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V. Hutson
W. Shen
G.T. Vickers

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Non-autonomous non-local dispersal - Spectral Theory.

V. Hutson
School of Mathematics and Statistics
The University of Sheffield
Sheffield S3 7RH, U.K.

W. Shen
Department of Mathematics and Statistics
Auburn University
Auburn, AL 36849, U.S.A.

and
G. T. Vickers
School of Mathematics and Statistics
The University of Sheffield
Sheffield S3 7RH, U.K.

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Abstract. In applications to spatial structure in biology and to the theory of phase transition, it has proved useful to generalize the idea of diffusion to a non-local dispersal with an integral operator replacing the Laplacian. We study the spectral problem for the linear scalar equation

$$u_t(x,t) = \int_\Omega K(x,y)u(y,t)dy + h(x,t)u(x,t),$$

and tackle the extra technical difficulties arising because of the lack of compactness for the evolution operator defined by the dispersal. Our aim is firstly to investigate the extent to which the idea of a periodic parabolic principal eigenvalue may be generalized. Secondly, we obtain a lower bound for this in terms of the corresponding averaged spatial problem, and then extend this to the principal Lyapunov exponent in the almost periodic case.

Key words: Non-local dispersal, Prinipal eigenvalue, Principal Lyapunov exponent, Non-autonomous reaction dispersal.

Mathematics subject classification: 37L05 47D06, 47G20

1 Introduction

Recently there has been extensive investigation into a class of models for non-local spatial dispersal, in which the dispersal operator D, say, involves an integral operator, for example

$$(Du)(x) = \int_\Omega K(x,y)[u(y) - u(x)]dy. \tag{1.1}$$

Such models occur in a number of applications, for example biology and the theory of phase transition, as a generalisation of classical diffusion where $D = \Delta$, the Laplacian with a suitable boundary condition. The derivation in the biological context is discussed in [22], [12] and [17], and for the theory of phase transition see [4] and [7, 8]. The non-linear theory has been investigated in a number of papers, of which a sample, in addition to the above, is [25], [6], [5], [11] and [16] .

We shall consider here aspects of spectral theory for linear evolution problems with non-local dispersal; this of course provides a basic technical tool in the non-linear theory, for example in a discussion of stability. Non-autonomous models have scarcely been considered in the dispersal context, and here we shall focus particularly on the periodic and almost-periodic cases. Consider then the linear evolution equation

$$u_t(x,t) = \int_\Omega K(x,y)u(y,t)dy + h(x,t)u(x,t), \tag{1.2}$$

where u_t denotes the derivative of u with respect to time t (with x constant) and $\Omega \subset \mathbb{R}^N$ is a compact spatial region; note that the second term in D in (1.1) has been incorporated into h. It is convenient to abbreviate the notation and write this as

$$u_t = Xu + Hu,$$

where X is the integral operator in (1.2) and H is multiplication by h.

If $D = \Delta$, for the autonomous case (h independent of t) and for the periodic case ($h(x,t) = h(x,t+T)$ for all $x \in \Omega$ and t) there is a well-known theory yielding the existence of a principal eigenvalue (PEV) and eigenfunction (PEF). For the theory, see [15]; in applications the idea has been used for example in studying the evolution of diffusion [18] and in permanence, see [2, Chapter 2]. This has important implications for the study of stability, rate of increase, invasion problems and sub/super solution methods for nonlinear models. The partial differentiation equation (PDE) technique of proof depends critically on compactness properties for the evolution operator based upon Δ.

Our first objective here is to enquire to what extent these results hold for the non-local case (1.2). The PDE technique is not applicable as the evolution operator generated by X does not appear to have compactness properties in convenient spaces. We employ a method based on using the evolution operator generated by the linear operator $(-\frac{\partial}{\partial t} + H)$ together with the compactness of X itself. It is proved that if $N = 1$, reasonable smoothness conditions on h are sufficient to ensure the existence of a PEV. However, if $N \geq 2$ then examples show that smoothness is not enough. This interesting issue is further discussed in Section 3.

An upper bound for the growth rate of the solution for a continuous initial condition is provided by the principal Lyapunov exponent λ_L. If a PEV λ exists, then $\lambda_L = \lambda$; it is also shown (Theorem 3.9) that if there is no PEV then $\lambda_L = s := \sup_{\lambda \in \sigma} \Re(\lambda)$, where σ is the spectrum of $(-\frac{\partial}{\partial t} + X + H)$ on the space of T-periodic functions.

If h is almost periodic (AP) only, there appears to be no analogue of a PEV. A partial analogue of the above is provided by using the dynamical spectrum (see Definition 2.2) Σ of $(-\frac{\partial}{\partial t} + X + H)$. Then $\lambda_L = \lambda_s := \sup_{\lambda \in \Sigma} \lambda$.

Our second main objective is to show that, for the periodic case, when a PEV exists it is always larger than the PEV for the associated time-average case; this extends a result of [19] for the PDE problem. In the biological context, this inequality shows that, perhaps rather counter intuitively, invasion by a new species (see [2], p. 220) is always easier in the periodic case. For the AP case, an analogous result holds, viz. that λ_L is always larger than the PEV for the time-averaged case.

An outline of the contents is as follows. In Section 2, the notation is described and some background results proved. The question of existence of a PEV for the periodic case is considered in Section 3 and in Section 4 the lower bound for the PEV is established. In Section 5, the AP case is discussed.

2 Definitions and Basic Properties

First the notation is described. Some of the spectral theory for the most general case to be considered here (h AP in t) is then outlined. We note that the theory is not as simple as for the well-known case where the dispersal operator is an elliptic partial differential operator.

The following conditions are assumed throughout.

(H1) (a) $\Omega \subset \mathbb{R}^N$ is compact.

(b) $K : \Omega \times \Omega \to \mathbb{R}$ is continuous and

$$K(x, y) \geq 0 \quad (x, y \in \Omega). \tag{2.1}$$

(c) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, is uniformly AP in its second argument, t, and is uniformly bounded.

Let $E = C(\Omega)$ be the Banach space of continuous, complex valued functions on Ω with the maximum norm $\|u\| = \max_{x\in\Omega} |u(x)|$. With the ordering induced by the positive cone

$$E_+ = \{u \in E | u(x) \geq 0 \ (x \in \Omega)\},$$

$(E, \|\cdot\|)$ is a complex Banach lattice, and we write $u \geq v$ if $u(x) \geq v(x)$ $(x \in \Omega)$. The notation $u > v$ if $u(x) > v(x)$ for all $x \in \Omega$ will be adopted. A linear operator $L : E \to E$ is said to be *positive* if $u \geq v \Rightarrow Lu \geq Lv$. For each t, define the (continuous) operators $X, H : E \to E$ as follows.

$$(Xu)(x, t) = \int_\Omega K(x, y)u(y, t)\, dy, \tag{2.2}$$

$$(Hu)(x, t) = h(x, t)u(x, t). \tag{2.3}$$

In writing down the governing equation, we treat $u(\cdot, t)$ as an E-valued function for each t and, for brevity, suppress the t-dependence. We study the equation

$$u_t = Xu + Hu. \tag{2.4}$$

Let $\Phi(s, t)$ $(s \leq t)$, defined by

$$\Phi(t, s)u_0 = u(t, \cdot; u_0, s) \quad (u_0 \in E),$$

be the evolution operator generated on E, where $u(t, x; s, u_0)$ is the solution of (2.4) with $u(s, x; s, u_0) = u_0(x)$. For given $\lambda \in \mathbb{R}$, define

$$\Phi_\lambda(t, s) = e^{-\lambda(t-s)}\Phi(t, s).$$

In general, the 'spectrum' of the evolution operator and the 'spectrum' of its generator may not be same, see [23, Chapter 2, example 2.1]. It is the spectrum of the evolution operator which characterizes the asymptotic behaviour of solutions. For completeness, this concept is defined here.

