1}(y))$ is a symplectic leaf in Z. It carries nondegenerate symplectic structure. The same is true in the other direction. Take any point $z \in Z$, then the subspace $\mu(\nu^{-1}(z))$ is a symplectic leaf in Y. Actually, in this paper we don't need the full machinery of dual pairs. Only one simple fact will be of importance for us.

Lemma 2 *Let the pair of mappings (μ, ν) (3.68) be a dual pair. Under these conditions the restriction of the symplectic form Ω on X to the subspace $\mu^{-1}(y)$ coincides with the pull back of the symplectic form ω_y on the symplectic leave $\nu(\mu^{-1}(y))$ along the projection ν:*

$$\Omega \mid_{\mu^{-1}(y)} = \nu^* \omega_y. \tag{3.71}$$

This lemma relates the symplectic structure of the reduced phase space with the symplectic structure of the global space X which is usually much simpler.

3.4 Theory of orbits.

In this subsection we describe reductions of the Heisenberg double D_+ which lead to Poisson-Lie analogues of coadjoint orbits.

The coordinates g, g^*, h, h^* introduced in subsection 3.2 will be quite convenient for this purpose. Let us remark that the relation

$$gg^* = h^* h \tag{3.72}$$

may be used to define the action of G on G^*

$$g : g^* \rightarrow g^{*'}(g, g^*) = h^*. \tag{3.73}$$

This action usually appears in literature with the name dressing transformation [14].

The decomposition

$$d = gg^* = h^* h \tag{3.74}$$

induces Poisson structures on the groups G and G^*. Indeed, let us consider for example the realization of the group $G^* : G^*_L \approx D/G$. This formula is not quite correct because the decomposition $D \approx G^*G$ is not global. However, Poisson and symplectic structures are local objects and we can ignore this subtlety. We have used the notation G^*_L to indicate that we treat G^* as a special quotient of D.

Functions on G^*_L may be regarded as functions on D invariant with respect to right action of G:

$$f(dg) = f(d) \quad . \tag{3.75}$$

The right derivative $\nabla_R f$ is orthogonal to $\mathcal{G}$ for functions on G^*_L:

$$< \nabla_R f, \mathcal{G} >= 0 \quad . \tag{3.76}$$

For a pair of invariant functions f and h the second term in the formula (3.19) vanishes because $r^* \in \mathcal{G}^* \otimes \mathcal{G}$. The first term is an invariant function because the left derivative ∇_L preserves the condition (3.75). So we conclude that the Poisson bracket

$$\{f, h\} = - < \nabla_L f \otimes \nabla_L h, r > \tag{3.77}$$

is well-defined on invariant functions and hence it can be treated as a Poisson bracket on G^*_L. This bracket is consistent with the group multiplication in G^* so that the group G^* equipped with such Poisson bracket becomes a Poisson-Lie group. The same is true for the other three quotients $G^*_R = G \backslash D$, $G_R = G^* \backslash D$ and $G_L = D/G^*$. The purpose of this subsection is to study the stratification of the space G^*_R into symplectic leaves and describe the corresponding symplectic forms on them.

It is instructive to consider the classical limit, when g^* and h^* are very close to the identity. Then formula (3.73) transforms into the coadjoint action of G on $\mathcal{G}^*$:

$$g^* = I + \gamma l + \dots \quad , \tag{3.78}$$

$$h^* = I + \gamma l' + \dots \quad , \tag{3.79}$$

$$l' = Ad^*(g) l \quad . \tag{3.80}$$

We denote the transformations (3.73) by AD^* to remind their relation to the coadjoint action:

$$h^*(g, g^*) = AD^*(g) g^* \quad . \tag{3.81}$$

In order to describe symplectic leaves in G^* let us consider the following pair of Poisson mappings:

$$\begin{array}{ccc} & D_0 & \\ \swarrow & & \searrow \\ G^*_L & & G^*_R \end{array} \quad . \tag{3.82}$$

This pair is a dual pair [14],[15].

Let us apply the general prescription of the previous subsection to the dual pair (3.82). In order to find a symplectic leaf in G^*_R one should pick up some element

$h^*G \in D/G$, consider its preimage in D and project it into $G \setminus D$. It is easy to see that we get an orbit of dressing transformations

$$\mathcal{O}_{h^*} = \{g^* \in G^*, \quad g^* = AD(g^{-1})h^*\}. \tag{3.83}$$

The definition (3.83) introduces at the same time the projection p from G to $\mathcal{O}_{h^*}$:

$$p : g \to g^* = AD(g^{-1})h^*. \tag{3.84}$$

So, the orbits of dressing transformations coincide with symplectic leaves in G^*. Our next task is to evaluate the corresponding symplectic forms. Due to *Lemma 2* the pull back of symplectic form on the orbit to its preimage in D coincides with the restriction of the symplectic form (3.49) on D_0 to this preimage. As h^* is set to be equal to constant, the first term in (3.49) disappears and we end up with the following formula for the symplectic form ϑ on the orbit:

$$p^*\vartheta = \frac{1}{2} < \theta_{g^*}, \mu_g > . \tag{3.85}$$

To consider the classical limit we can introduce a deformation parameter into the formula (3.85):

$$p^*\vartheta_\gamma = \frac{1}{2\gamma} < \theta_{g^*}, \mu_g > \quad . \tag{3.86}$$

In this way one can recover the classical Kirillov form (2.28) as we did it for T^*G in subsection 3.2.

3.5 Example: simple group

In this subsection we rewrite formulae for symplectic forms on D_+ and orbits of dressing transformations for the case of G being a simple Lie group. We begin with form (3.85) on the orbit. In order to make the expression for this form more transparent we need more detailed information about the group G^*. Let us introduce two Borel subgroups B_+ and B_- in the group G. In the case of $G = SL(n)$ these are subgroups of upper-triangular and lower-triangular matrices correspondingly. For both B_+ and B_- one can define a canonical projection to the Cartan subgroup in G. For $SL(n)$ the projection picks up a diagonal part of upper- or lower-triangular matrix. If we denote elements of B_+ or B_- by big letters, then the corresponding small letters always denote the diagonal parts. The group G^* is defined as follows [14]:

$$G^* = \{(L_+, L_-) \in B_+ \times B_-, \quad l_+ l_- = I\}. \tag{3.87}$$

Multiplication in G^* is component-wise:

$$(L_+, L_-)(M_+, M_-) = (L_+M_+, L_-M_-). \tag{3.88}$$

There is a natural mapping α from G^* to G which is given by Gauss decomposition formula:

$$\alpha : (L_+, L_-) \to L = L_+ L_-^{-1}. \tag{3.89}$$

The group structures of G and G^* are different and the mapping α is not a group homomorphism. However, we shall see in Section 4 that it may be useful to replace the requirements of group homomorphism by some weaker conditions. The mapping (3.89) provides an identification of the spaces $\mathcal{G}$ and $\mathcal{G}^*$. Then the pairing $<,>$ may be replaced by the invariant form Tr on $\mathcal{G}$.

