ESI **The Erwin Schrödinger International Institute for Mathematical Physics**
Pasteurgasse 6/7
A-1090 Wien, Austria

On the Functional Equation Related to the Quantum Three–Body Problem

V.M. Buchstaber
A.M. Perelomov

Vienna, Preprint ESI 143 (1994) October 17, 1994

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

In memory of F.A. Berezin

V.M. Buchstaber* and A.M. Perelomov**

On the functional equation related to the quantum three-body problem***

* Department of Mathematics and Mechanics, Moscow State University, 119899 Moscow, Russia
** Institute of Theoretical and Experimental Physics, 117 259 Moscow, Russia
*** submitted to Berezin Memorial Volume, Ed. R.L. Dobrushin, A.M. Vershik and M.A. Shubin (AMS 1994)

Abstract

In the present paper we give the general nondegenerate solution of the functional equation

$$(f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z),$$

$$x + y + z = 0,$$

which is related to the exact factorized ground-state wave function for the quantum one-dimensional problem of three different particles with pair interaction.

Functional equations connecting several functions and admitting a general analytic solution have recently attracted much attention of mathematicians and also physicists (for recent researches see for example [BC 1990], [BP 1993], [BK 1993], [BFV 1994]).

In modern mathematical physics such equations arise in considerations of integrable systems of classical and quantum mechanics (for review see for example [Pe 1990], [OP 1983]).

In the present paper we investigate one such equation. Namely, we investigate the functional equation connecting six unknown functions

$$(f(x)+g(y)+h(z))^2 = F(x)+G(y)+H(z), \tag{1}$$

$$x+y+z=0,$$

which generalize the well–known Frobenius–Stickelberger equation [FS 1880] and is related to the exact factorized ground-state wave function for the quantum one-dimensional problem of three different particles with pair interaction. We give the general nondegenerate solution of this equation.

1 Let us recall first that an analogous (but simpler) equation for the special case of three identical particles was considered earlier by B. Sutherland [Su 1975]] and F. Calogero [Ca 1975]. Namely, in the paper [Su 1975] the one-dimensional many-body problem of n identical particles with pair interaction was considered, whose exact ground-state wave function $\Psi_0(x_1, x_2, \ldots, x_n)$ is factorized

$$\Psi_0(x_1, x_2, \ldots, x_n) = \prod_{j<k} \psi(x_j - x_k). \tag{2}$$

It was shown that the logarithmic derivative of $\psi(x)$

$$f(x) = \psi'(x)/\psi(x) \tag{3}$$

should satisfy the functional equation

$$f(x)f(y)+f(y)f(z)+f(z)f(x) = F(x)+F(y)+F(z),$$

$$x+y+z=0, \tag{4}$$

where $f(x) \quad (F(x))$ is odd (even) function

$$f(-x) = -f(x), \quad F(-x) = F(x). \tag{5}$$

In [Su 1975], a partial solution of equations (4), (5) was also found.

The general solution of equations (4), (5) was found in [Ca 1975] (see also [OP 1983] for review of this and related problems). This solution has the form

$$f(x) = \alpha\zeta(x; g_2, g_3) + \beta x, \tag{6}$$

where $\zeta(x)$ is the Weierstrass zeta-function (see for instance [WW 1927]).

In the present paper we consider only the three-body problem, but in the general case when all three particles are different from each other.

In this case the ground-state wave function has the form

$$\Psi_0(x_1, x_2, x_3) = \psi_1(x_2 - x_3)\psi_2(x_3 - x_1)\psi_3(x_1 - x_2) \tag{7}$$

and satisfies the Schrödinger equation

$$-\Delta\psi_0 + U\psi_0 = E_0\Psi_0, \tag{8}$$

$$U = u_1(x_2 - x_3) + u_2(x_3 - x_1) + u_3(x_1 - x_2). \tag{9}$$

Substituting Ψ_0 from (7) into (8), we obtain

$$\begin{aligned}\Psi_0^{-1}\Delta\Psi_0 = U - E_0 = 3(f_1^2(x_2 - x_3) + f_2^2(x_3 - x_1) + f_3^2(x_1 - x_2))\\ -(f_1(x_2 - x_3) + f_2(x_3 - x_1) + f_3(x_1 - x_2))^2\\ +2(f_1'(x_2 - x_3) + f_2'(x_3 - x_1) + f_3'(x_1 - x_2));\\ f_j = \psi_j'/\psi_j.\end{aligned} \tag{10}$$

Hence, for the potential energy $U(x_1, x_2, x_3)$ to have the form of pair interactions (9), the three functions

$$f(x) = f_1(x), \quad g(y) = f_2(y), \quad h(z) = f_3(z) \tag{11}$$

must satisfy the functional equation

$$\begin{aligned}(f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z),\\ x + y + z = 0.\end{aligned} \tag{12}$$

The following expression for the potential energies results from (10)–(12).

$$\begin{aligned}u_1(x) = 3f^2(x) + 2f'(x) - F(x) + \varepsilon_1,\\ u_2(x) = 3g^2(x) + 2g'(x) - G(x) + \varepsilon_2,\\ u_3(x) = 3h^2(x) + 2h'(x) - H(x) + \varepsilon_3.\\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = E_0\end{aligned} \tag{13}$$

2. Let us consider the meromorphic solutions of the equation

$$(f(x)+g(y)+h(z))^2 = F(x)+G(y)+H(z) \tag{14}$$

satisfying the condition $x+y+z=0$.

Let us call the solution of Eq. (14) nondegenerate, if the functions $f(x), g(x)$ and $h(x)$ have the pole in finite domain of complex x-plane.

2. The main result of this paper is the following

Theorem. *The general nondegenerate solution of the equation (14) in the class of meromorphic functions has the form*

$$f(x) = \alpha\zeta(x-a_1; g_2, g_3) + \beta x + \gamma_1, \tag{15}$$

$$g(x) = \alpha\zeta(x-a_2; g_2, g_3) + \beta x + \gamma_2. \tag{16}$$

$$h(x) = \alpha\zeta(x-a_3; g_2, g_3) + \beta x + \gamma_3. \tag{17}$$

$$F(x) = \alpha^2\mathcal{P}(x-a_1; g_2, g_3) + 2\gamma\alpha\zeta(x-a_1; g_2, g_3) + \frac{1}{3}\gamma^2. \tag{18}$$

$$G(x) = \alpha^2\mathcal{P}(x-a_2; g_2, g_3) + 2\gamma\alpha\zeta(x-a_2; g_2, g_3) + \frac{1}{3}\gamma^2. \tag{19}$$

$$H(x) = \alpha^2\mathcal{P}(x-a_3; g_2, g_3) + 2\gamma\alpha\zeta(x-a_3; g_2, g_3) + \frac{1}{3}\gamma^2. \tag{20}$$

where

$$a_1 + a_2 + a_3 = 0 \tag{21}$$

Proof . The proof of the theorem is divided on several steps.