Definition 2.1. *Given $\lambda \in \mathbb{R}$, $\{\Phi_\lambda(t, s)\}_{s,t\in\mathbb{R},s\leq t}$ is said to admit an* exponential dichotomy (ED *for short) if there exist $\beta > 0$ and $C > 0$ and continuous projections $P(s) : E \to E$ $(s \in \mathbb{R})$ such that for any $s, t \in \mathbb{R}$ with $s \leq t$ the following holds:*

(1) $\Phi_\lambda(t,s)P(s) = P(t)\Phi_\lambda(t,s)$;

(2) $\Phi_\lambda(t,s)|_{R(P(s))} : R(P(s)) \to R(P(t))$ *is an isomorphism for* $t \geq s$ *(hence* $\Phi_\lambda(s,t) := \Phi_\lambda(t,s)^{-1} : R(P(t)) \to R(P(s))$ *is well defined);*

(3)

$$\|\Phi_\lambda(t,s)(I - P(s))\| \leq Ce^{-\beta(t-s)}, \quad t \geq s$$
$$\|\Phi_\lambda(t,s)P(s)\| \leq Ce^{\beta(t-s)}, \quad t \leq s.$$

Definition 2.2. *(1)* $\lambda \in \mathbb{R}$ *is said to be in the* dynamical spectrum, *denoted by* $\Sigma(X,H)$, *of (2.4) if* $\Phi_\lambda(t,s)$ *does not admit an ED.*

(2) $\lambda_s(X,H) := \sup\{\lambda \in \Sigma(X,H)\}$ *is called the* principal dynamical spectrum point *of (2.4).*

Definition 2.3. $\lambda_L(X,H) := \limsup_{t-s\to\infty} \dfrac{\ln \|\Phi(t,s)\|}{t-s}$ *is called the* principal Lyapunov exponent *of (2.4).*

Proposition 2.4. *Assume that* $u_0 \in C(\Omega)$ *and* $u_0 \geq 0$. *Assume also that* $p : \Omega \times \mathbb{R} \to \mathbb{R}$ *is continuous and uniformly bounded. Suppose that* $u : \Omega \times \mathbb{R} \to \mathbb{R}$ *is continuous and differentiable in its second argument with* u_t *continuous on* $\Omega \times \mathbb{R}$. *Then if* u *satisfies the following:*

$$\begin{aligned} u_t(x,t) &\geq \int_\Omega K(x,y)u(y,t)\,dy + p(x,t)u(x,t) \quad (t \geq s) \\ u(x,s) &= u_0(x) \end{aligned} \tag{2.5}$$

then

(1) $K(x,y) \geq 0$ $(x,y \in \Omega) \Rightarrow u(x,t) \geq 0$ $(x \in \Omega, t > s)$

(2) $K(x,y) > 0$ $(x,y \in \Omega)$ *and* $u_0 \not\equiv 0 \Rightarrow u(x,t) > 0$ $(x \in \Omega, t > s)$

Proof. Note first that from (2.5), $v(x,t) = e^{\lambda(t-s)}u(x,t)$ satisfies the inequality

$$v_t(x,t) \geq \int_\Omega K(x,y)u(y,t)\,dy + [p(x,t) + \lambda]v(x,t) \quad (t \geq s)$$

with $v(x,s) = u_0(x)$. Therefore it may be assumed without loss of generality that $p(x,t) > 0$ $(x \in \Omega, t \in \mathbb{R})$.

(1) Let

$$K_0 = \max_{x\in\Omega} \int_\Omega K(x,y)\,dy \quad \text{and} \quad p_0 = \sup_{x\in\Omega, t\in\mathbb{R}} p(x,t).$$

Take $\tau = \frac{1}{2}(K_0 + p_0)^{-1}$. Suppose that for some x and $t \in [s, s+\tau]$, $u(x,t) < 0$. Then there exist x_1 and $t_1 \in [s, s+\tau]$ such that

$$\min_{x\in\Omega, s\le t\le s+\tau} u(x,t) = u(x_1, t_1) < 0.$$

Integrating (2.5) with respect to t over $[s, t_1]$ and using the Mean Value Theorem for integrals, we deduce that

$$u(x_1, t_1) - u(x_1, s) \ge (t_1 - s)(K_0 + p_0)u(x_1, t_1).$$

But by assumption, $u(x_1, s) = u_0(x_1) \ge 0$. Therefore

$$u(x_1, t_1)[1 - (t_1 - s)(K_0 + p_0)] \ge 0.$$

Since $(t_1 - s) \le \tau = \frac{1}{2}(K_0 + p_0)^{-1}$ we have $u(x_1, t_1) \ge 0$ which is a contradiction. Therefore $u(x, y) \ge 0$ for all $x \in \Omega$ and $s \le t \le s + \tau$. The result follows on repeating the argument with initial times $s + \tau, s + 2\tau, \ldots$.

(2) ¿From the result just proved and (2.5), clearly $u_t(x,t) \ge 0$ $(x \in \Omega, t \ge s)$. Also, since $K > 0$ and $u_0 \not\equiv 0$ we see that $u_t(x, s) > 0$ $(x \in \Omega)$. Therefore, by continuity and compactness, there exist $\delta > 0$, $t_0 > s$ such that $u(x, t_0) \ge \delta$ $(x \in \Omega)$ and $u(x,t) > 0$ $(x \in \Omega, s < t \le t_0)$. If the assertion does not hold, there exist $x_1 \in \Omega$, $t_1 > t_0$ such that $u(x_1, t_1) = 0$. But $u(x_1, t_0) \ge \delta$ and $u_t(x_1, t) \ge 0$ for all $t \ge s$. This yields a contradiction.

□

Proposition 2.5. $\lambda_s(X, H) = \lambda_L(X, H)$.

Proof. The proposition may be proved by arguments similar to those in [24, Proposition 4.1]. For completeness, we provide a proof here.

First we note that there are $\bar{C}$ and $\omega \in \mathbb{R}$ such that

$$\|\Phi(t,s)\| \le \bar{C}e^{\omega(t-s)} \tag{2.6}$$

for any $s, t \in \mathbb{R}$ with $s \le t$.

Next, suppose that $\lambda_s(X, H)$, $\lambda_L(X, H) > -\infty$. By (2.6), $\lambda_s(X, H) < \infty$. Hence, for $\epsilon > 0$ and $\lambda^* = \lambda_s(X, H) + \epsilon$, there is $C > 0$ such that

$$\|e^{-\lambda^*(t-s)}\Phi(t,s)\| \le C$$

that is,

$$\|\Phi(t,s)\| \le Ce^{\lambda^*(t-s)}$$

for $s \le t$. It then follows that

$$\lambda_L(X, H) \le \lambda^* = \lambda_s(X, H) + \epsilon.$$

By taking $\epsilon \to 0$, we have $\lambda_L(X, H) \le \lambda_s(X, H)$. Conversely, since $\lambda_L(X, H) < \infty$, for any $\epsilon > 0$,

$$e^{-(\lambda_L(X,H)+\epsilon)(t-s)}\|\Phi(t, s)\| \to 0 \quad \text{as} \quad t - s \to \infty.$$

This implies that $\lambda_L(X, H) + \epsilon \in \mathbb{R} \setminus \Sigma(X, H)$ and $\lambda_s(X, H) \le \lambda_L(X, H) + \epsilon$. Since ϵ is arbitrary, $\lambda_s(X, H) \le \lambda_L(X, H)$. Therefore, $\lambda_s(X, H) = \lambda_L(X, H)$.

