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Approximation of the heat kernel on a Riemannian manifold based on the Smolyanov–Weizsäcker approach

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Abstract

Let M be a compact Riemannian manifold without boundary isometrically embedded into $\mathbb{R}^m$, $\mathbb{W}^x_{M,t}$ be the distribution of a Brownian bridge starting at $x \in M$ and returning to M at time t. Let $Q_t : \mathrm{C}(M) \to \mathrm{C}(M)$, $(Q_t f)(x) = \int_{\mathrm{C}([0,1],\mathbb{R}^m)} f(\omega(t))\, \mathbb{W}^x_{M,t}(d\omega)$, and let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of $[0,t]$. It was shown in [2] that

$$Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} f \to e^{-t\,\frac{\Delta_M}{2}} f\ , \quad \text{as } |\mathcal{P}| \to 0, \tag{1}$$

in $\mathrm{C}(M)$. Taking into consideration integral representations: $(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} f)(x) = \int_M q_{\mathcal{P}}(x,y) f(y) \lambda_M(dy)$ and $(e^{-t\,\frac{\Delta_M}{2}} f)(x) = \int_M h(x,y,t)\, f(y)\, \lambda_M(dy)$, where λ_M is the volume measure on M, $h(x,y,t)$ is the heat kernel on M, one interprets relation (1) as a weak convergence in $\mathrm{C}(M)$ of the integral kernels:

$$q_{\mathcal{P}}(x,y) \to h(x,y,t). \tag{2}$$

The present paper improves the result of [2], and shows that convergence in (2) is uniform on $M \times M$.

Keywords: Gaussian integrals on compact Riemannian manifolds, heat kernel, Smolyanov–Weizsäcker approach, Smolyanov–Weizsäcker surface measures

1 Introduction

Let M be a compact Riemannian manifold without boundary isometrically embedded into $\mathbb{R}^m$, $\dim M = d$. Define

$$q(x,y,t) = \frac{e^{-\frac{|x-y|^2}{2t}}}{\int_M e^{-\frac{|x-\bar{y}|^2}{2t}} \lambda_M(d\bar{y})}$$

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where λ_M is the volume measure on M. It was shown in [2] that the following limit exists relative to the family of bounded continuous functions, and defines a probability measure on $\mathrm{C}([0,1],\mathbb{R}^m)$:

$$\int_{\mathrm{C}([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,t}(d\omega) = \lim_{\varepsilon\to 0} \frac{\int_{\pi_t^{-1}(U_\varepsilon(M))} f(\omega)\,\mathbb{W}^x(d\omega)}{\mathbb{W}^x(\pi_t^{-1}(U_\varepsilon(M))}$$

where $x \in M$, $t \in [0,1]$, $\mathbb{W}^x$ is the Wiener measure on $\mathrm{C}([0,1],\mathbb{R}^m)$, $U_\varepsilon(M)$ is the ε-neighborhood of M, π_t is the evaluation mapping $\mathrm{C}([0,1],\mathbb{R}^m) \to \mathbb{R}^m$, $\varphi \mapsto \varphi(t)$. The measure $\mathbb{W}^x_{M,t}$ is the distribution of a Brownian motion on $\mathbb{R}^m$ conditioned to return to M at time t (Brownian bridge). We introduce operators Q_t as defined in [2]. If f is a cylinder function satisfying the relation $f(\omega) = f\big(\pi_t^{-1}\big(\omega(t)\big)\big)$, and $g : \mathbb{R}^m \to \mathbb{R}$ is such that $f(\omega) = g(\omega(t))$, then

$$(Q_t\, g)(x) = \int_M g(y)\, q(x,y,t)\lambda_M(dy) = \int_{\mathrm{C}([0,1],\mathbb{R}^m)} f(\omega)\mathbb{W}^x_{M,t}(d\omega). \tag{3}$$