Let us begin with

Lemma 1. *The functions $(f(x), g(y), h(z))$ satisfy equation (14) for the corresponding functions $(F(x), G(y), H(z))$ if and only if the equation*

$$\begin{pmatrix} f''(x) & g''(y) & h''(z) \\ f'(x) & g'(y) & h'(z) \\ 1 & 1 & 1 \end{pmatrix} = 0 \tag{22}$$

can be solved under condition: $x+y+z=0$.

Proof. Let us apply the operator $\partial_- \cdot \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial x}$, where $\partial_- = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ to eq.(1). This gives:

$$\frac{\partial}{\partial x}\colon 2(f'(x) - h'(z))(f(x) + g(y) + h(z)) = F'(x) - H'(z), \tag{23}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\colon 2h''(z)(f(x)+g(y)+h(z)) + 2(f'(x)-h'(z))(g'(y)-h'(z)) = H''(z), \tag{24}$$

$$\partial_- \frac{\partial}{\partial y}\frac{\partial}{\partial x}\colon h''(z)(f'(x)-g'(y)) + f''(x)(g'(y)-h'(z)) + g''(y)(h'(z)-f'(x)) = 0. \tag{25}$$

Here we use the fact that ∂_- is a differential operator , and that $\partial_- h'(z) = \partial_- h''(z) = 0$.

Hence, if functions $(f(x), g(y), h(z))$ satisfy equation (1), then these functions satisfy also equation (25), that can be obviously rewritten in the form (22).

Conversely, let the functions $(f(x), g(y), h(z))$ satisfy (22) and, consequently, (25). The equation (25) may be rewritten as

$$\partial_-[h''(z)(f(x) + g(y) + h(z)) + (f'(x) - h'(z))(g'(y) - h'(z))] = 0;$$

then there is a function $H_1(z)$ satisfying the following equation:

$$h''(z)(f(x) + g(y) + h(z)) + (f'(x) - h'(z))(g'(y) - h'(z)) = H_1(z) \tag{26}$$

Let us note that eq.(26) is equivalent to the equation

$$\frac{\partial}{\partial y}[(f'(x) - h'(z))(f(x) + g(y) + h(z))] = H_1(z)$$

Therefore, there are functions $F_1(x)$ and $H_2(z)$ such that $H_2'(z) = H_1(z)$, and

$$(f'(x) - h'(z))(f(x) + g(y) + h(z)) = F_1(x) - H_2(z). \tag{27}$$

On the other hand, equation (27) is equivalent to

$$\frac{\partial}{\partial x}(f(x) + g(y) + h(z))^2 = 2(F_1(x) - H_2(z)),$$

i. e. there are functions $F(x), G(y)$ and $H(z)$ such that $F'(x) = 2F_1(x), H'(z) = 2H_2(z)$, and

$$(f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z).$$

Thus Lemma 1 is proved.

Lemma 2. *Equation (14) is invariant under the following transformations:*

$$f(x) \to f_0 + a_1 x + a_2 f(a_3 x + \alpha_1),$$

$$F(x) \to F_0 + a_4 x + a_2^2 F(a_3 x + \alpha_1) + 2a_2 c f(a_3 x + \alpha_1),$$

$$g(y) \to g_0 + a_1 y + a_2 g(a_3 y + \alpha_2).$$

$$G(y) \to G_0 + a_4 y + a_2^2 G(a_3 y + \alpha_2) + 2a_2 c g(a_3 y + \alpha_2),$$

$$h(z) \to h_0 + a_1 z + a_2 h(a_3 z + \alpha_3),$$

$$H(z) \to H_0 + a_4 z + a_2^2 H(a_3 z + \alpha_3) + 2a_2 c \cdot h(a_3 z + \alpha_3),$$

where $a_k (k = 1, \dots, 4)$ *and* c *are free parameters*

$$f_0 + g_0 + h_0 = c, \quad F_0 + G_0 + H_0 = c^2, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{28}$$

This Lemma is proved by a direct calculation.

Corollary 3. *Taking corresponding values of the parameters* $\alpha_1, \alpha_2, \alpha_3$, *one can prove that all the functions* $(f(x), g(y), h(z))$, $(F(x), G(y), H(z))$ *are regular at* $x = 0, y = 0, z = 0$, *respectively.*

The proof follows from the fact that the set of poles of a meromorphic function of one complex variable is discrete. Thus in what follows we may suppose that all the functions are regular at $x = 0, y = 0, z = 0$.

Definition 4. *Let us call the solution of equation (14) degenerate, if at least one of functions* $f(x), g(x)$, *and* $h(x)$ *is linear.*

The next Lemma describes all the degenerate solutions of equation (14).

Lemma 5. *Let* $(f(x), g(y), h(z))$, $(F(x), G(y), H(z))$ *is some degenerate solution of the equation (14).*

Three cases are possible.

1. All three functions $f(x), g(y), h(z)$ are linear. Then

$$f(x) = f_0 + f_1 x, \quad g(y) = g_0 + g_1 y, \quad h(z) = h_0 + h_1 z,$$

$$F(x) = F_0 + F_1 x + (f_1 - g_1)(f_1 - h_1)x^2, \quad G(y) = G_0 + G_1 y + (g_1 - f_1)(g_1 - h_1)y^2,$$

$$H(z) = H_0 + H_1 z + (h_1 - g_1)(h_1 - f_1)z^2$$

Here $f_0, f_1, g_0, g_1, h_0, h_1$ are free parameters.

Let $f_0 + g_0 + h_0 = c$. Then

$$F_0 + G_0 + H_0 = c^2, \quad F_1 = b + 2cf_1, \quad G_1 = b + 2cg_1, \quad H_1 = b + 2ch_1$$

and b is a free parameter.