Now if $\lambda_s(X, H) = -\infty$ or $\lambda_L(X, H) = -\infty$, by the above arguments, for any $M > 0$, $\lambda_L(X, H) \le -M$ or $\lambda_s(X, L) \le -M$. Therefore, $\lambda_L(X, H) = -\infty$ or $\lambda_s(X, H) = -\infty$. □

Proposition 2.6. *$\lambda_L(X, H)$ is continuous in H with respect to the topology of uniform convergence, that is, if $h_n(x, t) \to h(x, t)$ as $n \to \infty$ uniformly in $x \in \Omega$ and $t \in \mathbb{R}$, then $\lambda_L(X, H_n) \to \lambda_L(X, H)$, where $H_n u = h_n u$ and $Hu = hu$.*

Proof. First, let $\Phi^{\pm\epsilon}(t, s)$ be the evolution operators generated by (2.4) with H being replaced by $H \pm \epsilon$. It is clear that

$$\Phi^{\pm\epsilon}(t, s) = \Phi_{\pm\epsilon}(t, s) = e^{\pm\epsilon(t-s)}\Phi(t, s).$$

Therefore,

$$\lambda_L(X, H \pm \epsilon) = \lambda_L(X, H) \pm \epsilon. \tag{2.7}$$

Next, for given h_1, h_2 with $h_1 \le h_2$, let $\Phi^i(t, s)$ $(i = 1, 2)$ be the evolution operators generated by (2.4) with $Hu = H_i u := h_i u$. We claim that

$$\|\Phi^1(t, s)\| \le \|\Phi^2(t, s)\|. \tag{2.8}$$

In fact, for any given $u_0 \in E$ with $u_0 \ge 0$, by Proposition 2.4 (1) with $p = h_i$, $\Phi^i(t, s)u_0 \ge 0$ for $s \le t$ and $i = 1, 2$. Let $v(x, t) = \Phi^2(t, s)u_0 - \Phi^1(t, s)u_0$. Then $v(x, t)$ satisfies

$$\begin{aligned} v_t &= \int_\Omega K(x, y)v(y, t)dy + h_2(x, t)v(x, t) + (h_2(x, t) - h_1(x, t))\Phi^1(t, s)u_0 \\ &\ge \int_\Omega K(x, y)v(y, t)dy + h_2(x, t)v(x, t) \end{aligned}$$

with $v(x, s) = 0$. By Proposition 2.4 (1) with $p = h_2 - h_1$, $v(x, t) \ge 0$ which implies (2.8) and this in turn gives

$$\lambda_L(X, H_1) \le \lambda_L(X, H_2). \tag{2.9}$$

The proposition then follows from (2.7) and (2.9). □

3 The Periodic Case.

Our objective is to enquire to what extent the well-known PDE theory for the existence of a PEV when h is periodic extends to the non-local dispersal case (2.4). It will be proved that, under the assumed smoothness condition, that is h is Lipschitz in x, the results extend if the dimension $N = 1$. If $N > 1$, in general a smoothness condition is not enough; this observation raises questions which are discussed at the end of this section. We also show that even when a PEV does not exist, the principal Lyapunov exponent $\lambda_L = s(\tilde{X}, \tilde{H})$, $s(\tilde{X}, \tilde{H}) := \sup_{\lambda \in \sigma(\tilde{X}, \tilde{H})} \Re(\lambda)$, where $\tilde{X}$, $\tilde{H}$ are X, $-\frac{\partial}{\partial t} + H$, respectively, restricted to a space of periodic functions and $\sigma(\tilde{X}, \tilde{H})$ is the spectrum of $\tilde{H} + \tilde{X}$.

It will be assumed in this section that h is T-periodic in t, that is $h(x,t) = h(x, t+T)$ for each $x \in \Omega$. Since the problem is linear and we shall be discussing the spectrum, we may assume $h(x,t) \le 0$ $(x \in \Omega, t \in \mathbb{R})$ without loss of generality, since only a shift in the spectrum is involved. By a solution of equation (2.4), or related equations, we shall mean a function $u \in C(\Omega \times \mathbb{R})$ which is continuously differentiable in the second variable with u_t continuous on $\Omega \times \mathbb{R}$.

Set

$$\tilde{E} = \{u \in C(\Omega \times \mathbb{R}) | u(x, t+T) = u(x,t)\},$$

equipped with the sup norm. Let $\tilde{H}, \tilde{X} : \tilde{E} \to \tilde{E}$ be the linear operators defined as follows:

$$(\tilde{H}u)(x,t) = -u_t(x,t) + h(x,t)u(x,t),$$

with domain

$$\mathcal{D}(\tilde{H}) = \{u \in \tilde{H} | u \text{ is } C^1 \text{ in } t \text{ and } u_t \in \tilde{E}\},$$

and

$$(\tilde{X}u)(x,t) = \int_\Omega K(x,y)u(y,t)\,dy.$$

The governing equation (2.4) restricted to $\tilde{E}$ becomes, in this notation,

$$(\tilde{H} + \tilde{X})u = 0, \tag{3.1}$$

and our object is to study the spectrum of $(\tilde{H} + \tilde{X})$. Denote by $\rho(\tilde{X}, \tilde{H})$, $\sigma(\tilde{X}, \tilde{H})$ its *resolvent set* and *spectrum* respectively: $s(\tilde{X}, \tilde{H}) = \sup_{\mu \in \sigma(\tilde{X}, \tilde{H})} \Re(\mu)$ will be called the *principal spectrum point* ($s(\tilde{X}, \tilde{H})$ is defined to be $-\infty$ if $\sigma(\tilde{X}, \tilde{H}) = \emptyset$). As usual, $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of $(\tilde{H} + \tilde{X})$ if there is a non-trivial solution $\phi \in \tilde{E}$, an *eigenfunction*, of the equation

$$(\tilde{H} + \tilde{X})\phi = \lambda\phi. \tag{3.2}$$

An eigenvalue λ is called the *principal eigenvalue* (PEV) if there is exactly one corresponding *principal eigenfunction* ϕ, with $\phi(x,t) \ge 0$ $(x \in \Omega, t \in \mathbb{R})$, and the inequality $\Re(\mu) \le \lambda$ $(\mu \in \sigma(\tilde{X}, \tilde{H}))$ holds. Obviously $\lambda = s(\tilde{X}, \tilde{H})$.

We use $\rho(\tilde{H})$, $\sigma(\tilde{H})$, and $s(\tilde{H})$ for $\rho(\tilde{X}, \tilde{H})$, $\sigma(\tilde{X}, \tilde{H})$, and $s(\tilde{X}, \tilde{H})$ respectively when $K(x,y) \equiv 0$.

The following additional conditions are imposed.

(H2) (a) $K(x,y) > 0$ $(x, y \in \Omega)$.

(b) For each x and t $(x \in \Omega,\ t \in \mathbb{R})$, $h(x,t) \le 0$ and $h(x,t) = h(x, t+T)$.

Theorem 3.1. *Assume (H1) and (H2) and take $N = 1$. Then $\lambda = s(\tilde{X}, \tilde{H})$ is the PEV of (3.2) and is an isolated point of $\sigma(\tilde{X}, \tilde{H})$. Furthermore, $\phi(x,t) > 0$ $(x \in \Omega, t \in \mathbb{R})$.*

To prove this result, several preliminary lemmas are needed. The basic approach is to use the idea that the linear operator $\tilde{H} = -\frac{\partial}{\partial t} + h(x,t)$ itself generates a nice semigroup on $\tilde{E}$; this is related to an approach used in [3].

Lemma 3.2. *$\tilde{H}$ generates a positive continuous semigroup of contractions on $\tilde{E}$. $\tilde{H}$ is closed with dense domain.*

Proof. Let $\phi(s) : \tilde{E} \to \tilde{E}$ $(s \in \mathbb{R})$ be defined by

$$(\phi(s)u)(x,t) = \exp\left\{\int_{t-s}^{t} h(x,\xi)d\xi\right\} u(x,t-s)$$

and let $U(s,t,x;u)$ be the solution of

$$\frac{\partial U}{\partial s} = -\frac{\partial U}{\partial t} + h(x,t)U$$

with $U(0,t,x;u) = u(x,t) \in \tilde{E}$.

