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Projective Invariance for Classical and Quantum Systems [1]

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Abstract

The Lie Group of projective transformations for different physical systems is considered. It is shown that many mathematically and physically relevant equations are projectively invariant. Classical and quantum systems of particles and their generalizations to kinetic theory and hydrodynamics are considered from this new view point. New invariant equations and corresponding Conservation laws are introduced. The specific role of these transformations and The potential $U(x) = \frac{\alpha}{|x|^2}$ with $x \in \mathcal{R}^n$ are discussed from physical and geometrical points of view. It is shown that all the considered examples are connected with a system of free particles.

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1 Introduction

Lie group analysis methods applied to mathematical models of relevant physical systems always lead to an unexplicitely distinction between trivial and non-trivial symmetries. Roughly speaking it means that we can easily guess the final result in the first case, not in the second one.

The term *trivial* can be also applied to the linear transformations of the both dependent and independent variables in the considered equation. We note that these *trivial* symmetries are often connected with very fundamental properties of the physical world, such as isotropy and uniformity of space or time.

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Analysing, from this point of view, the well-known *complete symmetry groups* of classical equations in mathematical physics, one can notice that in many cases (see below) one of a few non-trivial symmetries turns out to be linked with projective transformations. In the classical book of L.V.Ovsiannikov [1] this symmetry for the Euler gasdynamics equations is even used as an example of non trivial transformations which cannot be guessed from so-called *physical considerations.*

Apparently there is no serious comparative analysis of different projectively invariant physical systems in literature. Therefore we try to do it in this paper.

We shall also observe that in some sense all projectively invariant systems are closely connected with the free particle motion in a bigger *carrier* Euclidean space.

That is why we start from the simplest system (a particle moving along the x-axis with a constant velocity) using it to introduce all necessary notions.

Let us consider the free motion equation:

$$\frac{d^2x}{dt^2} = 0 \qquad x \in \mathcal{R}^1 \qquad t \in \mathcal{R}^1 \tag{1}$$

and introduce some standard notations

We are interested in a 1- parameter group of transformations:

$$x' = \varphi(x, t; a) \qquad t' = \psi(x, t; a) \tag{2}$$

where a is the so-called group parameter.

The group property of transformations (2) implies that they satisfy the following Cauchy problem.

$$\frac{dx'}{da} = X(x', t') \qquad \frac{dt'}{da} = T(x', t') \tag{3}$$

$$x'_{|a=0} = x \qquad t'_{|a=0} = t$$

where

$$X = \left(\frac{\partial \varphi}{\partial a}\right)_{a=o} \qquad T = \left(\frac{\partial \psi}{\partial a}\right)_{a=0}$$

at least for sufficiently small $| \, a \, |$. In a modern language, equations (3) are the differential equations for integral curves (2) of the vector field:

$$Z = X(x, t)\frac{\partial}{\partial x} + T(x, t)\frac{\partial}{\partial t} \tag{4}$$

belonging to the tangent bundle based on the manifold whose local coordinates are (x,t).

Using an older but perhaps a more familiar terminology, the invariance of our equation for transformations (2) is expressed saying that it *admits the operator* Z, the infinitesimal operator of the group (3).

It is well-known [1] , [2], [3] that the complete invariance of eq. (1) is accounted by a 8-parameters group whose corresponding operator Z has the following form

$$\begin{aligned} Z &= \alpha_1 \frac{\partial}{\partial t} + \alpha_2 t \frac{\partial}{\partial t} + \alpha_3 t \frac{\partial}{\partial x} + \alpha_4 \left(t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x}\right) + \\ &+ \beta_1 \frac{\partial}{\partial x} + \beta_2 x \frac{\partial}{\partial x} + \beta_3 x \frac{\partial}{\partial t} + \beta_4 \left(x^2 \frac{\partial}{\partial x} + xt \frac{\partial}{\partial t}\right) \end{aligned} \tag{5}$$

We notice that Z is constructed by using the operators

$$Z_1 = \frac{\partial}{\partial t} \quad Z_2 = t\frac{\partial}{\partial t} \quad Z_3 = t\frac{\partial}{\partial x} \quad Z_4 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} \tag{6}$$

and the exchange $\beta_i \leftrightarrow \alpha_i$ $x \leftrightarrow t$.