2. Two of the functions $f(x), g(y), h(z)$ are linear. For example, it is $g(y) = g_0 + g_1 y, \quad h(z) = h_0 + h_1 z$. Then $f(x)$ is an arbitrary function, $g(y) = g_0 + ay, \quad h(z) = h_0 + az, \quad G(y) = G_0 + by, \quad H(z) = H_0 + bz$ and

$$F(x) = [g_0 + h_0 - ax + f(x)]^2 - (G_0 + H_0 - bx).$$

Here g_0, h_0, a, b, G_0, H_0 are free parameters.

3. Only one of the functions $f(x), g(y), h(z)$ is linear. For example, $h(z) = h_0 + h_1 z$. Then

$$f(x) = f_0 + ax + c_1 \exp(\lambda x), \quad g(y) = g_0 + ay + c_2 \exp(\lambda y), \quad h(z) = h_0 + az,$$

$$F(x) = F_0 + bx + c_1 \exp(\lambda x)(2c + c_1 \exp(\lambda x)),$$

$$G(y) = G_0 + by + c_2 \exp(\lambda y)(2c + c_2 \exp(\lambda y)),$$

$$H(z) = H_0 + bz + 2c_1 c_2 \exp(-\lambda z).$$

Here $a, b, c, c_1, c_2, \lambda$ are free parameters, and

$$f_0 + g_0 + h_0 = c, \quad F_0 + G_0 + H_0 = c^2.$$

Proof.

Case 1. It follows from (22) that $f(x), g(y), h(z)$ are arbitrary linear functions. A form of the functions $F(x), G(y), H(z)$ can be reconstructed directly from (14), taking into account the identity $2xy = z^2 - x^2 - y^2$.

Case 2 . We obtain from (22)

$$f''(x)(g_1 - h_1) = 0$$

If $f''(x) \neq 0$, then $g_1 = h_1$ and $f(x)$ is arbitrary. The form of the functions $F(x), G(y), H(z)$ can be reconstructed immediately.

Case 3 . We get from (24):

$$2(f'(x) - h_1)(g'(y) - h_1) = H''(-x - y).$$

If $f'(x)$ and $g'(y)$ are not constants, then according to the classical Cauchy–Pexider result [Ca 1821] (see also [Ab 1823]) we obtain:

$$f'(x) - h_1 = \tilde{c}_1 \exp(\lambda x), \quad g'(y) - h_1 = \tilde{c}_2 \exp(\lambda x).$$

where $\tilde{c}_1, \tilde{c}_2$ and λ are free parameters.

Therefore

$$f(x) = f_0 + h_1 x + c_1 \exp(\lambda x), \quad g(y) = g_0 + h_1 y + c_2 \exp(\lambda y),$$

where $c_k = \tilde{c}_k/\lambda, k = 1, 2$. The form of the functions $F(x), G(y), H(z)$ can be reconstructed easily. The Lemma is proved.

The functions $f(x), g(x), h(x)$ from eq. (1), will considered as nondegenerate solutions of eq. (1).

Lemma 6. *On choosing the appropriate values of the parameters* $f_0, g_0, h_0, a_1, F_0, G_0$ *(see Lemma 2) we obtain*

$$f(0) = g(0) = h(0) = 0, \quad h'(0) = 0, \quad F(0) = G(0). \tag{29}$$

The proof is easy.

Lemma 7. *An appropriate choice of the parameters* α_1 *and* α_2 *leads to the relation* $f(x) \neq g(x)$.

Proof. Suppose on the contrary that

$$f(x + \alpha_1) - f(\alpha_1) \equiv g(x + \alpha_1) - g(\alpha_2) \tag{30}$$

for all α_1 and α_2 in any neighbourhood of the point $x = 0$. On differentiating (30), we obtain

$$\frac{\partial f(x + \alpha_1)}{\partial x} = \frac{\partial f(x + \alpha_1)}{\partial \alpha_1} = f'(\alpha_1),$$

i. e.

$$f(x + \alpha_1) = f'(\alpha_1)x + f(\alpha_1).$$

in contradiction to the assumption about the nondegeneracy of the solution. The Lemma is proved.

Hence, it is sufficient to find all the nondegenerate solutions of eq. (1) under the following additional conditions: $f(x) \neq g(x)$ and $f(0) = g(0) = h(0), \quad h'(0) = 0, \quad F(0) = G(0) = 0.$

Then, using the transformations from Lemma 2 we obtain the general nondegenerate solution. In what follows only the nondegenerate solutions of (14) satisfying the above additional conditions will be considered.

Interchanging x and y in eq. (14), we obtain

$$(f(y)+g(x)+h(z))^2 = F(y)+G(x)+H(z). \tag{31}$$

$$\varphi(x+y)=\eta(x)+\eta(y)-\frac{\gamma(x)-\gamma(y)}{\xi(x)-\xi(y)}, \tag{32}$$

where $\varphi(0)=\varphi'(0)=\eta(0)-\gamma(0)=\xi(0)=0$ and $\varphi''(x)\neq 0$.

Definition 8. *Let us call solution $(\varphi,\eta,\xi,\gamma)$ of the equation (32) normalized, if the following initial conditions are satisfied:*

$$\xi'(0)=1, \quad \eta'(0)=0.$$

Lemma 9. *The map*

$$(\varphi,\eta,\xi,\gamma)\to(\varphi,\eta+b_1\xi,b_2\xi,b_2(\gamma+b_1\xi^2)), \tag{33}$$

where b_1 and b_2 are parameters, $b_2\neq 0$, defines a group action. Each orbit of this group contains one and only one solution.

Proof. The first statement may be checked by a direct computation. To proof the second statement, let us differentiate eq. (32) with respect to y. At the point $y=0$ we have:

$$\varphi'(x)=\eta'(0)+\frac{\gamma'(0)}{\xi(x)}-\xi'(0)\frac{\gamma(x)}{\xi(x)^2},$$

Assuming $\varphi(x)$ is regular at $x=0$ and $\varphi''(x)\neq 0$, it is easy to check that $\xi'(0)\neq 0$.

Applying the transformation (33) with $b_2=(\xi'(0))^{-1}, b_1=-\eta'(0)/\xi'(0)$ to the solution $(\varphi,\eta,\xi,\gamma)$, we obtain a normalized solution, and the Lemma is proved.