Then by direct computation, we have

$$\{u \in \tilde{E} | \lim_{t\to 0+} \frac{\phi(s)u - u}{s} \quad \text{exists}\} = \mathcal{D}(\tilde{H})$$

and

$$U(s,t,x;u) = (\phi(s)u)(x,t)$$

for any $u \in \mathcal{D}(\tilde{H})$. Hence $\{\phi(s)\}_{s\in\mathbb{R}^+}$ is a continuous semigroup on $\tilde{E}$ with generator $\tilde{H}$.

By the definition of $\phi(s)$, for any $u \in \tilde{E}$ with $u(x,t) \geq 0$, $\phi(s)u \geq 0$ for any $s \geq 0$. Moreover, since $h(x,t) \leq 0$, we have

$$\|\phi(s)u\|_{\tilde{E}} \leq \|u\|_{\tilde{E}}$$

for $s \geq 0$. Therefore, $\{\phi(s)\}_{s\in\mathbb{R}^+}$ is a positive continuous semigroup of contractions on $\tilde{E}$ with generator $\tilde{H}$. It follows from [23, Chapter 1, Corollary 2.5] that $\tilde{H}$ is closed with dense domain. $\square$

Lemma 3.3. *$\tilde{X} : \tilde{F} \to \tilde{E}$ is positive and compact, where $\tilde{F} = \mathcal{D}(\tilde{H})$ with the graph norm.*

Proof. ¿From (H2)(a), the positivity is obvious. Let $\{u_n\}$ be a sequence in the unit ball of $\tilde{F}$ and let

$$v_n(x,t) = \int_{\Omega} K(x,y)u_n(y,t)\,dy.$$

Then $|\frac{\partial u_n}{\partial t}(x,t)| \leq 1$ $(n \geq 1, x \in \Omega, t \in \mathbb{R})$, and there is a constant $M > 0$ such that $|\frac{\partial v_n}{\partial t}(x,t)| \leq M$ $(n \geq 1, x \in \Omega, t \in \mathbb{R})$. Also, from the uniform continuity of K, given $\epsilon > 0$ there is a $\delta > 0$ such that

$$|v_n(x_1,t) - v_n(x_2,t)| < \epsilon \quad (x_1, x_2 \in \Omega,\ |x_1 - x_2| < \delta,\ n \geq 1,\ t \in \mathbb{R}).$$

It follows that the sequence $\{v_n\}$ is equicontinuous and the compactness then follows from the Arzela-Ascoli theorem. $\square$

To investigate the spectrum of $\tilde{X} + \tilde{H}$, take

$$\bar{E} = \{u \in C(\mathbb{R}) | u(t+T) = u(t)\},$$

and let $C(\mathbb{R})$ have the sup norm. For given $x_0 \in \Omega$ let $\bar{H}(x_0)$ be the linear operator on $\bar{E}$ defined by

$$(\bar{H}(x_0)u)(t) = -\frac{du}{dt}(t) + h(x_0, t)u(t)$$

with $\mathcal{D}(\bar{H})$ the set $C^1(\mathbb{R}) \subset \bar{E}$ of functions with continuous first derivatives. Denote by $\rho(\bar{H}(x_0))$ and $\sigma(\bar{H}(x_0))$ the resolvent set and spectrum respectively of $\bar{H}(x_0)$. Define

$$\lambda(x_0) = \frac{1}{T}\int_0^T h(x_0, s)\, ds,$$

and note that by (H2)(b), $\lambda(x_0) \leq 0$.

Lemma 3.4. *(1) For fixed $x_0 \in \Omega$ and $\lambda \in \mathbb{R}$, $\bar{H}(x_0)u - \lambda u = 0$ has a non-trivial solution $u \in \bar{E}$ if and only if $\lambda = \lambda(x_0)$.*

Choose any $\delta > 0$. Then there are constants $M_1, M_2 > 0$ such that the following hold for any $x_0 \in \Omega$.

(2)

$$\|(\bar{H}(x_0) - \lambda)^{-1}\| \geq \frac{M_1}{|\lambda - \lambda(x_0)|} \tag{3.3}$$

for $\lambda \in \mathbb{R}$ with $0 < |\lambda - \lambda(x_0)| \leq \delta$.

(3)

$$\|(\bar{H}(x_0) - \lambda)^{-1}\| \leq \frac{M_2}{|\Re(\lambda) - \lambda(x_0)|} \tag{3.4}$$

for $\lambda \in \mathbb{C}$ with $0 < |\Re(\lambda) - \lambda(x_0)| \leq \delta$.

Proof. (1) This follows from the Floquet theory for periodic ordinary differential equations.

In preparation for the proof of (2) and (3), we first note that by the Fredholm alternative (see Lemma 1.1 and Theorem 1.1 in Chapter IV of [13]), for any $x_0 \in \Omega$ and $\lambda \in \mathbb{C}$ with $\Re(\lambda) \neq \lambda(x_0)$, and for any $v \in \bar{E}$, the equation

$$[\bar{H}(x_0) - \lambda]u = v$$

has a unique solution $u \in \bar{E}$. Denote it by $[\bar{H}(x_0) - \lambda]^{-1}v$. We show that $[\bar{H}(x_0) - \lambda]^{-1}v = u(\cdot; v)$, where $u(\cdot; v)$ is defined by

$$u(t; v) = -\int_{-\infty}^{t} \exp\left\{\int_s^t [h(x_0, \tau) - \lambda]\, d\tau\right\} v(s)\, ds \quad \text{if} \quad \Re(\lambda) > \lambda(x_0) \tag{3.5}$$

and

$$u(t;v) = \int_t^\infty \exp\left\{-\int_t^s [h(x_0,\tau)-\lambda]\,d\tau\right\} v(s)\,ds \quad \text{if} \quad \Re(\lambda) < \lambda(x_0). \tag{3.6}$$

By direct computation, we have that $u(t;v)$ is a solution of

$$[\bar{H}(x_0)-\lambda]u = v.$$

We claim that $u(\cdot;v) \in \bar{E}$, i.e. $u(t+T;v) = u(t;v)$. We prove the claim for the case $\Re(\lambda) > \lambda(x_0)$. It can be proved similarly for the case $\Re(\lambda) < \lambda(x_0)$. By (3.5),

$$\begin{aligned} u(t+T;v) &= -\int_{-\infty}^{t+T} \exp\left\{\int_s^{t+T} [h(x_0,\tau)-\lambda]\,d\tau\right\} v(s)\,ds \\ &= -\int_{-\infty}^{t} \exp\left\{\int_{s+T}^{t+T} [h(x_0,\tau)-\lambda]\,d\tau\right\} v(s+T)\,ds \\ &= -\int_{-\infty}^{t} \exp\left\{\int_s^{t} [h(x_0,\tau)-\lambda]\,d\tau\right\} v(s)\,ds \\ &= u(t;v) \end{aligned}$$

Hence $u(\cdot;v) \in \bar{E}$. It then follows from the uniqueness of the solutions of $[\bar{H}(x_0)-\lambda]u = v$, $[\bar{H}(x_0)-\lambda]^{-1}v = u(t;v)$.