The first three operators reflect, respectively, the invariance of the free motion, described by equation (1), under time translations (the uniformity of time), time scaling (different time units choose freedom), Galileo transformations (the principle of relativity).

The corresponding 1-parameter subgroups on the (x, t) plane are given by:

$$\begin{array}{lll} x' = x & x' = x & x' = x + \alpha_3 t \\ t' = t + \alpha_1, & t' = e^{\alpha_2} t, & t' = t. \end{array} \tag{7}$$

The fourth operator in (6), i.e.

$$Z_4 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}, \tag{8}$$

generates the 1-parameter group

$$x' = \frac{x}{1 - at} \quad , \quad t' = \frac{t}{1 - at}, \tag{9}$$

which is a particular case of so-called projective transformations [1] , [2]. The invariance under transformations (9) will be called *the projective invariance*[1].

From our considerations, it follows that the projective transformations (9) belong to the list of the four basic groups (7)-(9) of the equation (1).

From a geometrical point of view we are studying the point transformations on the plane (x, t), which map straight lines into straight lines. So, we have four independent groups and know that three of them, the linear ones, have a clear physical interpretation. The fourth, nonlinear, group is, obviously, local ($at < 1$) as local in nature is the Lie Group Theory. The physical meaning of the projective transformations remains unclear. On the other side there exist some models of different physical nature in which the property of projective invariance is also present. How to connect these examples of projectively invariant equations from different fields of physics? How to look for new projectively invariant systems? What about the consequences of this invariance on the existence of additional conservation laws? The first step in answering these questions was made in [4], but some problems remain unsolved. This paper can be considered as an attempt to clarify the situation, also for quantum systems.

To avoid any possible misundertandings we would like to stress that the aim of our paper is far from usual aims of group analysis, although we shall indicate some new projectively invariant systems. The central problem here is to understand the role of these simple transformations in physics and try to expalin why we find projectively invariant systems in quite different fields. It is of course rather physical than mathematical problem

We start from well-known examples as projectively invariant systems.

2 Three examples.

The transformations (9) can be generalized obviously to the multidimensional case, the corresponding operator being:

[1]This terminology is not universally adopted (see, for example, [3]) but it is convenient for goals of our paper.

$$Z = t^2 \frac{\partial}{\partial t} + tx \cdot \frac{\partial}{\partial x} \quad , \quad x \in \mathcal{R}^n \quad , \quad t \in \mathcal{R}, \tag{10}$$

the *dot* $\cdot$, x and $\frac{\partial}{\partial x}$ denoting, respectively, the standard scalar product in $\mathcal{R}^n$, a vector of Euclidean space $\mathcal{R}^n$ $(n = 1, 2, ..)$ and the *gradient* operator.

The corresponding projective tranformations in $\mathcal{R}^{n+1}$ are:

$$x' = \frac{x}{1 - at} \quad , \quad t' = \frac{t}{1 - at} \quad , \quad x \in \mathcal{R}^n \quad , \quad t \in \mathcal{R} \tag{11}$$

We also consider the transformation at the points (x, v) of the phase space R^{2n}, where v denotes the velocity, i.e. $v = \frac{dx}{dt}$. If we require

the correspondence

$$v = \frac{d}{dt} x(t) \rightarrow v' = \frac{d}{dt'} x'(t'), \tag{12}$$

the extended operator $\hat{Z}$, the so-called *first extension of* Z, can be written as

$$\hat{Z} = t^2 \frac{\partial}{\partial t} + tx \cdot \frac{\partial}{\partial x} + (x - vt) \cdot \frac{\partial}{\partial v} \tag{13}$$

from which we get the following formulas for the extended transformations in the phase space-time R^{2n+1}

$$t' = \frac{t}{1 - at} \quad , \quad x' = \frac{x}{1 - at} \quad , \quad v' = v(1 - at) + ax$$
$$x \in \mathcal{R}^n \quad , \quad v \in \mathcal{R}^n \quad , \quad t \in \mathcal{R} \quad , \quad n = 1, ... \tag{14}$$

We are ready now to consider the three classical examples with similar symmetry properties [1] , [2].