It is remarkable that for the element L the dressing action simplifies and acquires the form of group conjugations:

$$AD(g)L = gLg^{-1}. \tag{3.90}$$

Let us choose the orbit of dressing transformations which contains a Cartan matrix C:

$$L = AD(g^{-1})C = g^{-1}Cg = L_+ L_-^{-1}. \tag{3.91}$$

Here we specify the definition (3.83) for the case of simple group G. In the notations (3.91) the symplectic form (3.85) may be represented as:

$$\begin{aligned}\vartheta(g, C) = \frac{1}{2}Tr(\delta L_+ L_+^{-1} - \delta L_- L_-^{-1}) \wedge g^{-1}\delta g = \\ = \frac{1}{2}Tr\{C\delta g g^{-1} \wedge C^{-1}\delta g g^{-1} + L_+^{-1}\delta L_+ \wedge L_-^{-1}\delta L_-\}.\end{aligned} \tag{3.92}$$

The second line may be obtained from the first by straightforward but lengthy calculation.

Now we have an efficient formula for symplectic forms on the orbits and the symplectic form on D_+ is in order. As we learn from formula (3.49), the symplectic form on D_+ consists of two terms. Each term resembles the symplectic form on the orbit of dressing transformations. Let us make this statement more precise. In the simple case one can rewrite the relation (3.72) as follows:

$$L' = gL^{-1}g^{-1}. \tag{3.93}$$

Here L' represents an analogue of right momentum in D_+. We have inverted matrix L in order to get similar Poisson brackets for L and L'. Following the pattern of the dressing orbits, we introduce the diagonal matrix C which consists of common eigenvalues of L and L'^{-1}:

$$\begin{aligned} L &= u^{-1}Cu, \\ L' &= v^{-1}C^{-1}v. \end{aligned} \tag{3.94}$$

The group variable g may be represented as a ratio of u and v:

$$g = v^{-1}u. \tag{3.95}$$

In the notations (3.94,3.95) the form Θ looks as

$$\Theta(u,v,C)=\vartheta(u,C)+\vartheta(v,C^{-1})+Tr\delta CC^{-1}\wedge(\delta uu^{-1}-\delta vv^{-1}). \tag{3.96}$$

The last term in (3.96) corresponds to the fact that the diagonal matrix C is dynamical in D_+. Speaking about the dressing orbits we have no analogue of this term because there C is constant.

In the next Section we shall be considering the symplectic form on the moduli space of flat connections on a Riemann surface with marked points. We shall find that the orbit symplectic structure ϑ may be naturally assigned to a marked point and the form Θ to a handle. Formula (3.96) demonstrates that in some sense one handle is equivalent to two marked points.

4 Symplectic structure of the moduli space

This section is devoted to symplectic geometry of the Chern-Simons theory. As we discussed in Introduction, this theory is defined by the canonical symplectic structure on the moduli space of flat connections on a Riemann surface. Surprisingly, this symplectic structure may be expressed in terms of Poisson-Lie symplectic forms introduced in the previous Section.

4.1 Chern - Simons model

The purpose of this subsection is to provide some physical motivations for study of the moduli space of flat connections starting from the Chern-Simons theory. We follow the approach of [6].

The Chern-Simons theory is a gauge theory in 3 dimensions (in principle the CS term exists in any odd dimension). It is defined by the action principle

$$CS(A)=-Tr\int_M(AdA+\frac{2}{3}A^3). \tag{4.1}$$

Here M is a 3-dimensional (3D) manifold, the gauge field A takes values in some simple Lie algebra $\mathcal{G}$

$$A=A_i^a t^a dx_i. \tag{4.2}$$

The generators t^a form a basis in $\mathcal{G}$ and satisfy the commutation relations

$$[t^a,t^b]=f_c^{ab}t^c. \tag{4.3}$$

We concentrate on the very special case of the CS theory. Suppose that the manifold M locally looks like a cylinder $\Sigma\times R$ (Cartesian product of a Riemann surface Σ and a segment of the real line). In this case we may interpret the theory in terms of Hamiltonian mechanics. We choose the direction parallel to the real line R to be the time direction. Two space–like components of the gauge field A become dynamical variables and we often denote by A the two component gauge field on the

Riemann surface Σ. As usual, the time-component A_0 is a Lagrangian multiplier. After the change of variables the action (4.1) acquires the form

$$CS(A) = Tr \int (A\partial_0 A - 2A_0 F)dt, \tag{4.4}$$

where the first term is a short action $\int pdq$ and the second term introduces a first class constraint

$$F = dA + A^2 = 0. \tag{4.5}$$

The first term in (4.4) determines the Poisson brackets of dynamical variables. In particular, the Poisson bracket of the constraints (4.5) may be easily calculated:

$$\{F^a(z_1), F^b(z_2)\} = f^{ab}_c F^c(z_1)\delta^{(2)}(z_1 - z_2). \tag{4.6}$$

As one expects, the constraints (4.5) generate gauge transformations

$$A^g = g^{-1}Ag + g^{-1}dg. \tag{4.7}$$

Thus, the phase space in the Hamiltonian CS theory is a quotient of the space $\Im$ of flat connections (4.5) over the gauge group ΣG (4.7). We see that the moduli space (we shall often refer to the moduli space of flat connections as to the moduli space) appears to be a phase space of the CS theory on the cylinder. The action principle (4.4) provides canonical Poisson brackets on the moduli space. The efficient description of this Poisson bracket was given in [16].

We continue our brief survey of the CS theory by consideration of possible observables. The CS model enjoys two important symmetries: gauge symmetry and the symmetry with respect to diffeomorphisms. The reparametrization symmetry appears due to the geometric nature of the action (4.1) which is written in terms of differential forms and automatically invariant with respect to diffeomorphisms of the manifold M. It is natural to require that the observables in the CS model respect the invariant properties of the theory. Some observables of this type may be constructed starting from the following data. Let us choose the closed contour Γ in M and a representation I of the algebra $\mathcal{G}$. Apparently the following functional of the gauge field A

$$W_I(\Gamma) = Tr_I Pexp(\int_\Gamma A^I) \tag{4.8}$$

is invariant with respect to both gauge and reparametrization symmetries. Usually the contour Γ and also the expression (4.8) are called a Wilson line and a Wilson line observable. The connection A^I is equal to

$$A^I = A^a T^a_I, \tag{4.9}$$

where matrices T^a_I represent the algebra $\mathcal{G}$ in the representation I.

In the Hamiltonian formulation we may choose two special classes of Wilson lines— vertical and horizontal.

We call a Wilson line horizontal if it lies on an equal time surface. The observable corresponding to a horizontal Wilson line is a functional of two-dimensional gauge field and after quantization it becomes a physical operator. It is important to stress that Wilson lines do not cover the whole set of observables in the CS model.