(2) We next prove (3.3) for the case $\lambda > \lambda(x_0)$; the case $\lambda < \lambda(x_0)$ may be proved similarly. Note that

$$\|[\bar{H}(x_0)-\lambda]^{-1}\| = \sup_{v\in\bar{E},\|v\|=1} \|[\bar{H}(x_0)-\lambda]^{-1}v\|.$$

By (3.5), since the exponential is positive, we have

$$\|[\bar{H}(x_0)-\lambda]^{-1}\| = \|[\bar{H}(x_0)-\lambda]^{-1}v^*\|,$$

where $v^*(t) \equiv 1$ and

$$[\bar{H}(x_0)-\lambda]^{-1}v^* = -\int_{-\infty}^{t} \exp\left\{\int_s^t [h(x_0,\tau)-\lambda]\,d\tau\right\} ds.$$

Let n_s be the largest integer less than or equal to $\frac{t-s}{T}$. Then

$$\left|\int_{-\infty}^{t} \exp\left\{\int_s^t [h(x_0,\tau)-\lambda]\,d\tau\right\} ds\right| =$$
$$\int_{-\infty}^{t} \exp\left\{\int_s^{t-n_sT} [h(x_0,\tau)-\lambda]\,d\tau\right\} \exp\left\{\int_{t-n_sT}^{t} [h(x_0,\tau)-\lambda]\,d\tau\right\} ds.$$

Hence

$$\begin{aligned}
\left|\int_{-\infty}^{t} \exp\left\{\int_{s}^{t}[h(x_0,\tau)-\lambda]\,d\tau\right\} ds\right| &\geq M_1 \int_{-\infty}^{t} \exp\left\{\int_{t-n_sT}^{t}[h(x_0,\tau)-\lambda]\,d\tau\right\} ds \\
&= M_1 \int_{-\infty}^{t} \exp\left\{[\lambda(x_0)-\lambda]n_sT\right\} ds \\
&\geq M_1 \int_{-\infty}^{t} \exp\left\{[\lambda(x_0)-\lambda]\left(\frac{t-s}{T}\right)T\right\} ds \\
&= \frac{M_1}{|\lambda-\lambda(x_0)|},
\end{aligned}$$

where

$$M_1 = \inf_{t-n_sT\geq s\geq t-(n_s+1)T, x_0\in\Omega, 0<|\lambda-\lambda(x_0)|<\delta} \left(\exp\left\{\int_{s}^{t-n_sT}[h(x_0,\tau)-\lambda]\,d\tau\right\}\right).$$

The length of the integration range is less than T, and since the integrand is independent of s, $M_1 > 0$ and (3.3) follows.

(3) We note that from (3.5)

$$\|[\bar{H}(x_0)-\lambda]^{-1}\| \leq \|[\bar{H}(x_0)-\Re(\lambda)]^{-1}v^*\|$$

where $v^*(t) \equiv 1$. A very similar argument yields (3.4) and we omit the details.

□

Lemma 3.5. *(1)* $\{\lambda(x_0)|x_0\in\Omega\} \subset \sigma(\tilde{H})$.

(2) $\mathbb{C}\setminus\{\lambda|\inf_{x_0\in\Omega}\lambda(x_0)\leq\Re(\lambda)\leq\sup_{x_0\in\Omega}\lambda(x_0)\}\subset\rho(\tilde{H})$.

Proof. (1) Given $\lambda=\lambda(x_0)$ for some $x_0\in\Omega$, if $\lambda\in\rho(\tilde{H})$, then for any $\bar{v}\in\bar{E}$,

$$-\frac{du}{dt}+[h(x,t)-\lambda(x_0)]u=v$$

has a unique solution $u\in\tilde{E}$, where $v(x,t)=\bar{v}(t)$. This implies that for any $\bar{v}\in\bar{E}$,

$$-\frac{du}{dt}+[h(x_0,t)-\lambda(x_0)]u=\bar{v}$$

has a solution $u\in\bar{E}$. Then by the Fredholm alternative (see Lemma 1.1 and Theorem 1.1 in Chapter IV of [13]), $\bar{H}(x_0)u-\lambda(x_0)u=0$ has no nontrivial solution in $\bar{E}$, which contradicts Lemma 3.4 (1). Hence $\lambda\in\sigma(\tilde{H})$.

(2) Take any $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \sup_{x_0 \in \Omega} \lambda(x_0)$ or $\Re(\lambda) < \inf_{x_0 \in \Omega} \lambda(x_0)$. By Lemma 3.4(3), $\lambda \in \rho(\bar{H})$. Also for each $x_0 \in \Omega$,

$$([\tilde{H} - \lambda]^{-1} u)(x_0, t) = ([\bar{H}(x_0) - \lambda]^{-1} u)(x_0, t).$$

Hence also $\lambda \in \rho(\tilde{H})$.

□

Lemma 3.6. *There is an $\alpha > s(\tilde{H})$ such that*

$$r(\tilde{X}(\tilde{H} - \alpha)^{-1}) > 1,$$

where $r(\cdot)$ denotes the spectral radius.

Proof. First note that $\lambda(x_0)$ is continuous in x_0, and it follows from Lemma 3.5 that there is an $x_0 \in \Omega$ such that $\lambda(x_0) = s(\tilde{H})$. By Lemma 3.4, $[\bar{H}(x) - \alpha]^{-1}$ exists for $\alpha > \lambda(x_0)$, and there is an $M_1 > 0$ such that if $\alpha \neq \lambda(x)$ then

$$\|[\bar{H}(x) - \alpha]^{-1}\| \geq \frac{M_1}{|\alpha - \lambda(x)|} \tag{3.7}$$

for all $x \in \Omega$ and α in a neighbourhood of $\lambda(x)$. Note also that

$$\lambda(x) = \frac{1}{T} \int_0^T h(x, s) ds. \tag{3.8}$$

¿From (H1)(c), there is an $M_3 > 0$ such that

$$|h(x, s) - h(x_0, s)| \leq M_3 |x - x_0| \quad (x \in \Omega, s \in \mathbb{R}).$$

Therefore

$$\begin{aligned} |\lambda(x) - \lambda(x_0)| &\leq \frac{1}{T} \int_0^T |h(x, s) - h(x_0, s)| ds \\ &\leq M_3 |x - x_0|. \end{aligned} \tag{3.9}$$

From (3.7) and (3.9)

$$\begin{aligned} \|(\bar{H}(x) - \alpha)^{-1}\| &\geq \frac{M_1}{|\alpha - \lambda(x)|} \\ &\geq \frac{M_1}{|\alpha - \lambda(x_0)| + M_3 |x - x_0|}. \end{aligned} \tag{3.10}$$

Noting that by (H2)(a), $\min_{(x,y) \in \Omega \times \Omega} K(x, y) > 0$, we deduce that for $u(x, t) \equiv 1$, there is a constant $M_4 > 0$ such that

$$\begin{aligned} |(\tilde{X}(\tilde{H} - \alpha)^{-1} u)(x, t)| &= \left| \int_\Omega K(x, y)((\tilde{H} - \alpha)^{-1} u)(y, t)\, dy \right|, \\ &\geq M_4 \int_\Omega \frac{dy}{|\alpha - \lambda(x_0)| + M_3 |y - x_0|} \end{aligned}$$

from (3.10). The right hand side of this inequality tends to infinity as $\alpha \to \lambda(x_0)$. This completes the proof. □

The proof of the existence of the PEV below is based on [1, Theorem 2.2 and Remark 2.1], and for convenience we give the part of this result needed here translated into the current notation. Let

$$r_\alpha = r(\tilde{X}(\tilde{H} - \alpha)^{-1}).$$

Theorem 3.7. *Assume that*

(1) $\tilde{X}$ is positive and bounded.

(2) $\tilde{H}$ is closed with dense domain and generates a positive continuous semigroup of contractions.

(3) $\tilde{X} : \tilde{F} \to \tilde{E}$ is compact.

Then if $r_\alpha > 1$ for some $\alpha > s(\tilde{H})$, there is a unique $\alpha_0(> s(\tilde{H}))$ with $r_{\alpha_0} = 1$ and $\alpha_0 = s(\tilde{X}, \tilde{H})$. Further, α_0 is an isolated eigenvalue of $(\tilde{X} + \tilde{H})$ of finite multiplicity with a positive eigenfunction.