In what follows solutions are assumed to be normalized, unless the contrary is asserted. Let us now consider the functional equation

$$\varphi(x+y)=\varphi(x)+\varphi(y)+\tau(x)\tau(y)A(x+y),$$

$$\varphi(0) = \varphi'(0) = \tau(0) = \tau''(0) = 0, \quad \tau'(0) = 1. \tag{34}$$

Lemma 10. *For any solution $(\varphi, \eta, \xi, \gamma)$ of the eq. (32), there is a unique solution (φ, τ, A) of the eq. (34) such that*

$$\xi(x) = \frac{\tau(x)}{\tau'(x) - b_3\tau(x)}, \tag{35}$$

$$\eta(x) = \varphi(x) - \varphi'(x)\xi(x), \tag{36}$$

$$\gamma(x) = -\varphi'(x)\xi(x)^2, \tag{37}$$

where $b_3 = \xi''(0)$ is a free parameter.

Proof. Let (φ, τ, A) is some solution of the eq. (34). Then acting on (34) by the operator $\partial_- = (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$ we obtain

$$0 = \varphi'(x) - \varphi'(y) + (\tau'(x)\tau(y) - \tau(x)\tau'(y))A(x+y),$$

i. e.

$$A(x+y) = -\frac{\varphi'(x) - \varphi'(y)}{\tau'(x)\tau(y) - \tau(x)\tau'(y)}. \tag{38}$$

Hence, we transform the eq. (34) to the equation

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \tau(x)\tau(y)\frac{\varphi'(x) - \varphi'(y)}{\tau(x)\tau'(y) - \tau'(x)\tau(y)}. \tag{39}$$

On the other hand,

$$\frac{\tau(x)\tau(y)}{\tau(x)\tau'(y) - \tau'(x)\tau(y)} = \frac{\tau(x)}{\tau'(x)}\frac{\tau(y)}{\tau'(y)}\frac{1}{(\frac{\tau(x)}{\tau'(x)} - b_3) - (\frac{\tau(y)}{\tau'(y)} - b_3)} = \frac{\xi(x)\xi(y)}{\xi(x) - \xi(y)},$$

where the function $\xi(x)$ may be expressed in terms of $\tau(x)$ by the formula (35) with a free parameter b_3.

Therefore,

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \xi(x)\xi(y)\frac{\varphi'(x) - \varphi'(y)}{\xi(x) - \xi(y)}.$$

On substituting the expressions for $\eta(x)$ and $\gamma(x)$ from (36) and (37) we obtain a solution $(\varphi, \eta, \xi, \gamma)$ of the eq. (32).

Let now $(\varphi, \eta, \xi, \gamma)$ be a solution of eq. (32). On substituting $y = 0$ in eq. (32) we obtain

$$\varphi(x) = \eta(x) - \frac{\gamma(x)}{\xi(x)},$$

i.e. $\gamma(x) = \xi(x)\delta(x)$, where $\delta(x) = \eta(x) - \varphi(x)$, and our initial conditions $\varphi'(0) = \eta'(0) = 0 = \varphi(0) = \eta(0)$ are satisfied.

Hence, $\gamma'(0) = 0$, and from the formula for $\varphi'(x)$ obtained in the course of the proof of Lemma 9 we have

$$\gamma(x) = -\varphi'(x)\xi^2(x),$$

$$\eta(x) = \varphi(x) - \varphi'(x)\xi(x),$$

as asserted in (36) and (37). Let us note that formula (35) may be considered as the differential equation for the function $\tau(x)$. Solving this equation at initial conditions $\tau(0) = 0, \tau'(0) = 1$ we obtain the function $\tau(x)$, if, moreover, we take $b_3 = \xi''(0)$ it will satisfy the condition $\tau''(0) = 0$.

Substituting now the expressions for $\xi(x), \eta(x), \gamma(x)$ into eq. (32) we obtain eq. (39).

Let us apply the operator ∂_- to the eq. (39); we obtain

$$\partial_-\left(\frac{\varphi'(x) - \varphi'(y)}{\tau(x)\tau'(y) - \tau'(x)\tau(y)}\right) \equiv 0.$$

Thus it is shown that the functions $\varphi(x)$ and $\tau(x)$ determine the function $A(x)$ given by the expression (38). The Lemma is proved.

So it was shown, how it is possible to construct all the solutions of the eq. (32) using the solutions of eq. (32).

Now we describe the general analytical solution of equation (14).

Lemma 11. *Let (φ, τ, A) be a solution of equation (34). (Let us remind that $\varphi(0) = \varphi'(0) = \tau(0) = \tau''(0) = 0$ and $\tau'(0) = 1$.) Then the function $u(x) = \varphi'(x)$ is a solution of the equation*

$$(u')^2 = c_3 u^3 + 4c_2 u^2 + 2c_1 u + c_0^2,$$

$$u(0) = 0, \quad u'(0) = c_0, \tag{40}$$

and if $c_0 = 0$ then $c_1 \neq 0$.

The functions $\tau(x)$ and $A(x)$ satisfy the following equations:

$$\frac{\tau'(x)}{\tau(x)} = \frac{1}{2} \quad \frac{u'(x) + c_0}{u(x)}, \tag{41}$$

$$\frac{A'(x)}{A(x)} = \frac{1}{2} \quad \frac{u'(x) - c_0}{u(x)}. \tag{42}$$

If $c_0 = 0$ *then* $u(x) = \frac{c_1}{2}\tau(x)^2$, *and* $A(x) = \frac{c_1}{2}\tau(x)$.

Proof. Let us consider the first three derivatives with respect to y of the eq. (34)

$$\varphi'(x+y) = \varphi'(y) + \tau(x)[\tau'(y)A(x+y) + \tau(y)A'(x+y)],$$

$$\varphi''(x+y) = \varphi''(y) + \tau(x)[\tau''(y)A(x+y) + 2\tau'(y)A'(x+y) + \tau(y)A''(x+y)],$$

$$\varphi'''(x+y) = \varphi'''(y)+\tau(x)[\tau'''(y)A(x+y)+3\tau''(y)A'(x+y)+3\tau'(y)A''(x+y)+\tau(y)A'''(x+y)].$$