The Wilson line is called vertical if the contour Γ is parallel to the time axis. In Hamiltonian picture we do not actually control the fact that vertical Wilson lines are closed. They come from the past through reality and disappear in the future. The vertical Wilson line is characterized by the representation I and the point z where it intersects the Riemann surface Σ. The choice of the time axis produces a big difference in the role of horizontal and vertical Wilson lines in the theory. Vertical Wilson lines *do not* correspond to observables in the Hamiltonian formulation. Instead they change the Hamiltonian system (4.4) so that both short action and the constraint get modified.

Using the formula (2.51), one may treat the CS correlator with n vertical Wilson lines inserted

$$Z_k(I_1,\dots,I_n)=\int DAe^{\frac{ik}{4\pi}CS(A)}W_{I_1}\dots W_{I_n} \tag{4.10}$$

as an expression where the gauge field is still classical, whereas some modes corresponding to the matrices T_I are already quantized. The original functional integral would be

$$Z=\int DADg_1\dots Dg_n e^{iS^{tot}}. \tag{4.11}$$

The action S^{tot} is defined by the formula

$$S^{tot}=\frac{k}{4\pi}CS(A)+\sum_{i=1}^{n}(S_{I_i}(v_i)+Tr\int dtA_0(z_i)T_i). \tag{4.12}$$

Here the first term coincides with the standard Chern-Simons action, the second term consists of two parts. The first part collects auxiliary orbit actions for each Wilson line, the second part represents contributions of the Wilson lines into the CS partition function (4.10).

We have reformulated the Hamiltonian Chern-Simons model with vertical Wilson lines as a theory of the 2D gauge field A interacting with a set of finite dimensional systems with coordinates T_i localized at the points z_i. As in the case of the pure CS theory, the Hamiltonian (4.12) is equal to zero. The action of the modified system may be rewritten as

$$S^{tot}=Tr(\frac{k}{4\pi}\int A\partial_0 A+\sum_{i=1}^{n}D_i\partial_0 v_i v_i^{-1})+Tr\int A_0(\sum_{i=1}^{n}T_i\delta(z-z_i)-\frac{k}{2\pi}F). \tag{4.13}$$

The first term in (4.13) is a short action of the Hamiltonian system. It is responsible for the Poisson brackets of dynamical variables. The second term gives the modified constraint

$$\Phi(z)=\sum_{i=1}^{n}T_i\delta(z-z_i)-\frac{k}{2\pi}F=0. \tag{4.14}$$

Let us remark that after quantization the formula (4.14) is still true if we shift the central charge k in the standard way $k\to k+h$ (h is the dual Coxeter number of the algebra $\mathcal{G}$). Actually, the shift of the parameter k is of the same nature as a shift of the highest weight in formula (2.50).

The constraints (4.14) satisfy the same algebra (4.6) as in the pure CS theory. They generate gauge transformations for the gauge field A and conjugations for the variables T_i:

$$A^g = g^{-1}Ag + g^{-1}dg, \quad T_i^g = g(z_i)^{-1}T_i g(z_i). \tag{4.15}$$

The analogue of flatness condition (4.14) together with the modified gauge transformations (4.15) lead to the definition of the moduli space of flat connections on a Riemann surface with marked points (see the next subsection).

So the moduli space of flat connections emerges naturally as a phase space in the Chern-Simons theory. The rest of the paper is devoted to the analysis of the symplectic structure of this space.

4.2 Definition of the symplectic structure on the moduli space

Let Σ be a Riemann surface of genus g with n marked points. Consider a connection A on Σ taking values in a simple Lie algebra $\mathcal{G}$. The canonical symplectic structure [17] on the space $\mathcal{A}$ of all smooth connections may be read from the action (4.4)

$$\Omega_{\mathcal{A}} = \frac{k}{4\pi} Tr \int_\Sigma \delta A \wedge \delta A. \tag{4.16}$$

The form (4.16) is obviously nondegenerate and invariant with respect to the action of the gauge group G_Σ:

$$A^g = g^{-1}Ag + g^{-1}dg. \tag{4.17}$$

We denote the exterior derivative on the Riemann surface by d, whereas the exterior derivative on the space of connections, moduli space or elsewhere is always δ. The action (4.17) is actually Hamiltonian and the corresponding momentum mapping is given (up to a multiplier) by the curvature:

$$\begin{aligned} \mu(A) &= -\frac{k}{2\pi}F; \\ F &= dA + A^2. \end{aligned} \tag{4.18}$$

Let us start with a case when there is no marked points.

Definition 2 *The space of flat connections $\Im_g$ on a Riemann surface of genus g is defined as a zero level surface of the momentum mapping (4.18):*

$$F(z) = 0. \tag{4.19}$$

Definition 3 *The moduli space of flat connections is a quotient of the space of flat connections $\Im_g$ over the gauge group action (4.17):*

$$\mathcal{M}_g = \Im_g / G_\Sigma. \tag{4.20}$$

The curvature being the momentum mapping for the gauge group, the moduli space may be obtained by Hamiltonian reduction from the space of smooth connections. General theory of Hamiltonian reduction [9],[18] ensures that the moduli space carries canonical nondegenerate symplectic structure induced from the symplectic structure (4.16) on $\mathcal{A}$.

Now we turn to more sofisticated case of the Riemann surface with marked points.

We have a coadjoint orbit assigned to each marked point z_i. As the gauge field A may develop a singularity in the vicinity of a marked point we have to choose a class of connections different from smooth connections on Σ. To this end we introduce a notion of decoration.

Definition 4 *A decorated Riemann surface with n marked points is a Riemann surface and a set of coadjoint orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n$ assigned to the marked points $z_1, \ldots, z_n$.*

In order to explain this definition let us introduce the local coordinate ϕ_i in the small neighborhood of the marked point z_i so that

$$\oint_{S_i} d\phi_i = 2\pi. \tag{4.21}$$

Here S_i is a closed contour which surrounds the marked point. Apparently, the coordinate ϕ_i measures the angle in the neighborhood of z_i. On the surface with marked points we admit connections which have singularities of the form

$$A(z)_{z\sim z_i} = A_i d(\frac{\phi_i}{2\pi}) + \tilde{A}(z), \tag{4.22}$$

where A_i are constant coefficients and $\tilde{A}(z)$ is a smooth connection. We call the coefficients A_i singular parts of A.