Proof of Theorem 3.1. Under the assumptions of Theorem 3.1, (1)-(3) and the condition on r_α of Theorem 3.7 are satisfied. In fact, (1) follows from (H1) and (2) is then a consequence of Lemma 3.2. The compactness (3) is given by Lemma 3.3 and the condition on r_α is Lemma 3.6. The assertions of Theorem 3.1 then follow except for the claims that the eigenfunction, ϕ say, is strictly positive and unique.

For the positivity, note first that for every t, there is an x_0 such that $\phi(x_0, t) > 0$. For otherwise, for some t_0, $\phi(x, t_0) = 0$ $(x \in \Omega)$, and by uniqueness for the initial value problem, $\phi(x, t) = 0$ $(x \in \Omega, t \in \mathbb{R})$, in which case ϕ is not an eigenfunction. It follows from Proposition 2.4(2) with $p(x, t) = h(x, y) - \lambda$ that $\phi(x, t) > 0$ $(x \in \Omega, t \in \mathbb{R})$.

The uniqueness is proved by a contradiction argument: suppose there is another eigenfunction ψ. Then one can choose $a \in \mathbb{R}$ with $a \neq 0$ such that $\omega = \phi - a\psi$ and

$$\omega(x, t) \geq 0 \ (x \in \Omega, t \in \mathbb{R}) \quad \text{and} \quad \omega(x_0, t_0) = 0$$

for some $x_0 \in \Omega,\ t_0 \in \mathbb{R}$. But this contradicts the conclusion of the previous paragraph and so yields uniqueness.

□

Remark 3.8. *If $N = 2$, a smoothness condition on h is not quite enough. However, we may prove the following by a slight extension of the above argument. Let Ω have a uniform cone property: there exist $a, b > 0$ such that for any $x \in \Omega$, there is a right circular cone V_x with vertex x, opening a, height b, such that $V_x \in \Omega$. Assume that $h(\cdot, t) \in C^1(\Omega)$ and $h_x(\cdot, t)$, the partial derivative of h with respect to x, is uniformly Lipschitz. Note that $\lambda(x_0) = \max_{x \in \Omega} \lambda(x)$, where $\lambda(x)$ is defined by (3.8).*

Then the conclusions of Theorem 3.1 hold if $x_0 \in \mathrm{Int}(\Omega)$.

In order to discuss the dimension issue further, let us rewrite the governing equation (2.4) slightly, by replacing X by ρX, where X is fixed and $\rho > 0$ is a parameter, obtaining

$$u_t = \rho X u + H u. \tag{3.11}$$

Here ρ is a dispersal rate, analogous to the diffusion rate for the corresponding reaction-diffusion case. To show that a PEV may *not* exist, let us consider the special case where h is independent of t, that is the stationary case, and the kernel $K \equiv 1$. It is then straightforward to show explicitly, by constructing a counter-example, that in general a PEV does not exist for small $\rho > 0$, even in the following cases.

(1) $N = 1$ and h is continuous.

(2) $N = 2$ and h satisfies the conditions of Remark 3.8 except for the restriction $x_0 \in \mathrm{Int}(\Omega)$.

(3) $N \geq 3$, Ω is the closed unit ball and $h = -r^2$ where r is the distance from the centre of Ω.

In each of these, a further condition is needed, and similar remarks of course broadly apply for general K. This condition is that the dispersal rate ρ in (3.11) is large enough. It is not apparent what the implications are in applications, for example in biology. This issue raises interesting questions about the invasion of species, and further investigation is warranted. For information on invasion and its relation to the PEV for classical diffusion see [2, p. 220].

The PEV λ and PEF are useful in providing estimates of rates of growth, and for the application of sub/super solution methods. The maximum growth rate for the initial value problem is indeed measured by $\lambda_L (= \lambda_s)$, the principal Lyapunov exponent. It may be shown that $\lambda_L = \lambda$, and of course $\lambda = s(\tilde{X}, \tilde{H})$, the principal spectrum point. However, as remarked above, in the dispersal case (as opposed to the case of classical diffusion) a PEV may not exist. Nonetheless we shall show that $\lambda_L = s(\tilde{X}, \tilde{H})$, thus providing a partial analogy.

Theorem 3.9. *Assume that (H1) and (H2) hold. Then* $\lambda_s(X, H) = \lambda_L(X, H) = s(\tilde{X}, \tilde{H})$.

To show this result, we first show

Lemma 3.10. $s(\tilde{X}, \tilde{H}) \geq s(\tilde{H})$.

Proof. We prove the lemma by contradiction. Assume that $s(\tilde{X}, \tilde{H}) < s(\tilde{H})$. Let $\lambda_0 = s(\tilde{H})$. Then $\lambda_0 I - (\tilde{X} + \tilde{H})$ is invertible and

$$\begin{aligned} \lambda_0 I - \tilde{H} &= \lambda_0 I - (\tilde{X} + \tilde{H}) + \tilde{X} \\ &= \Big(\lambda_0 I - (\tilde{X} + \tilde{H})\Big)\Big(I + \big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1} \tilde{X}\Big). \end{aligned}$$

By Lemma 3.5, $s(\tilde{H}) \in \sigma(\tilde{H})$. We then must have $-1 \in \sigma\Big(\big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1} \tilde{X}\Big)$. By Lemma 3.3, $\tilde{X}$ is compact. Hence $\big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1} \tilde{X}$ is compact and -1 is then an isolated eigenvalue of $\big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1} \tilde{X}$.

Let $u_0 \in \tilde{E}$ be a nontrivial solution of

$$\Big(I + \big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1} \tilde{X}\Big) u_0 = 0. \tag{3.12}$$

It follows that

$$(\lambda_0 I - \tilde{H})u_0 = 0,$$

i.e.

$$\frac{\partial u_0(x,t)}{\partial t} = (h(x,t) - \lambda_0)u_0(x,t). \tag{3.13}$$

This implies that

$$u_0(x,t) = u_0(x,0)e^{\int_0^t (h(x,\tau)-\lambda_0)d\tau}$$

for every $x \in \Omega$.

Let $v_0(x,t) = |u_0(x,t)|$. Clearly $v_0(x,t)$ is also a nontrivial solution of (3.13) and thus is a nontrivial solution of (3.12).

Note that $\tilde{X}$ is positive. By [10, Theorem 1.1], $(\lambda_0 I - (\tilde{X} + \tilde{H}))^{-1}$ is positive. Hence $(\lambda_0 I - (\tilde{X} + \tilde{H}))^{-1}\tilde{X}$ is positive. Therefore

$$0 = \Big(I + \big(\lambda_0 I - (\tilde{X} + \tilde{H})\big)^{-1}\tilde{X}\Big)v_0 \geq v_0 \geq 0$$

This implies that $v_0 \equiv 0$ and hence $u_0 \equiv 0$. This is a contradiction. The lemma then follows. □

Proof of Theorem 3.9. To clarify the proof, we shall slightly contract the notation and put $\lambda_s = \lambda_s(X,H)$, $\lambda_L = \lambda_L(X,H)$ and $s = s(\tilde{X},\tilde{H})$. From Proposition 2.5, $\lambda_s = \lambda_L$.

We claim that $s \leq \lambda_s$. For any λ with $\Re(\lambda) > \lambda_s$,

$$\|\Phi_\lambda(t,s)\| = \|e^{-\lambda(t-s)}\Phi(t,s)\| \to 0$$

as $t - s \to \infty$ exponentially. It then follows that for any $v \in C(\Omega \times \mathbb{R})$ with $v(x,t+T) = v(x,t)$,

$$u(x,t) = -\int_{-\infty}^{t} \Phi_\lambda(t,s)v(x,s)\,ds$$

is the unique periodic solution of (3.1) with period T. Therefore $\lambda \in \rho(\tilde{X},\tilde{H})$, and the claim follows.