Let $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a partition of the interval $[0,t]$, and let

$$\begin{aligned} q_{\mathcal{P}}(x,y) = \int_M dx_1\, q(x,x_1,t_1) \int_M dx_2\, q(x_1,x_2,t_2-t_1)\cdots \\ \int_M dx_{n-1}\, q(x_{n-2},x_{n-1},t_{n-1}-t_{n-2})\, q(x_{n-1},y,t_n-t_{n-1}). \end{aligned} \tag{4}$$

Taking into account the representation (3), we obtain:

$$(Q_{t_1-t_0}\cdots Q_{t_n-t_{n-1}}\, g)(x) = \int_M q_{\mathcal{P}}(x,y)\, g(y)\, \lambda_M(dy).$$

Let $h(x,y,t)$, $x,y \in M$, $t \in \mathbb{R}$, denote the heat kernel on the manifold M. We have

$$(e^{-t\,\frac{\Delta_M}{2}} g)(x) = \int_M h(x,y,t)\, g(y)\, \lambda_M(dy).$$

The paper [2] states that

$$(Q_{t_1-t_0}\cdots Q_{t_n-t_{n-1}}\, g)(x) \to (e^{-t\,\frac{\Delta_M}{2}} g)(x)$$

uniformly in $x \in M$. Theorem 1 below improves this result of [2].

2 Main Theorem

Theorem 1. *Let the partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ satisfy the following condition: there exists an integer k such that* $\min\{t_i - t_{i-1}\} > |\mathcal{P}|^k$, *where $|\mathcal{P}|$ denotes the mesh of $\mathcal{P}$. Then, for all $t \in [0,1]$,*

$$\lim_{|\mathcal{P}|\to 0} q_{\mathcal{P}}(x,y) = h(x,y,t)$$

uniformly in $x,y \in M$.

For the proof of the theorem we will need a few lemmas. For $x, y \in M$, $t \in [0,1]$, we define

$$p(x,y,t) = \frac{1}{(2\pi t)^{\frac{d}{2}}}\, e^{-\frac{|x-y|^2}{2t}},$$
$$\mathcal{E}(x,y,t) = \frac{1}{(2\pi t)^{\frac{d}{2}}}\, e^{-\frac{d(x,y)^2}{2t}},$$

where $d(x,y)$ is the geodesic distance between x and y.

Lemma 1. *There exist bounded functions* $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 : M \times M \times [0,1] \to \mathbb{R}$ *such that*

$$q(x,y,t) = p(x,y,t)(1 + \Theta_1(x,y,t)\, t), \tag{5}$$

and for all $x, y \in M$ *satisfying* $|x-y| < t^\alpha$, *where* $\frac{1}{4} < \alpha < \frac{1}{2}$, *the following relations hold:*

$$p(x,y,t) = \mathcal{E}(x,y,t)(1 + \Theta_2(x,y,t)\, t^{4\alpha-1}), \tag{6}$$
$$\mathcal{E}(x,y,t) = h(x,y,t)(1 + \Theta_3(x,y,t)\, t^{2\alpha}), \tag{7}$$
$$q(x,y,t) = h(x,y,t)(1 + \Theta_4(x,y,t)\, t^{4\alpha-1}), \tag{8}$$
$$h(x,y,t) = p(x,y,t)(1 + \Theta_5(x,y,t)\, t^{4\alpha-1}). \tag{9}$$

Proof. The proof of relation (5) follows from the asymptotic expansion [2]:

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M e^{-\frac{|x-y|^2}{2t}} \lambda_M(dy) = 1 - t\left(\frac{1}{6}\,\mathrm{scal}(x) + \frac{1}{16}\Delta_M \Delta_M \left.|x - \cdot|^2\right|_x\right) + tR(t,x),$$

where $|R(t,y)| < Kt^{1/2}$, K is a constant, and $\mathrm{scal}(y)$ is the scalar curvature at the point y. To prove (6), notice that

$$|x-y|^2 = d(x,y)^2 + \theta(x,y)d(x,y)^4,$$

where θ is bounded on $M \times M$. Applying the Taylor expansion to $e^{-\frac{\theta(x,y)d(x,y)^4}{2t}}$, we can easily see the existence of a bounded function $\Theta_2 : M \times M \times [0,1] \to \mathbb{R}$ such that

$$e^{-\frac{\theta(x,y)d(x,y)^4}{2t}} = 1 + \Theta_2(x,y,t)\, t^{4\alpha-1}$$

for $x, y \in M$ satisfying $|x-y| < t^\alpha$. This proves relation (6). Relation (7) follows from the following representation of $h(x,y,t)$ for y in a neighborhood of x [1]:

$$h(x,y,t) = \mathcal{E}(x,y,t)\left(\sum_{i=0}^{k} u_i(x,y)t^i + O(t^{k+1})\right)$$

where $u_i : M \times M \to \mathbb{R}$ are continuous, $u_0(x,x) = 1$, and $\nabla_M u_0(x,x) = 0$. Applying Taylor expansion to $u_0(x,y)$ for $y \in M$ satisfying $|x-y| < t^\alpha$, we obtain (7). Relation (8) is a consequence of (5), (6), and (7) if we notice that for $\frac{1}{4} < \alpha < \frac{1}{2}$, $4\alpha - 1 < 2\alpha$. Relation (9) is an immediate corollary of (8). $\square$

Lemma 2. *Let $\frac{1}{4} < \alpha < \frac{1}{2}$. Then, there exist a bounded functions $R : M \times M \times [0,1] \times [0,1] \to \mathbb{R}$, and $\theta : M \times M \times 0,1] \times [0,1] \to M$ such that for all $x, z \in M$, and $t_1, t_2 \in [0,1]$,*

$$\int_M q(x,y,t_1)h(y,z,t_2)\lambda_M(dy) = h(x,z,t_1+t_2)(1+\Theta_4(x,\theta(x,z,t_1,t_2),t_1)\, t_1^{4\alpha-1}) + \frac{R(x,z,t_1,t_2)}{(2\pi t_1)^{\frac{d}{2}}}\, e^{-\frac{1}{t_1^{1-2\alpha}}}.$$

Proof. Let $U_{t_1}(x) = \{y \in M : |y-x| < t_1^{\alpha}\}$. Then

$$\int_{M\setminus U_{t_1}(x)} p(x,y,t_1)h(y,z,t_2)\lambda_M(dy) < \frac{1}{(2\pi t_1)^{\frac{d}{2}}}\, e^{-\frac{1}{2t_1^{1-2\alpha}}}, \tag{10}$$

$$\begin{aligned}&\int_{M\setminus U_{t_1}(x)} h(x,y,t_1)h(y,z,t_2)\lambda_M(dy)\\ =&\int_{M\setminus U_{t_1}(x)} p(x,y,t_1)h(y,z,t_2)(1+\Theta_5(x,y,t_1)t_1^{4\alpha-1})\lambda_M(dy) < \frac{K_1}{(2\pi t_1)^{\frac{d}{2}}}\, e^{-\frac{1}{2t_1^{1-2\alpha}}}\end{aligned}$$

where K_1 is a constant independent of x, z, t_1, and t_2. Inequality (10) and relation (5) imply the existence of a constant K_2 such that

$$\int_{M\setminus U_{t_1}(x)} q(x,y,t_1)h(y,z,t_2)\lambda_M(dy) < \frac{K_2}{(2\pi t_1)^{\frac{d}{2}}}\, e^{-\frac{1}{2t_1^{1-2\alpha}}}. \tag{11}$$