Definition 5 *The space of connections $\mathcal{A}_{g,n}$ on a decorated Riemann surface with marked points is defined by the requirement that the singular parts of the connection belong to the coadjoint orbits assigned to the corresponding marked points:*

$$\frac{2\pi}{k} A_i \in \mathcal{O}_i. \tag{4.23}$$

It is remarkable that the symplectic structure (4.16) may be used for the space $\mathcal{A}_{g,n}$ as well. It is convenient to introduce one more symplectic space which is the direct product of $\mathcal{A}_{g,n}$ and its collection of coadjoint orbits:

$$\mathcal{A}^{tot}_{g,n} = \mathcal{A}_{g,n} \times \mathcal{O}_1 \times \ldots \times \mathcal{O}_n. \tag{4.24}$$

It carries the symplectic structure

$$\Omega^{tot}_{\mathcal{A}} = \Omega_{\mathcal{A}} + \sum_i^n \varpi_i, \tag{4.25}$$

The action of the gauge group may be defined on the space $\mathcal{A}^{tot}_{g,n}$ as follows:

$$A^g = g^{-1}Ag + g^{-1}dg :$$
$$T_i^g = g(z_i)^{-1}T_i g(z_i), \quad v_i^g = v_i g(z_i). \tag{4.26}$$

As we see, the modified gauge transformations are combined from the standard gauge transformations (4.17) and orbit conjugations (2.44). The momentum mapping is given by the coefficient before A_0 in the action (4.13):

$$\mu(z) = \sum_i^n T_i \delta(z - z_i) - \frac{k}{2\pi}F(z). \tag{4.27}$$

It is easy to see that the definition of $\mathcal{A}_{g,n}$ ensures that there is a lot of solutions of the zero level conditions.

Definition 6 *The space of flat connections on a decorated Riemann surface $\Im_{g,n}$ is defined as a space of solutions of the following equation which replaces the zero curvature condition:*

$$\mu(z) = 0. \tag{4.28}$$

Let us choose a loop S_i surrounding the marked point z_i. One can define the monodromy matrix (or parallel transport) M_i along this way. It is easy to check that if A and $\{T_i\}$ satisfy (4.28), the monodromy matrix M_i belongs to the conjugancy class of the exponent of D_i

$$M_i = u_i^{-1} exp(\frac{2\pi}{k}D_i)u_i. \tag{4.29}$$

Definition 7 *The moduli space of flat connections on a Riemann surface of genus g with n marked points $\mathcal{M}_{g,n}$ is defined as a quotient of the space of flat connection on a decorated Riemann surface over the gauge group action (4.26):*

$$\mathcal{M}_{g,n} = \Im_{g,n}/G_\Sigma. \tag{4.30}$$

It is important that the moduli space $\mathcal{M}_{g,n}$ is obtained by Hamiltonian reduction from the symplectic space $\mathcal{A}_{g,n}^{tot}$. This procedure provides the nondegenerate symplectic form on $\mathcal{M}_{g,n}$ which is the main object of this paper.

4.3 Combinatorial description of the symplectic structure on the moduli space

As it was explained in subsection 3.3, the symplectic structure on the reduced phase space obtained by Hamiltonian reduction from some symplectic space is easy to describe. More exactly, we get the reduced phase space as a projection of some constant momentum surface to the quotient of the global phase space over the group action. The pull-back of the reduced symplectic form to the constant momentum surface is equal to the restriction of the global symplectic form to the same subspace.

The moduli space of flat connections plays the role of the reduced phase space the global space being the space of smooth connections with the symplectic form (4.16). So our main concern is to restrict the form (4.16) to the space of flat connections efficiently. To this end it is convenient to introduce some intermediate finite dimensional space between the moduli space and the space of flat connections which admits an efficient parametrization.

Let us choose a point P on the Riemann surface which does not coincide with marked points z_i. One can define a subgroup of the gauge group $G_\Sigma(P)$ by the requirement:

$$G_\Sigma(P) = \{g \in G_\Sigma, \quad g(P) = I\}. \tag{4.31}$$

The quotient space

$$\mathcal{M}_{g,n}(P) = \Im_{g,n}/G_\Sigma(P) \tag{4.32}$$

is already finite dimensional and admits efficient parametrization.

Let us draw a bunch of circles on the Riemann surface so that there is only one intersection point P. In this bunch we have two circles for each handle (corresponding to a- and b- cycles) and one circle for each marked point. We shall denote the circles corresponding to the i's handle by a_i and b_i $(i = 1, \ldots, g)$ and we shall use symbols m_i $(i = 1, \ldots, n)$ for the circles surrounding marked points. We assume that the circles on Σ are chosen in such a way that the only defining relation in $\pi_1(\Sigma_{g,n})$ looks as

$$m_1 \ldots m_n (a_1 b_1^{-1} a_1^{-1} b_1) \ldots (a_g b_g^{-1} a_g^{-1} b_g) = id. \tag{4.33}$$

To each circle we assign the corresponding monodromy matrix defined by the flat connection A. Let us denote these matrices by A_i,B_i and M_i for a-, b- and m-circles. The set of monodromy matrices provides coordinates on $\mathcal{M}_{g,n}$ and a representation of the fundamental group $\pi_1(\Sigma_{g,n})$. It implies the relation

$$M_1 \ldots M_n (A_1 B_1^{-1} A_1^{-1} B_1) \ldots (A_g B_g^{-1} A_g^{-1} B_g) = I \tag{4.34}$$

imposed on the values of A_i,B_i and M_i. Actually, monodromies M_i are not arbitrary. They belong to conjugancy classes $C_i(G)$ defined by

$$M_i = u_i^{-1} C_i u_i, \tag{4.35}$$

where

$$C_i = \exp\left(\frac{2\pi}{k} D_i\right). \tag{4.36}$$

So the space $\mathcal{M}_{g,n}(P)$ is a subspace in

$$\mathcal{F}_{g,n} = G^{2g} \times \prod_{i=1}^{n} C_i(G) \tag{4.37}$$

defined by the relation (4.34).

The original moduli space may be represented as a quotient of $\mathcal{M}_{g,n}$ over the residual gauge group which is isomorphic to the group G:

$$\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(P)/G. \tag{4.38}$$

It is convenient to define some additional coordinates K_i on $\mathcal{F}_{g,n}$:

$$\begin{aligned} K_0 &= I, \\ K_i &= M_1 \dots M_i, 1 \le i \le n \\ K_{n+2i-1} &= K_{n+2i-2} A_i, \\ K_{n+2i} &= K_{n+2i-1} B_i^{-1} A_i^{-1} B_i. \end{aligned} \tag{4.39}$$

It follows from the equation (4.34) that

$$K_{n+2g} = K_0 = I. \tag{4.40}$$

Unfortunately, coordinates A, B, M and K are not sufficient for analysis of the symplectic form on the moduli space and we have to introduce a new space $\tilde{\mathcal{F}}$:

$$\tilde{\mathcal{F}} = G^{n+2g} \times H^{n+g}. \tag{4.41}$$

Here H is a Cartan subgroup of G. $\tilde{\mathcal{F}}$ may be parametrized by matrices $u_i, i = 1, \dots, n+2g$ from the group G and by Cartan elements $C_i, i = 1, \dots, n+g$. We define a projection from $\tilde{\mathcal{F}}$ to $\mathcal{F}$ by the formulae:

$$\begin{aligned} M_i &= u_i^{-1} C_i u_i, \\ A_i &= u_{n+2i-1}^{-1} C_{n+i} u_{n+2i-1}, \\ B_i &= u_{n+2i} u_{n+2i-1}^{-1}. \end{aligned} \tag{4.42}$$

Let us call $\tilde{\mathcal{M}}_{g,n}(P)$ the preimage of $\mathcal{M}_{g,n}(P)$ in $\tilde{\mathcal{F}}$.