Taking $y = 0$ and making use of the initial conditions for $\varphi(x)$ and $\tau(x)$, we obtain

$$\varphi'(x) = \tau(x)A(x), \tag{43}$$

$$\varphi''(x) = \varphi''(0) + 2\tau(x)A'(x), \tag{44}$$

$$\varphi'''(x) = \varphi'''(0) + \tau(x)[\tau'''(0)A(x) + 3A''(x)]. \tag{45}$$

Let $\varphi_k = \varphi^{(k)}(0)$ and $\tau_3 = \tau'''(0)$. ¿From (43) and (44) we obtain

$$\frac{\varphi''(x) - \varphi_2}{\varphi'(x)} = 2 \quad \frac{A'(x)}{A(x)}; \tag{46}$$

from (45) and (43) it follows that

$$\frac{\varphi'''(x) - \varphi_3}{\varphi'(x)} = \frac{\tau_3 A(x) + 3A''(x)}{A(x)}. \tag{47}$$

Making use of the identity

$$\frac{A''}{A} = (\frac{A'}{A})' + (\frac{A'}{A})^2$$

for the quantity $\varphi'(x) = u(x)$, we obtain the following equation (see eq. (46), (47)):

$$\frac{u'' - \varphi_3}{u} = \tau_3 + 3(\frac{1}{2} \quad \frac{u' - \varphi_2}{u})' + \frac{3}{4} \quad (\frac{u' - \varphi_2}{u})^2.$$

This equation may be rewritten as follows:

$$4(u'' - \varphi_3)u = 4\tau_3 u^2 + 6[uu'' - u'(u' - \varphi_2)] + 3(u' - \varphi_2)^2,$$

$$2uu'' - 3(u')^2 + 4\tau_3 u^2 + 4\varphi_3 u + 3\varphi_2^2 = 0. \tag{48}$$

Let

$$\tau_3 = c_2, \quad \varphi_3 = c_1, \quad \varphi_2 = c_0.$$

The equation (48) admit the integrable factor $u^{-4}u'$ and may be reduced to the equation

$$(u^{-3}(u')^2)' = 4c_2(u^{-1})' + 2c_1(u^{-2})' + c_0^2(u^{-3})'. \tag{49}$$

Integrating (49) and multiplying the result by u^3 we obtain eq. (40), where c_3 is the integration constant. Equation (42) follows from (46). Then from eq. (43) we obtain:

$$u'(x) = \tau'(x)A(x) + \tau(x)A'(x).$$

It follows from (44) that

$$\tau(x)A'(x) = \frac{u'(x) - c_0}{2}$$

Making use of this fact, we obtain

$$\tau'(x)A(x) = \frac{u'(x) + c_0}{2}.$$

Let us divide this eq. to eq.(43) : $\tau(x)A(x) = u(x)$ we come to the equation (41). Note that if $c_0 = 0$ equations (41),(42), and conditions $\tau(0) = 0, \tau'(0) = 1$ imply

$$u(x) = \frac{c_1}{2}\tau(x)^2, \quad A(x) = \frac{c_1}{2}\tau(x),$$

and it follows, in particular, that $c_1 \neq 0$ if $c_0 = 0$. The Lemma is proved.

Consider the Weierstrass function $\wp(x)$ with parameters g_2 and g_3. We have

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3.$$

Lemma 12. *The general solution of the equation (40) may be written in one of the following equivalent forms:*

$$u(x) = \frac{4}{c_3}(\wp(x + \alpha) \cdot \wp(\alpha)), \tag{50}$$

$$u(x) = c_1\psi(x) + \frac{c_0^2 c_3}{2}\psi(x)^2 + c_0\psi'(x), \tag{51}$$

where

$$\psi(x) = \frac{1}{2} \quad \frac{1}{\wp(x) - \frac{1}{3}c_2} \tag{52}$$

Here $\wp(x)$ *is the Weierstrass function with parameters:*

$$g_2 = 3(\frac{2c_2}{3})^2 - \frac{c_1 c_3}{2}, \quad g_3 = -(\frac{2c_2}{3})^3 + \frac{c_1 c_2 c_3}{6} - (\frac{c_0 c_3}{4})^2, \tag{53}$$

and

$$\wp(\alpha) = \frac{c_2}{3}, \quad \wp'(\alpha) = \frac{c_0 c_3}{4}.$$

Proof. Formula (50) gives:

$$(u'(x))^2 = \frac{16}{c_3^2}[4\wp(x+\alpha)^3 - g_2\wp(x+\alpha) - g_3].$$

On the other hand,

$$(u'(x))^2 = c_3[\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha))]^3 + 4c_2[\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha)]^2 + 2c_1[\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha))] + c_0^2.$$

Hence

$$16[4\wp(x+\alpha)^3 - g_2\wp(x+\alpha) - g_3] =$$

$$4^3[\wp(x+\alpha)-\wp(\alpha)]^3 + 4^3 c_2[\wp(x+\alpha)-\wp(\alpha)]^2 + 8c_1c_3[\wp(x+\alpha)-\wp(\alpha)] + c_0^2 c_3^2.$$

Let us compare the coefficients of the terms of the same degree in $\wp(x+\alpha)$. This shows that formula (50) with parameters g_2, g_3 follows from (53). To deduce (51) from (50) one makes use of the addition theorem for the $\wp$-function (cf., e.g., [WW 1927]).

$$\wp(x+\alpha) - \wp(\alpha) = -(\wp(x) + 2\wp(\alpha)) + \frac{1}{4}(\frac{\wp'(x) - \wp'(\alpha)}{\wp(x) - \wp(\alpha)})^2.$$

therefore

$$(\wp(x+\alpha)-\wp(\alpha))(\wp(x)-\wp(\alpha))^2 = -(\wp(x)+2\wp(\alpha))(\wp(x)^2 - 2\wp(x)\wp(\alpha) + \wp(\alpha)^2) +$$

$$\frac{1}{4}(4\wp(x)^3 - g_2\wp(x) - g_3 - 2\wp'(x)\wp'(\alpha) + \wp'(\alpha)^2) =$$

$$3\wp(x)\wp(\alpha)^2 - 2\wp(\alpha)^3 - \frac{g_2}{4}\wp(x) - \frac{1}{4}g_3 - \frac{1}{2}\wp'(x)\wp'(\alpha) + (\frac{\wp'(\alpha)}{2})^2 =$$

$$(3\wp(\alpha)^2 - \frac{1}{4}g_2)(\wp(x) - \wp(\alpha)) - \frac{1}{2}\wp'(x)\wp'(\alpha) + \frac{\wp'(\alpha)^2}{2}.$$

Hence,

$$\wp(x+\alpha)-\wp(\alpha)-\frac{1}{2}\frac{\wp'(x)}{(\wp(x)-\wp(\alpha))^2}\wp'(\alpha)+\frac{3\wp(\alpha)^2-\frac{1}{4}g_2}{\wp(x)-\wp(\alpha)}+\frac{1}{2}(\frac{\wp'(\alpha)}{\wp(x)-\wp(\alpha)})^2. \tag{54}$$

This gives:

$$\wp'(\alpha)=\frac{c_0c_3}{4}, \quad 3\wp(\alpha)^2-\frac{1}{4}g_2=\frac{c_1c_3}{8}.$$

Formula (51) follows from eq.(54) on dividing by $\frac{1}{4}c_3$. The Lemma is proved.