Further, using relation (8) of Lemma 1 and the two inequalities above, we obtain:

$$\begin{aligned}&\int_{U_{t_1}(x)} q(x,y,t_1)h(y,z,t_2)\lambda_M(dy)\\ =\;&\int_{U_{t_1}(x)} h(x,y,t_1)h(y,z,t_2)\lambda_M(dy)\\ &+t_1^{4\alpha-1}\int_{U_{t_1}(x)} h(x,y,t_1)h(y,z,t_2)\Theta_4(x,y,t_1)\lambda_M(dy)\\ =\;&\int_M h(x,y,t_1)h(y,z,t_2)\lambda_M(dy)\big(1+t_1^{4\alpha-1}\Theta_4(x,\theta(x,z,t_1,t_2),t_1)\big)\\ &+\frac{\bar{R}(x,z,t_1,t_2)}{(2\pi t_1)^{\frac{d}{2}}}\, e^{-\frac{1}{2t_1^{1-2\alpha}}},\end{aligned}$$

where $\bar{R} : M \times M \times [0,1] \times [0,1] \to \mathbb{R}$ is bounded. Applying inequality (11), we obtain:

$$\int_M q(x,y,t_1)h(y,z,t_2)\lambda_M(dy) = h(x,z,t_1+t_2)\big(1+t_1^{4\alpha-1}\Theta_4(x,\theta(x,z,t_1,t_2),t_1)\big) + \frac{R(x,z,t_1,t_2)}{(2\pi t_1)^{\frac{d}{2}}}\, e^{-\frac{1}{2t_1^{1-2\alpha}}},$$

where, again, $R : M \times M \times [0,1] \times [0,1] \to \mathbb{R}$ is bounded by $K = K_1 + K_2$. This proves the lemma. $\square$

We will again need the operators Q_s below, and so we recall their definition:

$$Q_s : \mathrm{C}(M) \to \mathrm{C}(M),\ f \mapsto \int_M q(\,\cdot\,, y, s) f(y) \lambda_M(dy).$$

LEMMA 3. *Let $\mathcal{P}$ be a partition of $[0,t]$ as above, and let $\tau = t_n - t_{n-1}$, the length of the last partition interval, be such that $\tau^{d+9} > |\mathcal{P} \setminus \{t_n\}|$. Then, as the mesh of $\mathcal{P}$ tends to zero,*

$$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} p\,(\,\cdot\,, y, \tau))(x) \to h(x,y,t), \tag{12}$$
$$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} h\,(\,\cdot\,, y, \tau))(x) \to h(x,y,t), \tag{13}$$

uniformly in $x, y \in M$.

Proof. Let y be fixed. From the paper [2], we have the following inequality:

$$\|(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} - e^{-\frac{t-\tau}{2}\Delta_M})\, p(\,\cdot\,, y, \tau)\| \leqslant K\, t\, \|p(\,\cdot\,, y, \tau)\|_4 \sqrt{|\mathcal{P} \setminus \{t_n\}|},$$

where the norm $\|\cdot\|_4$ is described in [2]. Note that

$$\|p(\,\cdot\,, y, \tau)\|_4 < \frac{\bar{K}}{\tau^{\frac{d}{2}+4}},$$

where $\bar{K}$ is a constant. Next, since we assumed that $|\mathcal{P} \setminus \{t_n\}| < \tau^{d+9}$, we obtain:

$$\|(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} - e^{-\frac{t-\tau}{2}\Delta_M})\, p(\,\cdot\,, y, \tau)\| < \tilde{K}\sqrt{\tau} \to 0, \quad |\mathcal{P}| \to 0$$

where $\tilde{K}$ is a constant. Further, note that

$$(e^{-\frac{t-\tau}{2}\Delta_M})\, p(\,\cdot\,, y, \tau)(x) = \int_M h(x,z,t-\tau)\, p(z,y,\tau) \lambda_M(dz).$$