After this lengthy preparations we are ready to formulate the main result of this subsection.

Theorem 3 *The pull-back of the canonical symplectic form on $\mathcal{M}_{g,n}$ to $\tilde{\mathcal{M}}_{g,n}(P)$ coincides with the restriction of the following two-form defined on $\tilde{\mathcal{F}}$:*

$$\Omega_{\mathcal{F}} = \frac{k}{4\pi} Tr[\sum_{i=1}^{n+2g} \delta u_i u_i^{-1} C_i \wedge \delta u_i u_i^{-1} C_i^{-1} - \sum_{i=1}^{n+2g} \delta K_i K_i^{-1} \wedge \delta K_{i-1} K_{i-1}^{-1} + \\ + \sum_{i=1}^{g} \delta C_{n+i} C_{n+i}^{-1} \wedge (\delta u_{n+2i} u_{n+2i}^{-1} - \delta u_{n+2i-1} u_{n+2i-1}^{-1})]. \tag{4.43}$$

The rest of the subsection is devoted to proof of *Theorem 3*.

Proof.

Let us cut the surface along every circle a_i, b_i, m_i. We get $n+1$ disconnected parts. The first n are similar. Each of them is a neighborhood of the marked point with the cycle m_i as a boundary. We denote these disjoint parts by P_i. The last one is a polygon. There is no marked points inside and the boundary is composed of a-,b-, and m-cycles as it is prescribed by formula (4.33). We denote the polygon by P_0.

Being restricted to P_0 a flat connection A becomes trivial:

$$A \mid_{P_0}= g_0^{-1} dg_0. \tag{4.44}$$

For any other part P_i we get a bit more complicated expression:

$$A \mid_{P_i}= \frac{1}{k} g_i^{-1} D_i g_i d\phi_i + g_i^{-1} dg_i. \tag{4.45}$$

We remind that D_i is a diagonal matrix which characterizes the orbit attached to the marked point z_i. There is a set of consistency conditions which tells that the connection described by formulae (4.44,4.45) is actually smooth on the Riemann surface everywhere except the marked points. It means that when one approaches the cuts from two sides, one always gets the same value of A. To be explicit, let us consider the m-cycle which surrounds the marked point z_i. Comparison of equations (4.44,4.45) gives:

$$g_0^{-1} dg_0 \mid_{m_i}= (\frac{1}{k} g_i^{-1} D_i g_i d\phi_i + g_i^{-1} dg_i) \mid_{m_i} . \tag{4.46}$$

This equation may be easily solved:

$$g_0 \mid_{m_i}= NMg_i \mid_{m_i}, \tag{4.47}$$

where N is an arbitrary constant matrix and M is equal to

$$M(\phi_i) = \exp{(\frac{1}{k} D_i \phi_i)}. \tag{4.48}$$

Now we turn to consistency conditions which arise when one considers a- or b-cycles. In this case both sides of the cut belong to the polygon P_0. Let us denote the restrictions of g_0 on the cut sides by $g^{'}$ and $g^{''}$. So we have:

$$g^{'-1} dg^{'} = g^{''-1} dg^{''}. \tag{4.49}$$

We conclude that the matrices $g^{'}$ and $g^{''}$ may differ only by a constant left multiplier:

$$g^{''} = Ng^{'}. \tag{4.50}$$

By now we considered connection A in the region of the surface where it is flat. However, it is not true at the marked points. We calculate the curvature in the region P_i and get a δ-function singularity:

$$F(z) \mid_{P_i}= \frac{2\pi}{k} g_i^{-1} D_i g_i \delta(z - z_i). \tag{4.51}$$

Equations (4.51,4.27,4.28) imply that the value $g_i(z_i)$ coincides with the matrix v_i:

$$g_i(z_i) = v_i. \tag{4.52}$$

Let us remind that v_i diagonalizes the matrix T_i attached to the marked point z_i by definition of the decorated Riemann surface.

Now we are prepared to consider the symplectic structure on the space of flat connections. First, let us rewrite the definition (4.25) in the following way:

$$\Omega^{tot} = \omega_0 + \sum_{i=1}^{n} \omega_i, \tag{4.53}$$

where the summands correspond to different parts of the Riemann surface:

$$\begin{aligned} \omega_0 &= \frac{k}{4\pi} Tr \int_{P_0} \delta A \wedge \delta A, \\ \omega_i &= \frac{k}{4\pi} Tr \int_{P_i} \delta A \wedge \delta A + \varpi_i. \end{aligned} \tag{4.54}$$

The next step must be to substitute (4.44,4.45) into formulae (4.54). The following lemma provides an appropriate technical tool for this operation.

Lemma 3 *Let A be a $\mathcal{G}$-valued connection defined in the region P of the Riemann surface Σ. Suppose that*

$$A = g^{-1} B g + g^{-1} dg. \tag{4.55}$$

Then the canonical symplectic form

$$\omega_P = Tr \int_P \delta A \wedge \delta A \tag{4.56}$$

may be rewritten as

$$\omega_P = Tr \int_P \{\delta B \wedge \delta B + 2\delta[F_B \delta g g^{-1}]\} + Tr \int_{\partial P} \{\delta g g^{-1} d(\delta g g^{-1}) - \delta[B \delta g g^{-1}]\}, \tag{4.57}$$

where F_B is a curvature of the connection B

$$F_B = dB - B^2. \tag{4.58}$$

One can prove *Lemma 3* by straightforward calculation.

Let us apply *Lemma 3* to the polygon P_0. In this case $B = 0$ and the answer reduces to

$$\omega_0 = \frac{k}{4\pi} Tr \int_{\partial P_0} \delta g_0 g_0^{-1} d(\delta g_0 g_0^{-1}). \tag{4.59}$$

The boundary of the polygon ∂P_0 consists of $n+4g$ cycles (4.33). So actually we have $n+4g$ contour integrals in the r.h.s. of (4.59).

Now we use formula (4.57) to rewrite symplectic structures ω_i:

$$\omega_i = \frac{k}{4\pi} Tr \int_{\partial P_I} \{\delta g_i g_i^{-1} d(\delta g_i g_i^{-1}) - \frac{2\pi}{k}\delta[D_i \delta g_i g_i^{-1}]\} -$$
$$-Tr \int_{P_I} \delta\{D_i \delta g_i g_i^{-1}\}\delta(z-z_i) + Tr D_i(\delta v_i v_i^{-1})^2. \tag{4.60}$$

The last term in (4.60) represents Kirillov form attached to the marked point z_i. Taking into account relation (4.52) we discover that this term together with the third term in (4.60) cancel each other.