We next prove that for any $\epsilon > 0$, $\lambda_L \leq s + \epsilon$. Consider the equation

$$-u_t + Xu + Hu - \lambda_\epsilon u = v, \tag{3.14}$$

where $\lambda_\epsilon = s + \epsilon$ and $v \in \tilde{E}$. Since $\lambda_\epsilon \in \rho(\tilde{X},\tilde{H})$, equation (3.14) has exactly one solution $u \in \tilde{E}$. Rewrite (3.14) as

$$\left(X - \left[\frac{\partial}{\partial t} - H + \lambda_\epsilon\right]\right) u = v \tag{3.15}$$

By Lemma 3.2, $\tilde{H} = -\frac{\partial}{\partial t} + H$ generates a positive continuous semigroup of contractions on $\tilde{E}$. By Lemma 3.10, $\lambda_\epsilon \in \rho(\tilde{H})$. Hence $\lambda_\epsilon - \tilde{H} = \frac{\partial}{\partial t} - H + \lambda_\epsilon$ is invertible and

$(\frac{\partial}{\partial t} - H + \lambda_\epsilon)^{-1}$ and $X(\frac{\partial}{\partial t} - H + \lambda_\epsilon)^{-1}$ are positive. Equation (3.14) can then be rewritten as

$$\left(X\left[\frac{\partial}{\partial t} - H + \lambda_\epsilon\right]^{-1} - I\right)\left(\frac{\partial}{\partial t} - H + \lambda_\epsilon\right) u = v \tag{3.16}$$

¿From [1, Theorem 2.2(ii)] if $\alpha = s$,

$$r\left(X\left[\frac{\partial}{\partial t} - H + \alpha\right]^{-1}\right) = 1.$$

By Lemma 3 of [1], $r(\cdot)$ above is a strictly decreasing continuous function of α. Hence,

$$r\left(X\left[\frac{\partial}{\partial t} - H + \lambda_\epsilon\right]^{-1}\right) < 1.$$

Therefore, by [21, Proposition 4.1.1], $I - X(\frac{\partial}{\partial t} - H + \lambda_\epsilon)^{-1}$ is invertible and has a positive inverse. Now (3.14) can be rewritten as

$$u = \left(\frac{\partial}{\partial t} - H + \lambda_\epsilon\right)^{-1}\left(X\left[\frac{\partial}{\partial t} - H + \lambda_\epsilon\right]^{-1} - I\right)^{-1} v. \tag{3.17}$$

By the positivity of $(\frac{\partial}{\partial t} - H + \lambda_\epsilon)^{-1}$ and $[I - X(\frac{\partial}{\partial t} - H + \lambda_\epsilon)^{-1}]^{-1}$, if $v \le 0$, then $u \ge 0$.

Take $v^*(x,t) \equiv -1$, and let u^* be the (unique) solution of (3.14), given by (3.17) in $\tilde{E}$ with $v^* = v$. Clearly $u^* \not\equiv 0$, and by the conclusion of the last paragraph $u^* \ge 0$, so there are an s and an x such that $u^*(x,s) > 0$. Since u^* satisfies

$$\begin{aligned} u^*_t &= Xu^* + (H - \lambda_\epsilon)u^* + 1 \\ &> Xu^* + (H - \lambda_\epsilon)u^*, \end{aligned}$$

by Proposition 2.4(2) with $p(x,t) = h(x,t) - \lambda$, $u^* > 0$. From periodicity and the compactness of Ω, there exists $\delta > 0$ such that $u^*(x,t) \ge \delta$ $(x \in \Omega\ , t \in \mathbb{R})$.

Fix some $s \in \mathbb{R}$ and define

$$\begin{aligned} \theta(x,t) &= \Phi(t,s)u^*(x,s), \\ \omega(x,t) &= e^{\lambda_\epsilon(t-s)}u^*(x,t). \end{aligned}$$

Simple calculations show that θ is the solution of

$$\theta_t - X\theta - H\theta = 0 \tag{3.18}$$

with $\theta(x,s) = u^*(x,s)$, and ω is the solution of

$$\omega_t - X\omega - H\omega = e^{\lambda_\epsilon(t-s)} \tag{3.19}$$

with $\omega(x,s) = u^*(x,s)$. Therefore $(\omega - \theta)(x,s) = 0$. Also, subtracting (3.18) from (3.19), we see that by Proposition 2.4(1) applied to $p = \omega - \theta$, $\theta(x,t) \le \omega(x,t)$, $(x \in \Omega,\ t \in \mathbb{R})$. Thus for all x, s, t with $s \le t$,

$$0 \le \Phi(t,s)u^*(x,s) \le e^{\lambda_\epsilon(t-s)}u^*(x,t). \tag{3.20}$$

Note next that for any $u_0 \in C(\Omega)$ with $\|u_0\| = 1$,

$$-\frac{u^*(x,t)}{\delta} \le u_0(x) \le \frac{u^*(x,s)}{\delta}.$$

Applying again Proposition 2.4 and using (3.20), we conclude that

$$\|\Phi(t,s)\| \le e^{\lambda_\epsilon(t-s)}\|u^*\|/\delta.$$

It follows from Definition 2.3 that

$$\lambda_L \le \lambda_\epsilon,$$

and since ϵ is arbitrary, $\lambda_L \le s$. Together with the opposite inequality, this proves the result. □

4 A Bound for the Principal Eigenvalue

Here it is assumed that a strictly positive PEF ϕ exists for the periodic case. We show that a lower bound for the PEV is the PEV for the stationary case obtained by taking the time average of h. Again it is assumed that this time-averaged problem does have a strictly positive PEF, ψ.

Theorem 4.1. *Assume that (H1) holds and that $h(x,t)$ is periodic in t. Define*

$$\hat{h}(x) = \frac{1}{T}\int_0^T h(x,t)\,dt \tag{4.1}$$

and let λ, λ^* and ϕ, ψ be the PEVs and PEFs for the original problem and the time-averaged autonomous problem respectively. That is

$$-\phi_t(x,t) + \int_\Omega K(x,y)\phi(y,t)\,dy + h(x,t)\phi(x,t) = \lambda\phi(x,t) \tag{4.2}$$

and

$$\int_\Omega K(x,y)\psi(y)\,dy + \hat{h}(x)\psi(x) = \lambda^*\psi(x). \tag{4.3}$$

Then $\lambda \ge \lambda^*$. Also, if $\lambda = \lambda^*$ and $K(x,y) > 0 \forall x, y \in \Omega$, then

$$h(x,t) = \hat{h}(x) + g(t).$$

The following corollary follows directly from Theorems 3.9 and 4.1.

Corollary 4.2. *If $K(x,y) > 0$ for $x, y \in \Omega$, then $\lambda_s(X,H) = s(\tilde{X},\tilde{H}) \geq s(X,\hat{H}) = \lambda^*$, where $\hat{H}u = \hat{h}u$.*

The proof of the theorem will depend upon a preliminary lemma, and the proof of that depends upon a Jensen inequality, viz. if f is a positive, continuous function defined on $[0,T]$ then,

$$\frac{1}{T}\int_0^T f(t)\,dt \geq \exp\left\{\frac{1}{T}\int_0^T \ln[f(t)]\,dt\right\} \tag{4.4}$$

with equality if and only if f is a constant function.

Lemma 4.3. *Let $w(x,t)$ be a positive, continuous function defined on $\Omega \times [0,T]$. Let*

$$\theta(x,y) = \frac{1}{T}\int_0^T \frac{w(y,t)}{w(x,t)}\,dt.$$

Then either *$w(x,t)$ is independent of x* or *there exists $x^* \in \Omega$ such that*

$$\theta(x^*, y) \geq 1 \;\forall\; y \in \Omega$$

with strict inequality for some y.