Now, in the last integral, we apply the asymptotic expansion [2] relative to the small parameter τ to the function $h(x,z,t-\tau)$. We obtain:

$$\begin{aligned} & \int_M h(x,z,t-\tau)\, p(z,y,\tau) \lambda_M(dz) \\ = \; & h(x,y,t-\tau) - \frac{\tau}{2}\Delta_M h(x,y,t-\tau) \\ & + \tau h(x,y,t-\tau)\left(\frac{1}{6}\,\mathrm{scal}(x) + \frac{1}{16}\Delta_M \Delta_M \left.|x - \cdot\,|^2\right|_x\right) + \tau R(t\tau, x). \end{aligned}$$

Clearly, as $\tau \to 0$,

$$\int_M h(x,z,t-\tau) p(z,y,\tau) \lambda_M(dz) \to h(x,y,t)$$

uniformly in $x, y \in M$. This proves (12). Relation (9) shows that (13) also holds. $\square$

LEMMA 4. *Let $\lambda_i \in \mathbb{R}$ be such that $\sum_{i=1}^{k} \lambda_i = \tau$, and let $\tau^p < K \max\{\lambda_i\}$, $p > 1$, K a constant. Further, assume that there exists an integer $q > 1$ such that $\min\{\lambda_i\} > (\max\{\lambda_i\})^q$. Then there exists a sufficiently small number $x > 0$ such that $\sum_{i=1}^{k} \lambda_i^{1-x} \to 0$, as $\max\{\lambda_i\} \to 0$.*

Proof. We have

$$\lambda_1^{1-x} + \lambda_2^{1-x} + \cdots + \lambda_k^{1-x} \leqslant \frac{\tau}{(\max\{\lambda_i\})^{qx}} \leqslant K\, \tau^{1-pqx}.$$

Choosing $x < \frac{1}{pq}$ proves the lemma. □

Proof of Theorem 1. We have

$$q_{\mathcal{P}}(x,y) = (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}}\, q(\,\cdot\,, y, t_n - t_{n-1}))(x).$$

Applying relation (8), we obtain

$$q_{\mathcal{P}}(x,y) = \\ \Big(1 + (t_n - t_{n-1})\, \Theta_4(x_{\mathcal{P}}^{(n)}, y, t_n - t_{n-1})\Big)\ (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}}\, h\,(\,\cdot\,, y, t_n - t_{n-1}))(x), \tag{14}$$

where $x_{\mathcal{P}}^{(n)} \in M$ is a point on M depending on all points of the partition $\mathcal{P}$. Continuing transformations of the last term in (14), we obtain:

$$(Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}}\, h(\,\cdot\,, y, t_n - t_{n-1}))(x) = \\ \big(Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}} \int_M q(\,\cdot\,, y_{n-1}, t_{n-1} - t_{n-2})\, h(y_{n-1}, y, t_n - t_{n-1}) \lambda_M(dy_{n-1})\big)(x).$$

Applying Lemma 2, we obtain

$$\begin{aligned} &\big(Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}}\, h(\,\cdot\,, y, t_n - t_{n-1})\big)(x) \\ &= \big(Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}}\, h(\,\cdot\,, y, t_{n-2} - t_n) \\ &\quad \times (1 + (t_{n-1} - t_{n-2})^{4\alpha-1}\, \Theta_4(\,\cdot\,, y^{(n-1)}_{t_{n-1}t_{n-2}}, t_{n-1} - t_{n-2})\,\big)(x) \\ &+ \frac{R_{n-2}(x,y,\mathcal{P})}{(2\pi(t_{n-1} - t_{n-2}))^{\frac{d}{2}}}\, e^{-\frac{1}{2(t_{n-1}-t_{n-2})^{1-2\alpha}}}, \end{aligned}$$

where $R_{n-2}(x,y,\mathcal{P}) = \big(Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}}\, R(\,\cdot\,, y, t_{n-1}, t_{n-2})\big)(x)$, where the function $R(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is as described in Lemma 2. The function R_{n-2} is obviously bounded by the same constant K as the function R. Finally, applying the mean value theorem to the function Θ_4, we obtain