At this point it is convenient to denote the values of g_0 at the corners of the polygon. We enumerate the corners by the index $i = 0, \ldots, n+4g-1$ so that the end-points of the cycle m_i are labeled by $i-1$ and i. One can easily read from formula (4.33) the enumeration of the ends of a- and b-cycles (see Fig. 1). For example, the end-points of a_i are labeled by $n+4(i-1)$ and $n+4(i-1)+1$, whereas the end-points of a_i^{-1} entering in the same word are labeled by $n+4(i-1)+2$ and $n+4(i-1)+3$. We denote the value of g_0 at the i's corner by h_i.

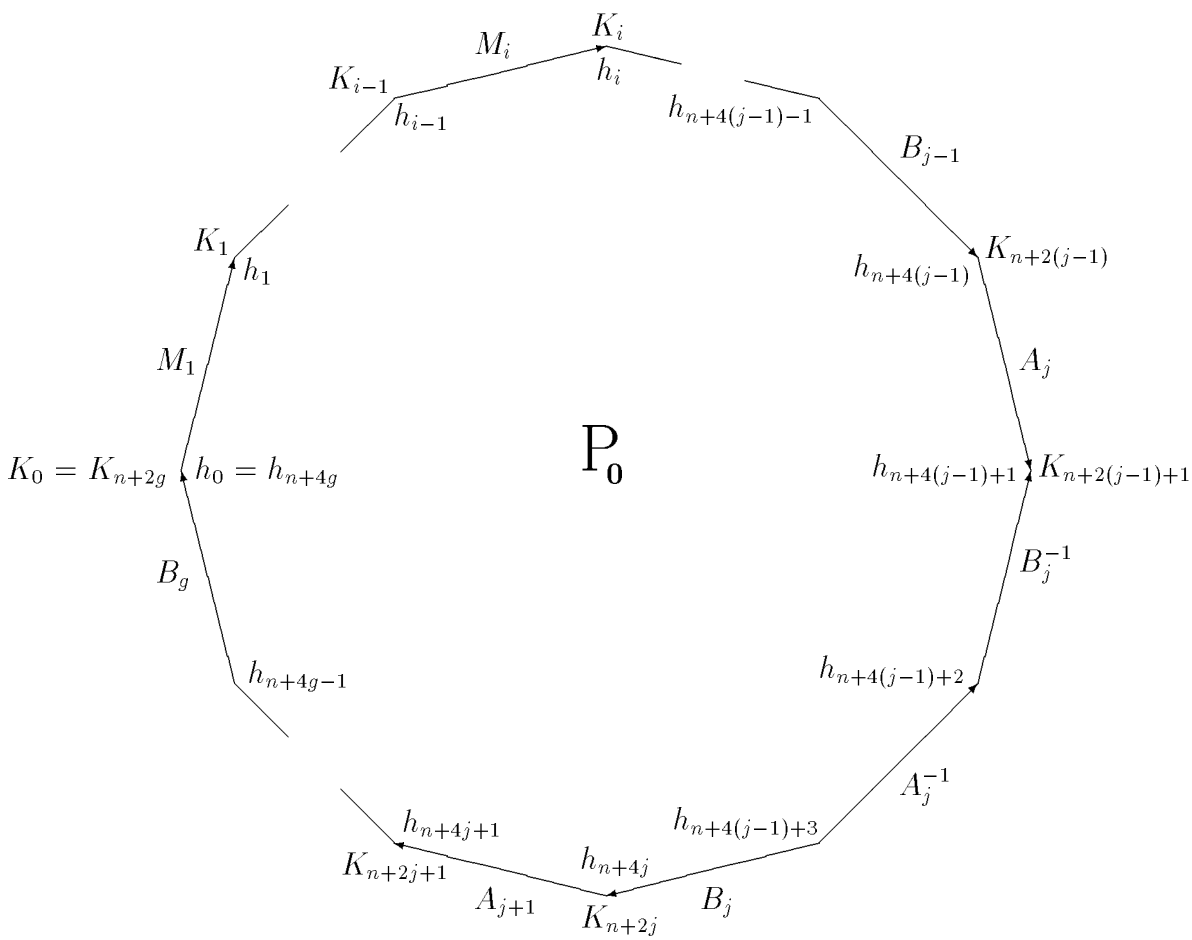

Fig. 1

Monodromies A_i,B_i and M_i may be expressed in terms of h_i as

$$M_i = h_{i-1}^{-1} h_i, \tag{4.61}$$

$$A_i = h_{n+4(i-1)}^{-1} h_{n+4(i-1)+1} = h_{n+4(i-1)+3}^{-1} h_{4(i-1)+2}, \tag{4.62}$$

$$B_i = h_{n+4(i-1)+1}^{-1} h_{n+4(i-1)+2} = h_{n+4i}^{-1} h_{4(i-1)+3}. \tag{4.63}$$

Let us remark that without loss of generality we can choose g_0 in such a way that its value h_0 is equal to unit element in G. After that some of the corner values h_i may be identified with K_i;

$$K_i = \begin{cases} h_i & \text{for } 1 \leq i \leq n \\ h_{2i-n-1} & \text{for } (i-n) \text{ odd} \\ h_{2i-n} & \text{for } (i-n) \text{ even} \end{cases} \tag{4.64}$$

Our strategy is to adjust notations to the description of Poisson-Lie symplectic forms (see subsection 3.2). Using formula (4.47) one can diagonalize M_i

$$M_i = u_i^{-1} C_i u_i. \tag{4.65}$$

Here u_i is the value of the variable g_i at the point P.

Let us rewrite formula (4.59) in the following way:

$$\omega_0 = \sum_{i=1}^{n} \varphi_i + \sum_{i=1}^{g} \psi_i. \tag{4.66}$$

Here φ_i is a contribution corresponding to the marked point:

$$\varphi_i = \frac{k}{4\pi} Tr \int_{m_i} \delta g_0 g_0^{-1} d(\delta g_0 g_0^{-1}), \tag{4.67}$$

and ψ_i is a contribution of the handle:

$$\psi_i = \frac{k}{4\pi} Tr \int_{a_i b_i^{-1} a_i^{-1} b_i} \delta g_0 g_0^{-1} d(\delta g_0 g_0^{-1}). \tag{4.68}$$

First, we are going to evaluate the total contribution of the given M-cycle which is equal to a sum of two terms:

$$\Omega_i = \omega_i + \varphi_i. \tag{4.69}$$

Actually, each summand in (4.69) includes an integral over the m-cycle. However, this sum of integrals is an integral of exact form and it depends only on some finite number of boundary values. This situation is typical and will repeat when we consider a contribution of a handle.

Lemma 4 *The form ω_i depends only on finite number of parameters and may be written as*

$$\omega_i = \frac{k}{4\pi} Tr[C_i \delta u_i u_i^{-1} \wedge C_i^{-1} \delta u_i u_i^{-1} - \delta K_i K_i^{-1} \wedge \delta K_{i-1} K_{i-1}^{-1}]. \tag{4.70}$$

To prove *Lemma 4* one should substitute formula (4.47) into expression for φ_i, integrate by parts and compare the result with the expression for ω_i. The integrals in φ_i and ω_i cancel each other and after rearrangements the boundary terms reproduce formula (4.70).