Proof. Let

$$\chi(x) = \exp\left(\frac{1}{T}\int_0^T \ln[w(x,t)]\,dt\right), \;\; (x \in \Omega)$$

then from the inequality (4.4)

$$\begin{aligned}
\theta(x,y) &= \frac{1}{T}\int_0^T \frac{w(y,t)}{w(x,t)}\,dt \\
&\geq \exp\left\{\frac{1}{T}\int_0^T \ln\frac{w(y,t)}{w(x,t)}\,dt\right\} \\
&= \exp\left\{\frac{1}{T}\int_0^T \ln[w(y,t)]\,dt - \frac{1}{T}\int_0^T \ln[w(x,t)]\,dt\right\} \\
&= \frac{\exp\left\{\frac{1}{T}\int_0^T \ln[w(y,t)]\,dt\right\}}{\exp\left\{\frac{1}{T}\int_0^T \ln[w(x,t)]\,dt\right\}} \\
&= \frac{\chi(y)}{\chi(x)}.
\end{aligned} \tag{4.5}$$

Now χ is a continuous function defined on the compact set Ω and so is bounded and attains its bounds. Let its least value occur at x_0. Then

$$\theta(x_0, y) \geq 1 \;\forall\; y \in \Omega.$$

If χ is not a constant then the inequality will be strict for some y and the required result (with $x^* = x_0$) is established. Otherwise χ is a constant and so

$$\theta(x, y) \geq 1 \; \forall \; (x, y) \in \Omega \times \Omega.$$

If this last inequality is somewhere strict, say at (x_1, y_1), then the theorem is proved with $x^* = x_1$. Suppose therefore that there is equality everywhere, i.e. $\theta(x, y) \equiv 1$. This implies that there is equality in (4.5) and so $w(y, t)/w(x, t)$ is independent of t. Let

$$w(0, t) = \gamma(t) \quad \text{and} \quad \frac{w(y, t)}{w(0, t)} = F(y),$$

then

$$w(y, t) = F(y)\gamma(t)$$

and so

$$1 = \frac{1}{T}\int_0^T \frac{w(y, t)}{w(x, t)}\, dt = \frac{1}{T}\int_0^T \frac{F(y)}{F(x)}\, dt = \frac{F(y)}{F(x)}.$$

Thus F is a constant and $w(x, t)$ depends only upon t. □

Proof of Theorem 4.1. ¿From equation (4.3)

$$\lambda^* = \hat{h}(x) + \frac{\int_\Omega K(x, y)\psi(y)\, dy}{\psi(x)} \quad \forall \; x \in \Omega$$

and from equation (4.2)

$$\lambda = h(x, t) + \frac{\int_\Omega K(x, y)\phi(y, t)\, dy}{\phi(x, t)} - \frac{\phi_t(x, t)}{\phi(x, t)} \quad \forall \; x, t. \tag{4.6}$$

When integrated over t, this last equation, because of equation (4.1) and ϕ being periodic in t, implies

$$\lambda = \hat{h}(x) + \frac{1}{T}\int_\Omega K(x, y)\int_0^T \frac{\phi(y, t)}{\phi(x, t)}\, dt\, dy \quad \forall \; x \in \Omega$$

and so

$$\lambda^* - \lambda = \int_\Omega K(x, y)\left\{\frac{\psi(y)}{\psi(x)} - \frac{1}{T}\int_0^T \frac{\phi(y, t)}{\phi(x, t)}\, dt\right\} dy \quad \forall \; x \in \Omega.$$

Set $w(y, t) = \phi(y, t)/\psi(y)$ to give

$$\lambda^* - \lambda = \int_\Omega K(x, y)\frac{\psi(y)}{\psi(x)}\left\{1 - \frac{1}{T}\int_0^T \frac{w(y, t)}{w(x, t)}\, dt\right\} dy \quad \forall \; x \in \Omega.$$

From the lemma, we know that there exists an x^* such that when $x = x^*$ the expression within $\{\}$ is non-positive for all y. Since $K(x, y)$ is non-negative and u is positive, it follows that $\lambda^* \leq \lambda$.

If K is strictly positive everywhere, then the lemma implies that either $w(y,t)$ varies with y (and hence $\lambda^* < \lambda$) or $w(y,t)$ depends only upon t (and hence $\lambda = \lambda^*$). In this latter case

$$\phi(y,t) = \psi(y)\gamma(t)$$

and so, from equation (4.6),

$$\begin{aligned} \lambda \quad &= \quad h(x,t) + \frac{\int_\Omega K(x,y)\psi(y)\,dy}{\psi(x)} - \frac{1}{\gamma}\frac{d\gamma}{dt} \\ = \lambda^* \quad &= \quad \hat{h}(x) + \frac{\int_\Omega K(x,y)\psi(y)\,dy}{\psi(x)} \\ \Longrightarrow h(x,t) \quad &= \quad \hat{h}(x) + \frac{1}{\gamma}\frac{d\gamma}{dt} \end{aligned}$$

and so h has the required form.

□

5 The Almost-Periodic Case

In this section, we consider (2.4) with $h(x,\cdot)$ being almost periodic. Our aim is to obtain a lower bound for the principal dynamic spectrum point λ_s, or equivalently the principal Lyapunov exponent. This provides the natural extension of the previous section from the periodic to the AP case. Define

$$\hat{h}(x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t h(x,s)ds. \tag{5.1}$$

Of course $\rho(X,\hat{H})$, $\sigma(X,\hat{H})$ and $s(X,\hat{H})$ are defined in the obvious way for the stationary problem on $C(\Omega)$ for the operator $(X+\hat{H})$, where X is the integral operator and $\hat{H}$ is multiplication by $\hat{h}$.

Theorem 5.1. *Suppose that (H1) and (H2) hold. Assume that a PEV λ of finite multiplicity exists for the stationary case with $\hat{h}$ defined by (5.1), and suppose moreover that λ is an isolated point of $\sigma(X,\hat{H})$. Then*

$$\lambda_L(X,H) = \lambda_s(X,H) \geq s(X,\hat{H}) = \lambda.$$

Proof. By the properties of almost-periodic functions (see [9]), there are periodic functions $h_n(x,t)$ such that

$$h_n(x,t) \to h(x,t) \quad \text{as} \quad n\to\infty$$

uniformly for $x\in\Omega$ and $t\in\mathbb{R}$. Then

$$\frac{1}{t}\int_0^t \big(h(x,s) - h_n(x,s)\big)ds \to 0 \quad \text{as} \quad n\to\infty$$

uniformly in $x \in \Omega$, that is,

$$\hat{h}_n(x) \to \hat{h}(x) \quad \text{as} \quad n \to \infty.$$

By Proposition 2.6,

$$\lambda_s(X, \tilde{H}_n) \to \lambda_s(X, H) \quad \text{as} \quad n \to \infty \tag{5.2}$$

and by Proposition 2.6 and Theorem 3.9,

$$s(X, \hat{H}_n) = \lambda_s(X, \hat{H}_n) \to \lambda_s(X, \hat{H}) = s(X, \hat{H}) \quad \text{as} \quad n \to \infty. \tag{5.3}$$

¿From the assumed condition on λ, by perturbation theory for the spectrum, see [20, IV Section 3.5], $\lambda_s(X, \hat{H}_n) = s(X, \hat{H}_n)$ is an isolated PEV of $(X + \hat{H}_n)$ for $n \gg 1$. Hence by Theorem 4.1,

$$\lambda_s(X, \tilde{H}_n) \geq \lambda_s(X, \hat{H}_n) \tag{5.4}$$

for $n \gg 1$. It then follows from (5.2)-(5.4) that

$$\lambda_s(X, H) \geq s(X, \hat{H}).$$

□

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