2.1 Classical Mechanics

Let $x \in R^n$ be a coordinate of a particle (mass $m \equiv 1$) moving in the central field with the potential

$$U(|x|) = \frac{\alpha}{|x|^2}, \ \alpha > 0, \ x \in R^n \tag{15}$$

The equation of the motion:

$$\frac{d^2 x}{dt^2} = -\frac{\partial}{\partial x} U(|x|) \tag{16}$$

is, for any α, invariant under projective transformations (11). The free motion (case $\alpha = 0$) is now included in the more general equation (15)-(16). On the other hand this equation can be obtained, for any fixed positive α, from the one of the free motion.

2.2 The Heat Transfer Equation

The heat transfer equation is written:

$$u_t = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \quad , \quad n = 1, \ldots \tag{17}$$

the function $u(x,t)$ denoting the temperature field at the point $x \in \mathcal{R}^n$, at time instant $t > 0$. The equation (17) admits the operator

$$Z = t^2 \frac{\partial}{\partial t} + tx \cdot \frac{\partial}{\partial x} - \left(\frac{nt}{2} + |x|^2 \right) u \frac{\partial}{\partial u} \ , \tag{18}$$

i.e. it is invariant under the projective transformations (11) of independent variables x and t and the following linear transformations of dependent variable $u(x,t)$:

$$u' = u(1-at)^{n/2} \exp\left(-\frac{a|x|^2}{4(1-at)} \right) \tag{19}$$

So,the symmetry property of the dissipative heat transfer (or diffusion) equation is similar to the one of a particle in the central field (15). This symmetry property can be obviously generalized to the Burgers equation [2] and to the Burgers Hierarchy [5].

2.3 Euler Gas Dynamics

The Euler gas dynamics equations for a monatomic gas can be written in the standard form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho div u = 0 \ , \quad x \in \mathcal{R}^n \ , \quad t > 0 \\ \frac{\partial u}{\partial t} + \left(u \cdot \frac{\partial}{\partial x} \right) u + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \\ \frac{\partial p}{\partial t} + u \cdot \frac{\partial p}{\partial x} + \frac{n+2}{n} p div u = 0 \ , \end{aligned} \tag{20}$$

where the scalar functions $\rho(x,t)$, $p(x,t)$ and the n-component vector function $u(x,t)$ respectively denote the density, the pressure and the bulk velocity of the gas at the space-time point (x,t). These equations admit the operator

$$Z = t^2 \frac{\partial}{\partial t} + tx \cdot \frac{\partial}{\partial x} + (x - ut) \cdot \frac{\partial}{\partial u} - nt\rho \frac{\partial}{\partial \rho} - (n+2)tp\frac{\partial}{\partial p} \tag{21}$$

So, the Euler compressible equations (20) are also invariant under the projective transformations (11) of independent variables (x,t) and the linear transformations of dependent variables

$$\begin{aligned} \rho' &= \rho(1-at)^n \\ u' &= u + a(x - ut) \\ p' &= p(1-at)^{n+2} \end{aligned} \tag{22}$$

We notice that the bulk velocity $u(x,t)$ is transformed similarly to the time-derivative of $x(t)$ in eq.s (12)-(14).

Remark. In the examples 1 and 3 two additional conservation laws appear due to this symmetry. These conservation laws and related questions will be considered in Sec. 4.

It is an interesting task to understand the coincidence of the symmetry properties for so different physical systems. The connection between examples 1 and 3, which was partially clarified in [4] (published only in Russian), will be discussed in next section.