Now we turn to the contribution of a handle ψ_i into the symplectic form on the moduli space. One can see that each a-cycle and each b-cycle enter twice into expression (4.66). These two contributions correspond to two sides of the cut. As usual, the result simplifies if we combine the contributions of two cut sides together.

Lemma 5 *Let $g^{'}, g^{''}$ be two mappings from the segment $[x_1, x_2]$ into the group G with boundary values $g^{'}_{1,2}, g^{''}_{1,2}$. Suppose that these mappings differ by the x-independent left multiplier*

$$g^{''} = Ng^{'}. \tag{4.71}$$

Then the following equality holds:

$$\Omega_{[x_1,x_2]} = Tr\int_{x_1}^{x_2} \delta g^{''} g^{''-1} d(\delta g^{''} g^{''-1}) - Tr\int_{x_1}^{x_2} \delta g^{'} g^{'-1} d(\delta g^{'} g^{'-1}) =$$
$$= Tr(g_1^{'-1}\delta g_1^{'} \wedge g_1^{''-1}\delta g_1^{''} - g_2^{'-1}\delta g_2^{'} \wedge g_2^{''-1}\delta g_2^{''}). \tag{4.72}$$

Proof is straightforward.

Let us parametrize A_i and B_i as in (4.42):

$$A_i = u^{-1}_{n+2i-1} C_{n+i} u_{n+2i-1}, \quad u_{n+2i} = B_i u_{n+2i-1}. \tag{4.73}$$

One of the motivations for such notations is the following identity:

$$B_i^{-1} A_i^{-1} B_i = u^{-1}_{n+2i} C^{-1}_{n+i} u_{n+2i}. \tag{4.74}$$

In principle, one can introduce the following uniformal variables

$$M_{n+2i-1} = A_i = u^{-1}_{n+2i-1} C_{n+i} u_{n+2i-1},$$
$$M_{n+2i} = B_i^{-1} A_i^{-1} B_i = u^{-1}_{n+2i} C^{-1}_{n+i} u_{n+2i}. \tag{4.75}$$

so that the defining relation (4.34) looks as

$$M_1 \dots M_n M_{n+1} \dots M_{n+2g} = I. \tag{4.76}$$

In these variables we treat handles and marked points in the same way. Roughly speaking, one handle produces two marked points which have the inverse values of C: $C_1 = C_{n+i}$, $C_2 = C^{-1}_{n+i}$. It resembles the relation between the double D_+ and two orbits of dressing transformations (see subsection 3.4). Using the definition of M (4.75) we can clarify the definition of K_i:

$$K_i = M_1 \dots M_i. \tag{4.77}$$

Now we turn to the contribution ψ_i of a handle into symplectic form (4.66).

Lemma 6 *The handle contribution into symplectic form depends only on the values of g_0 at the end-points of the corresponding a- and b-cycles and may be written as*

$$
\begin{aligned}
\psi_i = \frac{k}{4\pi} Tr[C_{n+i}\delta u_{n+2i-1}u^{-1}_{n+2i-1} \wedge C^{-1}_{n+i}\delta u_{n+2i-1}u^{-1}_{n+2i-1} - \\
-\delta K_{n+2i-1}K^{-1}_{n+2i-1} \wedge \delta K_{n+2(i-1)}K^{-1}_{n+2(i-1)} + \\
+C^{-1}_{n+i}\delta u_{n+2i}u^{-1}_{n+2i} \wedge C_{n+i}\delta u_{n+2i}u^{-1}_{n+2i} - \delta K_{n+2i}K^{-1}_{n+2i} \wedge \delta K_{n+2i-1}K^{-1}_{n+2i-1} + \\
+\delta C_{n+i}C^{-1}_{n+i} \wedge (\delta u_{n+2i-1}u^{-1}_{n+2i-1} - \delta u_{n+2i}u^{-1}_{n+2i})].
\end{aligned}
\tag{4.78}
$$

If we take into account *Lemma 5*, the proof of *Lemma 6* becomes straightforward but long calculation. Let us remark that the terrible formula (4.78) contains two copies of the marked point contribution (4.70) with parameters C_{n+i} and C^{-1}_{n+i}. The last term includes $\delta C_{n+i}C^{-1}_{n+i}$ and coincides with the corresponding additional term in formula (3.96) for the symplectic form on the double D_+.

Summarizing *Lemma 4* and *Lemma 6* we get the proof of *Theorem 3* completed.

4.4 Equivalence to Poisson-Lie symplectic structure

Formula (4.43) contains cross-terms with different indices i. In this subsection we represent the canonical symplectic structure as a direct sum of several terms. Using the results of Section 3, each term may be identified with either Kirillov form for the Poisson-Lie group G^* or symplectic form on the Heisenberg double D_+ of the Poisson-Lie group G. To achieve this result we have to make a change of variables. The new set of variables is designed to "decouple" contributions of different handles and marked points.

The following remark is important for understanding of the construction of decoupled variables. Monodromy matrices M_i, A_i and B_i are elements of the group G. In accordance with this fact we use G-multiplication to define the variables K_i (4.77) and to constraint monodromies (4.34). On the other hand, natural variables for description of orbits of dressing transformations or double D_+ must belong to G^*. In subsection 3.5 we defined the mapping $\alpha : G^* \to G$. Unfortunately, α is not a group homomorphism. So, we would face difficulties applying α to identities (4.77,4.34). This is a motivation to introduce a notion of a weak group homomorphism.

Definition 8 *Let G and $G^{'}$ be two groups. A set of mappings*

$$\alpha^{(n)} : G^n \to G'^n \tag{4.79}$$

is called a weak homomorphism if the following diagram is commutative for any i:

$$
\begin{array}{ccc}
G^n & \xrightarrow{\alpha^{(n)}} & G'^n \\
\mathbf{m}_i \downarrow & & \mathbf{m}_i^{'} \downarrow \\
G^{n-1} & \xrightarrow{\alpha^{(n-1)}} & G'^{n-1}\ .
\end{array}
\tag{4.80}
$$

Here $\mathbf{m}_i$ and $\mathbf{m}_i^{'}$ are multiplication mappings in G and $G^{'}$ correspondingly which map the product of n copies of the group into the product of $n-1$ copies:

$$\begin{aligned}\mathbf{m}_i &: (g_1,\ldots,g_i,g_{i+1},\ldots,g_n) \to (g_1,\ldots,g_ig_{i+1},\ldots,g_n) :\\ \mathbf{m}_i^{'} &: (g_1^{'},\ldots,g_i^{'},g_{i+1}^{'},\ldots,g_n^{'}) \to (g_1^{'},\ldots,g_i^{'}g_{i+1}^{'},\ldots,g_n^{'}) :\end{aligned} \tag{4.81}$$

The mapping α (3.89) may be considered as a first mapping of a weak homomorphism from G^* to G. To define the other mappings $\alpha^{(n)}$ we introduce the products

$$K_{\pm}(i) = L_{\pm}(1)\ldots L_{\pm}(i). \tag{4.82}$$

The action of $\alpha^{(n)}$ looks as follows. A tuple $(L_+(i), L_-(i)) \in G^*, i = 1,\ldots n$ is mapped into the tuple $M_i \in G, i = 1,\ldots n$:

$$M_i = K_-(i-1)L_iK_-(i-1)^{-1}. \tag{4.83}$$

Here L_i is the image of the pair $(L_+(i), L_-(i))$ under the action of α:

$$L_i = L_+(i)L_-(i)^{-1}. \tag{4.84}$$

One can easily check that the set of mappings (4.83) satisfies the requirements of a weak homomorphism.