$$\begin{aligned} &\big(Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}}\, h(\,\cdot\,, y, t_n - t_{n-1})\big)(x) \\ &= \big(1 + (t_{n-1} - t_{n-2})^{4\alpha-1}\, \Theta_4(x_{\mathcal{P}}^{(n-1)}, y_{\mathcal{P}}^{(n-1)}, t_{n-1} - t_{n-2})\,\big) \\ &\times \big(Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}}\, h(\,\cdot\,, y, t_{n-2} - t_n)\big)(x) \\ &+ \frac{R_{n-2}(x,y,\mathcal{P})}{(2\pi(t_{n-1} - t_{n-2}))^{\frac{d}{2}}}\, e^{-\frac{1}{2(t_{n-1}-t_{n-2})^{1-2\alpha}}}, \end{aligned} \tag{15}$$

where $x_{\mathcal{P}}^{(n-1)}$ and $y_{\mathcal{P}}^{(n-1)}$ are points on the manifold M. Let N be the smallest number satisfying $\tau = t_n - t_{n-N} > |\mathcal{P}|^{\frac{1}{d+9}}$. Also, this implies that $\tau - (t_{n-N+1} - t_{n-N}) < |\mathcal{P}|^{\frac{1}{d+9}}$, and hence, $\tau < |\mathcal{P}|^{\frac{1}{d+9}} + |\mathcal{P}| < 2\,|\mathcal{P}|^{\frac{1}{d+9}}$. Repeating the argument used in (15) $N-2$ times, we obtain

$$
\begin{aligned}
q_{\mathcal{P}}(x,y) = \\
&\big(1 + (t_n - t_{n-1})\,\Theta_4(x_{\mathcal{P}}^{(n)}, y, t_n - t_{n-1})\big) \\
&\times\big(1 + (t_{n-1} - t_{n-2})^{4\alpha-1}\,\Theta_4(x_{\mathcal{P}}^{(n-1)}, y_{\mathcal{P}}^{(n-1)}, t_{n-1} - t_{n-2})\big)\cdots \\
&\times\big(1 + (t_{n-N+1} - t_{n-N})^{4\alpha-1}\,\Theta_4(x_{\mathcal{P}}^{(n-N+1)}, y_{\mathcal{P}}^{(n-N+1)}, t_{n-N+1} - t_{n-N})\big) \\
&\times\big(Q_{t_1-t_0}\cdots Q_{t_{n-N}-t_{n-N-1}}\,h(\,\cdot\,, y, t_n - t_{n-N})\big)(x) \\
&+\sum_{k=2}^{N}\frac{R_{n-k}(x,y,\mathcal{P})\prod_{j=1}^{k-2}(1 + (t_{n-j} - t_{n-j-1})^{4\alpha-1})}{(2\pi(t_{n-k+1} - t_{n-k}))^{\frac{d}{2}}}\,e^{-\frac{1}{2(t_{n-k+1}-t_{n-k})^{1-2\alpha}}},
\end{aligned}
$$

where all functions R_{n-k} are bounded by the same constant. Now we just have to prove that as $|\mathcal{P}| \to 0$,

$$
\big(Q_{t_1-t_0}\cdots Q_{t_{n-N}-t_{n-N-1}}h\,(\,\cdot\,, y, t_n - t_{n-N})\big)(x) \to h(x,y,t), \tag{16}
$$

$$
\begin{aligned}
&\big(1 + (t_n - t_{n-1})\,\Theta_4(x_{\mathcal{P}}^{(n)}, y, t_n - t_{n-1})\big) \\
\times&\big(1 + (t_{n-1} - t_{n-2})^{4\alpha-1}\,\Theta_4(x_{\mathcal{P}}^{(n-1)}, y_{\mathcal{P}}^{(n-1)}, t_{n-1} - t_{n-2})\big)\cdots \\
\times&\big(1 + (t_{n-N+1} - t_{n-N})^{4\alpha-1}\,\Theta_4(x_{\mathcal{P}}^{(n-N+1)}, y_{\mathcal{P}}^{(n-N+1)}, t_{n-N+1} - t_{n-N})\big) \to 1,
\end{aligned} \tag{17}
$$