3 Classical Dynamics, Hydrodynamics and Kinetic Theory

We do not consider the example 2 (heat-transfer equation) in this section

and concentrate our attention on examples 1 and 3 only. To understand the connection between these examples (a particle in the central field and the Euler equations of gas dynamics) we have to remind the formal connection between dynamics and hydrodynamics through the kinetic theory of gases:

Two-body problem	→	N-body problem	→	Boltzmann equation	→	Hydrodynamics, Euler equations

It has been proved in [4] that, for particles interacting with the potential (15), the projective invariance is conserved along all this series: from the starting two-body problem to the finishing Euler equations.

It is worthwhile to observe that N-body dynamics is invariant under the transformations (11) or (14) for every particle coordinate and velocity. The exact formulation of this symmetry property is the following:

Proposition 1. The dynamical equations

$$\frac{dx_i}{dt} = V_i, \quad \frac{dv_i}{dt} = -\frac{\partial}{\partial x_i} \sum_{j=1, j\neq i}^{N} U(|x_i - x_j|) \tag{23}$$

$x_i \in \mathcal{R}^n, V_i \in \mathcal{R}^n, t \in \mathcal{R}, i = 1, \ldots, N$

for the N-particle system with the potential (15) [2] are invariant under the projective transformations:

$$t' = \frac{t}{1-at} \quad , \quad x_i' = \frac{x_i}{1-at} \quad , \quad v_i' = v_i(1-at) + a x_i \quad i = 1, \ldots, n. \tag{24}$$

We notice also that the transformations (24) preserve the phase volume, i.e.

$$dx' \wedge dv' = dx \wedge dv \tag{25}$$

So, the Liouville equation for this N-particle problem is also projectively invariant. It is easy now to guess the corresponding properties of the Boltzmann equation and compressible Navier- Stokes equation.

Proposition 2. Let $f(x, v, t)$ be a distribution function, x, v and t respectively denoting the position, the velocity and the time, satisfying the Boltzmann equation for the intermodular potential (15), i.e. for $x \in \mathcal{R}^n \;\; v \in \mathcal{R}^n, \;\; t > 0$

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} = \int_{R^n \times S^{n-1}} dw \; d\omega |u| \sigma_n \left(|u| \frac{u \cdot \omega}{|u|} \right) \{ f(v')f(w') - f(v)f(w) \} \tag{26}$$

where $u = v - w, \;\; S^{n-1}$ is the unit sphere in $\mathcal{R}^n$, $\omega \in S^{n-1}, \;\; |\omega| = 1$, v' and w' denote the velocities after collision:

$$v' = \frac{1}{2}(v + w + |u|\omega) \quad , \quad w' = \frac{1}{2}(v + w - |u|\omega).$$

[2] The N-body potential (15) is well known as the *Calogero-Moser potential*. It was firstly considered by Calogero [6] and its complete integrability, in one space dimension, was shown by Moser [7]. Furthermore new invariance properties have been recently found [8],[9].

In (26) $\sigma_n(|u|, \cos\theta)$ denotes the (generalized for $n \neq 3$) differential cross-section for the scattering angle $0 < \theta < \pi$. For the potential (15) it can be written in the form

$$\sigma_n(|u|, \cos\theta) = A_n(\cos\theta)|u|^{1-n}, n = 2, 3 \ldots \tag{27}$$

The equation (26)-(27) is invariant under the following transformation:

$$f' = f,\ t' = \frac{t}{1-at},\ x' = \frac{x}{1-at},\ v' = v(1-at) + ax \tag{28}$$

At last we can formulate the same properties for Navier Stokes equations.

Proposition 3. With notations of ref. [4] the standard Navier-Stokes equations for a dilute gas, with the potential (15), are written:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + div\ \rho u &= 0, u = u(u_1, ...u_n) \\ \left(\frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x}\right) u_k + \frac{1}{\rho}\frac{\partial p}{\partial x_k} &= \frac{1}{\rho}\frac{\partial}{\partial x_i}\eta_n(T)\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{n}\delta_{ik} div\ u\right] \\ \frac{\partial p}{\partial t} + u \cdot \frac{\partial p}{\partial x} + \gamma p\ div\ u &= \frac{2}{n}\frac{\partial}{\partial x} \cdot \chi_n(T)\frac{\partial T}{\partial x} + \frac{2}{n}\eta_n(T)\frac{\partial u_i}{\partial x_k} \times \\ & \times \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{n}\delta_{ik}\ div\ u\right), \end{aligned} \tag{29}$$