The next step is to implement the definition (4.83) to the space $\tilde{\mathcal{F}}$. Let us introduce a set of variables on $\tilde{\mathcal{F}}$ which consists of $v_i, i = 1,\ldots,n+2g$ taking values in G and $C_i^{'}, i = 1,\ldots,n+g$ taking values in H. In addition we introduce the elements of G^*:

$$\begin{aligned} L_i &= v_i^{-1}C_i^{'}v_i \quad for 1 \le i \le n;\\ L_{n+2i-1} &= v_{n+2i-1}C_{n+i}^{'}v_{n+2i-1} \quad for 1 \le i \le g;\\ L_{n+2i} &= v_{n+2i}C_{n+i}^{'-1}v_{n+2i} \quad for 1 \le i \le g.\end{aligned} \tag{4.85}$$

together with their Gauss components (3.89). So, we have natural variables to describe n copies of the orbit of dressing transformations in G^* and g copies of the Heisenberg double. The canonical symplectic form on this object is equal to the sum of symplectic forms for each copy of the orbit (3.92) and each copy of double (3.96):

$$\Omega_{PL} = \sum_{i=1}^{n}\vartheta(u_i, C_i^{'}) + \sum_{i=1}^{g}\Theta(u_{n+2i-1}, u_{n+2i}, C_{n+i}^{'}). \tag{4.86}$$

Let us compare the forms (4.43) and (4.86). Motivated by the definition (4.83) we introduce the mapping $\sigma : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$ defined by the relations:

$$u_i = v_iK_-^{-1}(i-1), \quad C_i = C_i^{'}. \tag{4.87}$$

Here $K_-(i)$ are defined as in (4.82). It is easy to see that the mapping σ induces the mapping $\alpha^{(n+2g)}$ from the set of pairs $(L_+(i), L_+(i))$ into the set of monodromies

M_i. It is guaranteed by the definition of weak homomorphism that G-product in the relation (4.34) is now replaced by G^*-product:

$$K_{\pm}(n+2g) = L_{\pm}(1)\dots L_{\pm}(n+2g) = I. \tag{4.88}$$

Equation (4.88) defines the preimage of $\tilde{\mathcal{M}}_{g,n}$ in $\tilde{\mathcal{F}}$ with respect to the mapping σ. It is worth mentioning that the matrices K_i from the previous subsection may be represented as

$$K_i = K_+(i)K_-(i)^{-1}. \tag{4.89}$$

This also a consequence of the definition of weak homomorphism. Indeed, K_i has been defined as a product in G of the first i monodromies. Formula (4.82) defines a product in G^* of i first elements $(L_+(i), L_+(i))$. Using the basic property of weak homomorphism $(i-1)$ times we check (4.89).

The mapping σ provides a possibility to compare two-forms $\Omega_{\mathcal{F}}$ and Ω_{PL}.

Lemma 7 *The two-forms $\Omega_{\mathcal{F}}$ is proportional to the pull-back of the form Ω_{PL} along the mapping σ:*

$$\Omega_F = \frac{k}{4\pi}\sigma^*(\Omega_{PL}). \tag{4.90}$$

Lemma 7 may be proved by straightforward calculation. *Theorem 3* and *Lemma 7* imply the following theorem which is the main result of this paper.

Theorem 4 *Being restricted to the subset (4.88), the direct sum of n copies of Kirillov symplectic form on the orbit of dressing transformations in G^* and g copies of the canonical form on the Heisenberg double of the group G coincides up to a scalar multiplier with the pull-back of the canonical symplectic form on the moduli space of flat connections on the Riemann surface of genus g with n marked points.*

5 Conclusions

We have started in Section 4 from the correlator of the Chern-Simons theory on the cylinder with inserted vertical Wilson lines. This system may be represented as a 3D gauge field interacting to a finite number degrees of freedom living on the Wilson lines. As there is no Hamiltonian, the system is completely defined by the symplectic form on the phase space. We proved that this symplectic form may be decomposed into the direct sum of Poisson-Lie symplectic structures subject to constraint (4.34). So, the functional integral for the correlator (4.10) may be rewritten as:

$$Z_k(I_1,\dots,I_n) = \int \prod_{i=1}^{n+2g} \mathcal{D}u_i \prod_{i=n+1}^{n+g} \mathcal{D}C_i \times$$
$$\times e^{\frac{k+h}{2\pi}\int \delta^{-1}\{\sum_{i=1}^{n}\vartheta(u_i,C_i)+\sum_{i=1}^{g}\Theta(u_{n+2i-1},u_{n+2i},C_{n+i})\}}\delta(M_1\dots B_g). \tag{5.91}$$

Here we took into account the standard shift $k \to k+h$ in the Chern-Simons action and used the symbol δ^{-1} as in Wess-Zumino action.. If we compare expression (5.91) with original formula (4.10), we find that the gauge field disappeared and the Wilson line insertions got modified. One can say that the Chern-Simons theory in the bulk quantizes the group variables living on the Wilson lines. In addition to the modified Wilson lines one finds in the partition function (5.91) the finite number of degrees of freedom which carry topological information about genus of the Riemann surface.

If we turn to operator approach, each Wilson line multiplier in (5.91) presents a deformed analogue of the orbit quantum mechanics considered in subsection 2.2. It is natural to expect that quantization leads to the Hilbert space which coincides with the space of certain irreducible representation of the quantum group with the highest weight w given by the formula (2.50). Each multiplier corresponding to a handle gives a regular representation $\Re$ of the same quantum group. The parts of the system corresponding to the different summands in the action (5.91) are related only by the constraint (4.34). It prescribes that the Hilbert space of the whole system is equal to the space of invariants in the tensor product.

$$\mathcal{H} = Inv_q(I_1 \otimes \ldots I_n \otimes \Re^{\otimes g}). \tag{5.92}$$

in agreement with the known results.

More detailed information about operator formalism and the corresponding representation theory may be found in [19], [20]. It would be interesting to work out the functional integral (5.91) by direct calculation and generalize this procedure for 3D topologies different from the cylinder.

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