Corollary 13. *The general solution of eq. (40) takes the shape*

$$u_*(x)=c_1(\frac{\cosh 2\sqrt{c_2}x-1}{(2\sqrt{c_2})^2})+c_0\frac{\sinh 2\sqrt{c_2}x}{2\sqrt{c_2}}. \tag{55}$$

as $c_3\to 0$.

Proof. Let

$$u_*(x)=\lim_{c_3\to 1}u(x), \quad \psi_*(x)=\lim_{c_3\to 0}\psi(x), \quad \wp_*(x)=\lim_{c_3\to 0}\wp(x).$$

By Lemma 12, the function $\wp_*(x)$ satisfies the equation

$$(\wp_*'(x))^2=4\wp_*(x)^3-3(\frac{2c_2}{3})^2\wp_*(x)+(\frac{2c_0}{3})^3=4(\wp_*(x)-\frac{c_2}{3})^2(\wp_*(x)+\frac{2}{3}c_2).$$

Therefore

$$(\psi_*'(x))^2=\frac{1}{4}\quad(\frac{-\wp_*'(x)}{(\wp_*(x)-\frac{1}{3}c_2)^2})^2=\frac{\wp_*(x)+\frac{2}{3}c_2}{(\wp_*(x)-\frac{1}{3}c_2)^2}=2\psi_*(x)+4c_2\psi_*(x)^2. \tag{56}$$

Differentiating (56) with respect to x, one obtains

$$\psi_*''(x)=4c_2\psi_*(x)+1,$$

$$\psi_*(0)=0,$$

$$\psi_*'(0)=0.$$

Therefore

$$\psi_*(x)=\frac{\cosh 2\sqrt{c_2}x-1}{(2\sqrt{c_2})^2}$$

In view of (51), it follows that

$$u_*(x) = c_1\psi_*(x) + c_0\psi'_*(x).$$

The Lemma 13 is proved.

Note that according to the Lemma 11, if the functions (φ, τ, A) satisfy equation (14) then the function $\tau(x)$ is determined uniquely by the equation

$$\frac{\tau'(x)}{\tau(x)} = \frac{1}{2} \quad \frac{u'(x) + c_0}{u(x)},$$

subject to the initial conditions $\tau(0) = 0, \tau'(0) = 1$, and the function $A(x)$ is determined by the equation (43):

$$A(x) = \frac{u(x)}{\tau(x)}.$$

Hence we may reqard the functions $\varphi(x)$ as solutions of the equation (14).

Theorem 14. *The general solution of the equation (14)*

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \tau(x)\tau(y)A(x+y)$$

is given by the function

$$\varphi(x) = \frac{4}{c_3}(\zeta(\alpha) - \zeta(x+\alpha) - \wp(\alpha)x),$$

$$\varphi(0) = \varphi'(0) = 0$$

where $\zeta(x)$ and $\wp(x)$ are the Weierstrass ζ-function and $\wp$-function with the parameters g_2 and g_3 (see Lemma 12).

Proof. According to the Lemmas (11) and (12) it is sufficient to prove that any function $\varphi(x)$ given by the formula (57) is a solution of equation (14). It is convenient to consider two different cases.
Case 1. $c_3 = 0$.

$$\varphi_*(x) = \lim_{c_3 \to 0} \varphi(x)$$

In this case $\varphi_*(x) = \int_0^\infty u_*(x)dx$ and hence, using the Corollary 13,we obtain

$$\varphi_*(x) = c_1 \frac{\sinh 2\sqrt{c_2}x - 2\sqrt{c_2}x}{(2\sqrt{c_2})^3} + c_0 \frac{\cosh 2\sqrt{c_2}x - 1}{(2\sqrt{c_2})^2}. \qquad (58)$$

Using the elementary identity

$$(e^{(x+y)}-1)=(e^{x}-1)+(e^{y}-1)+(e^{\frac{x}{2}}-e^{\frac{-x}{2}})(e^{\frac{y}{2}}-e^{\frac{-y}{2}})e^{\frac{x+y}{2}}. \qquad (59)$$

we obtain

$$\sinh 2\sqrt{c_2}(x+y)=\sinh 2\sqrt{c_2}x+\sinh 2\sqrt{c_2}y+4\sinh\sqrt{c_2}x\sinh\sqrt{c_2}y\sinh\sqrt{c_2}(x+y),$$

$$\cosh 2\sqrt{c_2}(x+y)=\cosh 2\sqrt{c_2}x+\cosh 2\sqrt{c_2}y+4\sinh\sqrt{c_2}x\sinh\sqrt{c_2}y\cosh\sqrt{c_2}(x+y)$$

Hence

$$\varphi_*(x+y)=\varphi_*(x)+\varphi_*(y)+\tau_*(x)\tau_*(y)A_*(x+y),$$

where

$$\tau_*(x)=\frac{\sinh\sqrt{c_2}x}{\sqrt{c_2}},\quad A_*(x)=\frac{c_1}{2}\quad\frac{\sinh\sqrt{c_2}x}{\sqrt{c_2}}+c_0\cosh\sqrt{c_2}x. \qquad (60)$$

Case 2. $c_3\neq 0$.
Then without any restriction we may take $C_3=2$. According to Frobenius-Stickelberger formula [FS 1880] the functions $f(x),g(y),h(z)$ are the solution the eq. (1):

$$f(x)=\zeta(\alpha_1-\frac{\alpha}{2}-x)-\gamma(\alpha)x-\zeta(\alpha_1-\frac{\alpha}{2}), \qquad (61)$$

$$g(y)=\zeta(-\alpha_1-\frac{\alpha}{2}-y)-\gamma(\alpha)y+\zeta(\alpha_1+\frac{\alpha}{2}), \qquad (62)$$

$$h(z)=\zeta(\alpha-z)-\gamma(\alpha)z-\zeta(\alpha). \qquad (63)$$

Using the considered above reduction of the (1) to the eq.(14) we obtain

$$\varphi(x)=-2h(-x)=2(\zeta(\alpha)-\zeta(x+\alpha)-\gamma(\alpha)x)$$

that gives the solution of eq. (14). The theorem is proved.