$$\gamma = \frac{n+2}{n} \quad , \quad p = \rho T\ , i, k = 1, ...n,$$

where T denotes the temperature and the sum over repetead indices is meant. For this potential the viscousity $\eta_n(T)$ and the heat-transfer $\chi_n(T)$ coefficients have the form

$$\eta_n(T) = \eta_0 T^{n/2},\ \chi(T) = \chi_0 T^{n/2}, \tag{30}$$

η_0 and χ_0 denoting constant values.

The equations (29)-(30) are invariant under the projective transformations (11), (22), i.e., these equations have the same symmetry property as the Euler equations (20).

These propositions can be easily proved by simple verification .

Thus we get the projective invariance of the Boltzmann, Navier-Stokes and Euler equations from the one of a very simple mechanical system. It is remarkable that the connection with the specific potential (15) appears to be lost in Euler (continuous media) limit because the Euler equations (20) are independent on the intermolecular potential.

4 Additional Conservation Laws

We describe briefly the additional invariants for previous three examples. In the example 1, the conservation of the energy can be written in the form

$$E = \frac{|v|^2}{2} + \frac{\alpha}{|x|^2} = const. \tag{31}$$

We can conclude, from the symmetry, that for any $a \gtrless 0$ ($|a|$ is sufficiently small)

$$E' = \frac{|v'|^2}{2} + \frac{\alpha}{|x'|^2} = const', \tag{32}$$

where the formulas (14) are used. Here E' is a polinomial (quadratic) function of parameter a

$$E' = E - aI_1 + \frac{a^2}{2} I_2 = const,$$

So we get two new invariants

$$I_1 = 2tE - x \cdot v = const$$

$$I_2 = 2t^2 E - x \cdot (2tv - x) = const \ , \quad E = \frac{|v|^2}{2} + \frac{\alpha}{|x|^2} = const \tag{33}$$

In the case of N-particle system we get similarly the same invariants in the form

$$\begin{aligned} I_1^{(N)} &= 2tE - \sum_{k=1}^{N} x_k \cdot v_k = const \\ I_2^{(N)} &= 2t^2 E - \sum_{k=1}^{N} x_k \cdot (2tv_k - x_k) = const \\ E &= \sum_{k=1}^{N} \frac{v_k^2}{2} + \sum_{1 \le i < j \le N} \frac{\alpha}{|x_i - x_j|^2} \ , \ N = 2, 3, \ldots \end{aligned} \tag{34}$$

The physical (or geometrical) meaning of these conservation laws is clarified by the following formula

$$\frac{d^2}{dt^2} \sum_{k=1}^{N} x_k^2 = 4E = const, \tag{35}$$

which is equivalent to the two conservation laws in (34).

It is remarkable that the analogous conservation laws for the Boltzmann equation

$$I_1^{(B)} = \int_{\mathcal{R}^n \times R^n} dx dv f(x, v, t) v \cdot (vt - x) = const$$

$$I_2^{(B)} = \int_{\mathcal{R}^n \times R^n} dx dv f(x, v, t) (vt - x)^2 = const \tag{36}$$

hold in the general case independently on the intermolecular potential. The same conservation laws in terms of density

$$\rho(x, t) = \int_{\mathcal{R}^n} dv \ f(x, v, t),$$

bulk velocity

$$u(x, t) = \frac{1}{\rho} \int_{\mathcal{R}^n} dv \ f(x, v, t)$$

and pressure

$$p(x, t) = \frac{1}{n} \int_{\mathcal{R}^n} dv f(x, v, t) (v - u)^2$$

also hold in the general case of the Navier- Stokes and Euler equations. Thus the projective invariance is valid only for special potential (15) (except the Euler equation) but the additional conservation laws are valid in the general case.