$$
\sum_{k=2}^{N}\frac{R_{n-k}(x,y,\mathcal{P})\prod_{j=1}^{k-2}(1 + (t_{n-j} - t_{n-j-1})^{4\alpha-1})}{(2\pi(t_{n-k+1} - t_{n-k}))^{\frac{d}{2}}}\,e^{-\frac{1}{2(t_{n-k+1}-t_{n-k})^{1-2\alpha}}} \to 0 \tag{18}
$$

uniformly in $x, y \in M$. Note that $|\mathcal{P}|^{\frac{1}{d+9}} < t_n - t_{n-N} < 2\,|\mathcal{P}|^{\frac{1}{d+9}}$. By Lemma 3,

$$
\big(Q_{t_1-t_0}\cdots Q_{t_{n-N}-t_{n-N-1}}h\,(\,\cdot\,, y, \tau)\big)(x) \to h(x,y,t),
$$

and the convergence is uniform in $x, y \in M$. Further, for simplicity introduce the notation $\tau_i = t_n - t_{n-i}$, for $i = 1, \ldots, N$, and $\Theta^{(i)} = \Theta_4(x_{\mathcal{P}}^{n-i+1}, y_{\mathcal{P}}^{n-i+1}, t_{n-i+1} - t_{n-i})$. Relation (17) holds if and only if

$$
\sum_{i=1}^{N}\log(1 + \tau_i^{4\alpha-1}\Theta^{(i)}) \to 0, \quad \text{as } |\mathcal{P}| \to 0.
$$

To prove this, we use the inequality

$$
\log(1 + \tau_i^{4\alpha-1}\Theta^{(i)}) < \tau_i^{4\alpha-1}\Theta^{(i)}.
$$

To treat negative numbers $\tau_i^{4\alpha-1}\Theta^{(i)}$, we consider the mesh of $\mathcal{P}$ small enough, so that $|\tau_i^{4\alpha-1}\Theta^{(i)}| < C|\mathcal{P}|^{4\alpha-1} < \varepsilon$, where C is a constant, and ε is sufficiently small, so that inequality

$$\frac{1}{2}\,\tau_i^{4\alpha-1}\Theta^{(i)} < \log(1+\tau_i^{4\alpha-1}\Theta^{(i)})$$

holds. Considering both cases of a positive and a negative value of $\tau_i^{4\alpha-1}\Theta^{(i)}$, we write down this inequality in the form

$$\frac{1}{2}\,\min\{0,\tau_i^{4\alpha-1}\Theta^{(i)}\} < \log(1+\tau_i^{4\alpha-1}\Theta^{(i)}) < \tau_i^{4\alpha-1}\Theta^{(i)}.$$

From this and from the fact that all $\Theta^{(i)}$ are bounded by the same constant C, it follows that the uniform convergence in $x, y \in M$ in (17) will hold if

$$\sum_{i=1}^{N}\tau_i^{4\alpha-1} \to 0, \quad \text{as } |\mathcal{P}| \to 0.$$

This will follow from Lemma 4 if we choose $\alpha < \frac{1}{2}$ sufficiently close to $\frac{1}{2}$. Thus, (17) is proved. Relation (18) is obvious if we notice that all functions R_k are bounded by the same constant, and the products by which R_k are being multiplied converge uniformly to 1. Hence, we have proved that

$$\lim_{|\mathcal{P}|\to 0} q_{\mathcal{P}}(x,y) = h(x,y,t)$$

uniformly in $x, y \in M$. The theorem is proved. □

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