Corollary 15. *The general normalized solution of eq. (12) is given by the formulae*

$$\varphi(x)=\frac{4}{c_3}(\zeta(\alpha)-\zeta(x+\alpha)-\gamma(\alpha)x),$$

$$\xi(x) = \frac{2u(x)}{c_0 - 2b_3 u(x) + u'(x)}, \tag{64}$$

where $u(x) = \varphi'(x) = \frac{4}{c_3}(\wp(x+\alpha) - \wp(\alpha))$ *and* b_3 *is free parameter.*

$$\eta(x) = \varphi(x) - \varphi'(x)\xi(x),$$

$$\gamma(x) = -\varphi'(x)\xi(x)^2.$$

The proof is follows from the theorem 14, formula (61) and Lemma 10. Let us remind that at the proof of Lemma 10 we will give the explicit construction of the solution eq. (12) on the solution of eq. (14).

So,it was proved already that if $(f(x), g(y), h(z))$ is the nondegenerate solution of eq. (1) with additional conditions

$$f(x) = g(x). \quad f(0) = g(0) = h(0) = h'(0). \tag{65}$$

than it is necessary that

$$h(x) = \frac{2}{c_3}(\zeta(\alpha - x) - \gamma(\alpha)x - \zeta(\alpha)), \tag{66}$$

where c_3, α and the parameters g_2, g_3 of the $\wp$- Weierstrass function should satisfy the condition of the Lemma (12). Moreover, if $c_3 \neq 0$, then for the functions

$$f(x) = \frac{2}{c_3}(\zeta(\alpha_1 - \frac{\alpha}{2} - x) - \gamma(\alpha)x - \zeta(\alpha_1 - \frac{\alpha}{2})), \tag{67}$$

$$g(x) = \frac{2}{c_3}(\zeta(-\alpha_1 - \frac{\alpha}{2} - x) - \gamma(\alpha)x + \zeta(\alpha_1 + \frac{\alpha}{2})), \tag{68}$$

where α is free parameter, the function $h(x)$ of type (66) is given the solution of eq. (1). Hence, there are two unsolved problems.

1. Does the functions $f(x)$ and $g(x)$ at $c_3 \neq 0$ are unique functions, which give the solution of eq. (1) at fixed function $h(x)$?

2. To find the sufficient conditions at $c_3 = 0$ on the parameters of the function $h_*(x) = \lim_{c_3 \to 0} h(x)$, that there are the functions $f(x)$ and $g(x)$, such that $(f(x), g(x), h_*(x))$ are the solution of the equation (1) and to find all such functions $(f(x), g(x))$.

Let us note, that at the case $c_3 = 0$ the main problem is that we cannot consider the limit $c_3 \to 0$ at the formulae (67),(68) (in the distinction of function (66)).

To solve these two problems we will consider first the reduction of the eq. (1) to the eq. (12) and will use the general analytic solution of eq. (12) (see Lemma 9 and Corollary 15).

Let us begin to the consideration of the case $c_3 \neq 0$.

Lemma 16. *Let the functions $(f_1(x), g_1(x), h_1(x))$ satisfy the equation (1) and the initial conditions under cinsideration. Then, if $h_1(x) = H(x)$ is the function from eq. (66), then*

$$f_1(x) = s_1 f(x) + s_2 g(x), \tag{69}$$

$$g_1(x) = t_1 f(x) + t_2 g(x), \tag{70}$$

where $f(x)$ and $g(x)$ are given by eqs. (47) and (48), and $s_1+s_2 = 1, t_1+t_2 =$ 1. **Proof.**
For the functions, given by eqs. (47) and (48) we have:

$$\xi(x) = f(x)-g(x) = \frac{2}{c_3}[\zeta(\alpha_1-\frac{\alpha}{2}-x)+\zeta(\alpha_1+\frac{\alpha}{2}+x)-\zeta(\alpha_1-\frac{\alpha}{2})-\zeta(\alpha_1+\frac{\alpha}{2})]. \tag{71}$$

Then

$$\xi'(x) = \frac{2}{c_3}[\gamma(\alpha_1 - \frac{\alpha}{2} - x) - \gamma(\alpha_1 + \frac{\alpha}{2} + x)],$$

$$\xi''(x) = \frac{2}{c_3}[-\gamma'(\alpha_1 - \frac{\alpha}{2} - x) - \gamma'(\alpha_1 + \frac{\alpha}{2} + x)]$$

We have that if the quantities α and α_1 are sufficiently close to the point $x = 0$, then $\xi'(0) \neq 0$, and the quantity $\xi''(0)$ gives the free parameter b_3 to construct the general normalized solution of eq. (12). Therefore, in this case the general solution of the equation has the form

$$\varphi(x) = -2h(-x), \quad \eta(x) + b_1\xi(x), \quad b_2\xi(x)$$

where $h(x)$ is the function (46), $\xi(x)$ is the function (49) and $\eta(x) = f(x) + g(x)$ for the functions (47) and (48).

Let us introduce now

$$f_1(x) + g_1(x) = \eta(x) + b_1\xi(x),$$

$$f_1(x) - g_1(x) = b_2\xi(x),$$

we have

$$f_1(x) = \frac{1}{2}\eta(x) + \frac{b_1 + b_2}{2}\xi(x) = s_1 f(x) + s_1 g(x),$$

$$g_1(x) = \frac{1}{2}\eta(x) + \frac{b_1 - b_2}{2}\xi(x) = t_1 f(x) + t_2 g(x),$$

where

$$s_1 = \frac{1}{2} + \frac{b_1 + b_2}{2}, \quad s_2 = \frac{1}{2} - \frac{b_1 + b_2}{2}, \quad t_1 = \frac{1}{2} + \frac{b_1 - b_2}{2}, \quad t_2 = \frac{1}{2} - \frac{b_1 - b_2}{2}$$

The Lemma is proved.

Now we need just find the values of parameters s_1 and t_1, for which the set of functions $(f_1(x), g(x), h(x))$ from the Lemma 16 gives the solution of eq. (1).