The explicit formulas of the hydrodynamical invariants are:

$$I_1^{(H)} = I_1^{(B)} = \int_{\mathcal{R}^3} dx \left\{ t \left(\rho u^2 + np \right) - \rho x \cdot u \right\} \tag{37}$$

$$I_2^{(H)} = I_2^{(B)} = \int_{\mathcal{R}^3} dx \left\{ t^2 \left(\rho u^2 + np \right) + \rho x \cdot (x - 2tu) \right\}$$

One can find in [4] a more complete information concerning the additional conservation laws. We notice only that these laws for the Euler equations were found in [10] and for the Boltzmann equation they were apparently firstly published in [11], without any connection with group symmetries. The special transformation of the Boltzmann and the Navier-Stokes equations for the potential (15) was introduced in [12].

The conservation laws (34)-(37) can be easily interpreted as a simple equations of motion for the first moments in phase space. Let $f(x, v, t)$ be normalized as probability density

$$\int_{\mathcal{R}^n \times \mathcal{R}^n} dx dv f(x, v, t) = 1$$

Introducing the standard notations for average values of a given function $h(x, v)$

$$\langle h(x, v) \rangle = \int_{\mathcal{R}^n \times \mathcal{R}^n} dx dv f(x, v, t) h(x, v),$$

or for N-particle system

$$\langle h(x, v) \rangle = \frac{1}{N} \sum_{k=1}^{N} h \left[x_k(t), v_k(t) \right],$$

the equations for the first moments are:

$$\frac{d}{dt} \langle x \rangle = \langle v \rangle \ , \ \frac{d}{dt} \langle v \rangle = 0 \ .$$

The ones for the second moments, in the general case, are not so evident. It follows from (36) that

$$\frac{d}{dt} \langle x^2 \rangle = 2 \langle x \cdot v \rangle$$

not only for N-particle system but also for the Boltzmann equation.

The energy conservation law can be written in different forms for the Boltzmann equation

$$\frac{\langle v^2 \rangle}{2} = \varepsilon = const$$

and for N-particle system interacting with pair potential $U(|x|)$

$$\frac{\langle v^2 \rangle}{2} + \frac{1}{N} \sum_{1 \le i < j \le N} U(|x_i - x_j|) = \frac{E}{N} = \tilde{\varepsilon} = const$$

For last scalar second moment we get from (36)

$$\frac{d}{dt} \langle x \cdot v \rangle = 2\varepsilon = const \ ,$$

The analogous equation for the interacting N-particle system can be written only in the case of the special potential (15)

$$\frac{d}{dt} \langle x \cdot v \rangle = 2\tilde{\varepsilon} = \frac{1}{N} \left[\sum_{k=1}^{N} v_k^2 + 2 \sum_{1 \le i < j \le N} \frac{\alpha}{|x_i - x_j|^2} \right] = const \ ,$$

which includes for $\alpha \equiv 0$ the system of non-interacting particles.

So we have a parabolic law

$$\langle x^2 \rangle = At^2 + Bt + C \tag{38}$$

in the three following cases:

1. non-interacting particles;
2. particles interacting *via* the Calogero- Moser potential (15).
3. Boltzmann (Navier-Stokes, Euler) gas.

Thus we observed the connection between our examples 1 and 3. In sec. 5 we consider the example 2, i.e. heat-transfer equation.

5 Heat Transfer and Schrödinger equations.

The projective invariance of heat transfer equation can be understood observing that the free motion equation for the velocity $v(x,t)$ of continuous media

$$v_t + vv_x = 0 \qquad x \in R, \quad t \in R$$

is invariant under the transformation (14).

A more general equation:

$$v_t + vv_x - F(v_{xx}, v_{xxx}, ...) = 0,$$

F denoting an arbitrary function, after the transformation (14), takes the form:

$$v_t + vv_x - (1+at)^{-3} F((1+at)^3 v_{xx}, (1+at)^4 v_{xxx}, ...) = 0.$$

It turns out the projective invariance of the Burgers equation ($F = v_{xx}$) and so the one of heat-transfer equation, because of the Hopf-Cole transformation.

As a by-product of last formula, some new projectively invariant systems as

$$(v_t + vv_x)^2 - v_{xxxxx} = 0,$$

$$(v_t + vv_x)^4 - (v_{xxx})^3 = 0,$$

can be obtained.

The projective symmetry of heat-transfer equation can be also understood from its formal analogy with the Schrödinger equation for the wave function $\Psi(x,t)$ of the free motion

$$i\frac{\partial \psi}{\partial t} = \Delta \psi \quad x \in \mathcal{R}^n \ , \ t \in R \ , \tag{39}$$

where the corresponding dimensionless variables have been used.

After our observation in Sec. 3-4 on the projectively invariant classical systems, it is natural to look for symmetry of quantum mechanical systems too. The first step is to study from this point of view the simplest quantum mechanical system, that corresponds to the equation (39) for a free particle. We know that the projective transformations

$$t' = \frac{t}{1-at} \quad , \quad x' = \frac{x}{1-at} \tag{40}$$

do not change the equation of the free motion

$$\frac{d^2 x}{dt^2} = 0$$

in classical mechanics. At quantum level it is easy to verify that the Schrödinger equation (39) is invariant under the transformations (40) if we simultaneously transform the dependent variable Ψ according to:

$$\Psi' = \Psi(1-at)^{n/2} \exp\left(-\frac{ia|x|^2}{4(1-at)}\right) \ , \tag{41}$$

similar to (19). So, the Schrödinger equation (39) admits the operator

$$Z = t^2\frac{\partial}{\partial t} + tx\cdot\frac{\partial}{\partial x} - \left(\frac{nt}{2} + i|x|^2\right)\frac{\partial}{\partial\Psi} \ , \tag{42}$$

that corresponds to the projective transformations (40-41).

We are able to understand from the physical point of view the origin of the transformation (41): it means that the motion of a free particle is projectively invariant in quantum mechanics, i.e. the projective invariance is preserved in the transition from classical to quantum mechanics.

Actually, the physical containt of the wave function $\Psi(x,t)$ is expressed *via* the probabilistic measure $\mu(A) = \int_A dx|\Psi(x,t)|^2, \quad A \subset \mathcal{R}^n$.

The physical picture does not change under some transformations if the measure is invariant. In our case (41) the probability density transforms as:

$$|\Psi'|^2 = |\Psi|^2(1-at)^n$$

Observing that $(1-at)^n$ is the inverse Jacobian of transformations (10) from x' to x, we get:

$$|\Psi'|^2 dx' = |\Psi|^2 dx \ .$$

Thus the free motion equation is projectively invariany in the both classical and quantum mechanics. This fact also reflects the fundamental nature of the projective transformations.

So, in the framework of our approach the projective invariance of the heat-transfer equation (17) is connected with the formal coincidence of this equation and the Schrödinger equation (39). We shall try now to generalize this symmetry property to more complex quantum systems.

6 Quantum systems with interaction

It is naturaly, at this point, to consider the Schrödinger equation for N particles interacting with pair potential (15). We have for the wave function $\Psi(x_1,\ldots,x_N,t)$ with $x_i \in \mathcal{R}^n, i = 1,\ldots,N$, the equation:

$$i\Psi_t = \sum_{i=1}^{N}\Delta_i\Psi - U_n\Psi, \tag{43}$$

where

$$U_N = \sum_{1\le i\le j\le N}\frac{\alpha}{|x_i-x_j|^2} \quad , \quad \alpha = 0$$

It is easy to verify the validity of the following

Proposition 4. The equation (43) admits the operator

$$Z = t^2\frac{\partial}{\partial t} + \sum_{k=1}^{N} tx_k\cdot\frac{\partial}{\partial x_k} - \left(\frac{N}{2}nt + i\sum_{k=1}^{N}|x_k|^2\right)\Psi\frac{\partial}{\partial\Psi} \ ,$$

i.e. the equation (43) is invariant under the transformations

$$t' = \frac{t}{1-at} \ , \ x'_k = \frac{x_k}{1-at} \ , \ \Psi' = \Psi(1-at)$$

So, we get the quantum version of the *Proposition 1* of Sec. 3. It is obvious that we can generalize the *Proposition 2* to quantum Ueling-Uhlenbeck kinetic equation [13] in the both cases of bosons and fermions. Thus the *Proposition 2* also holds replacing in it the Boltzmann equation by the Ueling-Uhlenbeck equation with the same potential.