Let us introduce the notation

$$\det(f, g, h) = \begin{pmatrix} f''(x) & g''(y) & h''(z) \\ f'(x) & g'(y) & h'(z) \\ 1 & 1 & 1 \end{pmatrix}$$

and let us use the following formula (see [WW 1927], p.458)

$$\frac{1}{2}\det(\wp(x), \wp(y), \wp(z)) = \frac{\sigma(x+y+z)\sigma(x-y)\sigma(y-z)\sigma(z-x)}{\sigma^3(x)\sigma^3(y)\sigma^3(z)}$$

If the conditions of the Lemma 16 are sutisfied, we have

$$\det(s_1 f(x) + s_2 g(x), \quad t_1 f(x) + t_2 g(x), \quad h(z)) =$$

$$s_1 t_1 \det(f(x), f(y), h(z)) + s_2 t_2 \det(g(x), g(y), h(z)) \tag{72}$$

Therefore,

$$\frac{c_3^3}{8}\det(f(x), f(y), h(z)) = \frac{c_3^3}{8}\det(\wp(\alpha_1 - \frac{\alpha}{2} - x), \quad \wp(\alpha_1 - \frac{\alpha}{2} - y), \quad \wp(\alpha - z))$$

$$= \frac{c_3^3}{4} \quad \frac{\sigma(2\alpha_1)\sigma(y-x)\sigma(z - y + \alpha_1 - \frac{3}{2}\alpha)\sigma(x - z + \frac{3}{2}\alpha - \alpha_1)}{\sigma^3(\alpha_1 - \frac{\alpha}{2} - x)\sigma^3(\alpha_1 - \frac{\alpha}{2} - y)\sigma^3(\alpha - z)}, \tag{73}$$

$$\frac{c_3^3}{8}\det(g(x), g(y), h(z)) = \frac{c_3}{8}\det(\wp(-\alpha_1 - \frac{\alpha}{2} - x), \quad \wp(-\alpha_1 - \frac{\alpha}{2} - y), \quad \wp(\alpha_1 - 2)$$

$$= \frac{c_3^3}{4} \quad \frac{\sigma(2\alpha_1)\sigma(y-x)\sigma(y - z + \alpha_1 + \frac{3}{2}\alpha)\sigma(x - z + \alpha_1 + \frac{3}{2}\alpha)}{\sigma^3(\alpha_1 + \frac{\alpha}{2} + x)\sigma^3(\alpha_1 + \frac{\alpha}{2} + y)\sigma^3(\alpha - z)}. \tag{74}$$

Comparison of expressions (71) and (72) shows that if $\alpha_1 = \omega_k$ is the one of the three halfperiods of the Weierstrass-function $\wp(x)$, then det ( ), given by the formula (70) is equal to zero identically for any values s_1 and t_1. If, however, $\alpha_1 \neq \omega_k, k = 1.2.3$, then the det is equal to zero if and only if $s_1 = t_1 = 0$.

Let us consider now the case $c_3 = 0$.

The general normalized solution in this case is given by the function (78).

Let us denote

$$\varphi_{**}(x) = \lim_{c_2 \to 0} \varphi_*(x), \quad \tau_{**}(x) = \lim_{c_2 \to 0} \tau_*(x), \quad A_{**}(x) = \lim_{c_2 \to 0} A_*(x)$$

¿From (78) we obtain

$$\varphi_{**}(x) = c_1 \frac{x^3}{3!} + c_0 \frac{x^2}{2}, \tag{75}$$

and according to the formulae (40) we have

$$\tau_{**}(x) = x, \quad A_{**}(x) = c_1 \frac{x}{2} + c_0.$$

We have:

$$\xi_{**}(x) = \frac{x}{1 - b_3 x}, \tag{76}$$

$$\eta_{**}(x) = c_1 \frac{x^3}{3!} + c_0 \frac{x^2}{2} - (c_1 \frac{x^2}{2} + c_0 x) \frac{x}{1 - b_3 x}. \tag{77}$$

Hence, the general solution of the equation (12) is given by the functions

$$\varphi_{**}(x), \quad \eta_{**} + b_1 \xi_{**}, \quad b_2 \xi_{**}.$$

Acknowledgements

One of authors (A.P.) would like to thank Erwin Schrödinger Institute for warm hospitality.

REFERENCES

[Ab 1823] Abel N.H., *Méthode geńérale pour trouver des fonctions d'une seul quantité variable, lorsqu'une propriété de ces fonctions est exprimee par uneéquation entre deux variables.*
Magazin for Naturvidenskaberne, Aargang I, Bind 1, Christiania,
Oeuvres completes, **1**, Christiania, 1881, 1-10

[BC 1990] M.Bruschi and F.Calogero *General analytic solution of certain functional equations of addition type*, SIAM J. Math. Anal. **21**, 1019–1030

[BFV 1994] V.Buchstaber, G.Felder and A.Veselov *Elliptic Dunkl operators, root systems and functional equations*, Preprint hep-th/9403178

[BK 1993] V.Buchstaber and I.Krichever *Vector addition theorems and Baker-Akhiezer functions*, Teor.Mat.Fiz. **94**, No.2, 200–212

[BP 1993] V.Buchstaber and A.Perelomov *On the functional equation related to the quantum three-body problem*, preprint MPI/93–17

[Ca 1821] Cauchy A.L., *Cours d'Analyse de l'Ecole Poilyt,* **1***. Analyse algebraique,* **103**
Oeuvres completes (2) **3**, 98-105

[Ca 1975] Calogero F., *One-dimensional many-body problems with pair interactions whose exact ground-state wave function is of product type,*
Lett. Nuovo Cimento **13**, 507-511

[FS 1880] Frobenius G. and Stickelberger L., *Über die Addition und Multiplication der elliptischen Functionen,*
Jour. Reine Angew. Math. **88**, 146-184

[OP 1983] Olshanetsky M.A. and Perelomov A.M., *Quantum integrable systems related to Lie algebras,*
Phys. Reports **94**, 313-404

[Pe 1990] Perelomov A.M. *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhauser

[Su 1975] Sutherland B., *Exact ground-state wave function for a one-dimensional plasma,*
Phys. Rev. Lett. **34**, 1083-1085

[WW 1927] Whittaker E. and Watson G., *Course of Modern Analysis,* Cambridge Univ. Press