To the best of our knowledge these proerties of quantum systems with interaction were never discussed in the literature.

7 Conclusion

We have considered the specific role of the projective transformations

$$x' = \frac{x}{1-at} \quad , \quad t' = \frac{t}{1-at} \quad , \quad x \in \mathcal{R}^n \quad , \quad t \in \mathcal{R} \tag{44}$$

of space-time in the non-relativistic classical and quantum mechanics of particles and in the theories of continuous media: the Boltzmann, the Navier-Stokes and the Euler equations. We also clarified the specific role of the potential:

$$U(|x|) = \frac{\alpha}{|x|^2}, \ \alpha > 0, \ x \in \mathcal{R}^n \tag{45}$$

in connection with these transformations. It is a remarkable fact that in the one-dimensional case U is the so-called Calogero-Moser potential, for which the N-body problem is completely integrable . We also observe that the Calogero-Moser system can be obtained by reduction from a free motion [8], and so it is the same for its projective invariance. In the two-dimensional case this potential corresponds to the Maxwell molecules for the Boltzmann equation. Finally in four dimensions it is the fundamental solution of Laplace equation:

$$\Delta \frac{\alpha}{|x|^2} = 0, \ |x| > 0, \ x \in \mathcal{R}^4 \tag{46}$$

i.e. the four-dimensional Coulomb potential. It is clear from our consideration that also the Vlasov kinetic equation [13] for the potential (45) are projectively invariant in $\mathcal{R}\mathcal{R}^n$ for any $n \geq 4$ [3]. This property is of interest in the case $n = 4$ too, because of the identity (46). Unfortunately, in the most important case $n = 3$ the applications of the potential (45) are unclear. Obviously, the transformations (44) are of interest not only for the potential (45), because they transform the initial equations (in all the above considered cases, i.e. for the Newton, Schrödinger, Boltzmann and other equations) to the same equations with the new time-dependent potential (or the cross-section for the Boltzmann equation and the viscousity and heat-transfer coefficients for the Navier-Stokes equations). Thus, in the general case the projective transformations are, for the above mentioned equations, the so-called equivalence (not invariance) transformations [1] , [2]. This clarifies the fact that the additional conservation laws in kinetic theory and hydrodynamics appear to be valid in the general case too (see section 4).

Furthermore, the postulate that the projective invariance of the free motion of a particle is preserved in the transition from classical (Newton) to quantum (Schrödinger) mechanics, immediately

[3] In the case $n = 3$ the integral term in the Vlasov equation, with the potential (45) is not correctly defined.

gives only to the measure $d\mu = |\psi(x,t)|^2 dx$ a physical meaning, the wave function $\psi(x,t)$ being not invariant. So the physical interpretation of the wave function can be obtained, at least at *physical level* of rigour, from the postulate of the projective invariance of a free particle motion in quantum mechanics.

Resuming, the projective transformations (44) seem to play a relevant role in physics. At least they are transformations of invariance (or of equivalence) for a wide class of systems going from those described by ordinary differential equations to the integro-differential Boltzmann and Vlasov kinetic equations. We were trying in the paper to collect them and to explain their mutual connection. Further, some new projectively invariant systems and the additional conservation laws, following from the projective symmetry, were founded. We hope that this paper will attract the attention to the role of the projective symmetry and of its physical meaning. At least it is clear that this symmetry arises from the geometrical properties of the space-time in the non- relativistic physics. All the considered examples are connected in some way with a system of free particles moving along geodesical (straight) trajectories in the Euclidean space-time. Observing that many completely integrable interacting systems actually turn out to be *reduced system* of free ones [14], one is tempted to use the *projective invariance* as an additional powerful symmetry in the study of an *unfolding procedure* [8].

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