Quantization of Higher Abelian Gauge Theory in Generalized Differential Cohomology

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Abstract

We review and elaborate on some aspects of the quantization of certain classes of higher abelian gauge theories using techniques of generalized differential cohomology. Particular emphasis is placed on the examples of generalized Maxwell theory and Cheeger–Simons cohomology, and of Ramond–Ramond fields in Type II superstring theory and differential K-theory.

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1 Introduction

This paper is devoted to a survey of some topics in the mathematical formulation of generalized abelian gauge theories as they arise in string theory and M-theory. It has been realised that the proper mathematical treatment of the quantization of such systems involves techniques from generalized differential cohomology theories, and this has sparked a wealth of activity in both physics and mathematics in recent years, as well as intensive interactions between the two disciplines. Our presentation is neither complete nor is it exhaustive in the treatment of topics we have chosen to cover. Instead, we broadly overview various topics, presenting some aspects with a different emphasis compared to other treatments of the subject, and also elucidate certain calculational details which are probably well-known to the experts but which we have not found in the literature. Our presentation is intentionally mathematical but with all motivations, discussions and prejudices towards certain features inspired by physical considerations from string theory; we have attempted to define all pertinent mathematical concepts for the non-experts whilst describing physical concepts more formally.

Before describing the precise contents of this article, let us explain the general setting we shall consider and which important omissions the reader can anticipate in the following sections. The different theories we are interested in all have the same low-energy limit, which is ten-dimensional supergravity defined on a manifold M; we can generalize M to include groupoids which enables us to formulate the theory on e.g. orbifolds. The relevant field content includes a riemannian metric g, a dilaton field $\phi : M \to \mathbb{R}$, a closed three-form H, and the Ramond–Ramond gauge fields F which are inhomogeneous differential forms on M that are H-twisted closed, i.e. $(\mathrm{d} + H\wedge)F = 0$. The relevant string theories are quantized by certain generalized differential cohomology theories $\check{\mathrm{E}}^\bullet(M)$ depending on the physical constraints imposed, which gives a suitable lattice in de Rham cohomology $\mathrm{H}^\bullet(M;\mathbb{R})$ where the fields should live in order to enforce charge quantization; in this paper these theories will always be certain smooth refinements of ordinary cohomology and K-theory. Examples of theories with the correct low-energy limit include bosonic string theory which is quantized by ordinary differential cohomology $\check{\mathrm{H}}^3(M)$, heterotic string theory where H is not closed and which is thereby described by differential cochains, and Type I string theory which involves self-dual fields that are quantized by differential KO-theory $\check{\mathrm{KO}}(M)$. For Type II superstring theory, the quantization of the B-field is somewhat more involved and has been studied thoroughly in [23, 24]; the issue is to reconcile the worldsheet and target space descriptions of the B-field, which also serves as a differential twisting of K-theory in which the suitable refinements of the Ramond–Ramond fields F to classes or cocycles should live. This latter construction includes Type I string theory as a special case of the Type II theory via an orientifold construction. Various generalizations of this theme can also be considered: For instance in Kaluza–Klein compactifications one sets $M = X \times K$ with K a compact "internal" manifold and regards the gauge fields as forms $F \in \Omega^\bullet(X; \mathtt{harm}^\bullet(K))$ where $\mathtt{harm}^\bullet(K)$ is the vector space of harmonic forms on K, while if M is

unoriented then the dual gauge fields live in $\Omega^\bullet(M; \texttt{or}(M))$ where $\texttt{or}(M)$ is the orientation bundle of M. With this setting in mind, we can now spell out a few of our main omissions of topics.

Firstly, we do not discuss in detail the appearence or role of anomalies, which are a physical driving force for the uses of K-theory in string theory and should be properly addressed within some framework of categorified index theory; see [26, 28, 30] for extensive discussions on the importance of generalized differential cohomology theories in this context.

Secondly, we do not consider twistings of our cohomology theories. The ingredients of twisted generalized differential cohomology theories are described in [39]. Twisted differential K-theory is defined in [18] using the twisting groupoid of abelian bundle gerbes with 2-connection; this construction is then based on sections of bundles of Fredholm operators.

Finally, we do not consider orbifolds. They represent a broad class of consistent string backgrounds with rich interesting features for which equivariant geometric versions of K-theory seem to have the appropriate features; here one should use the suggestion [50] that string orbifolds are best regarded as quotient stacks, at least from the perspective of the worldsheet sigma-model. Differential equivariant complex K-theory is developed in [54, 47, 15], while an account of orbifold Ramond–Ramond fields in the path integral framework and with more general backgrounds can be found in [23, 24]. The original model of [54] is defined using classifying spaces for equivariant K-theory and Bredon cohomology with coefficients in the real representation ring of the orbifold group; this approach nicely captures salient features of Ramond–Ramond fields on Type II orbifolds, such as flux quantization. The subsequent construction of [47] is rooted in homotopy theory and utilizes delocalized de Rham equivariant cohomology; in this approach a ring structure and push-forwards are readily constructed by taking real-valued forms as fixed points of a suitable real structure. Finally, the model of [15] is set in the framework of equivariant local index theory and differentiable étale stacks, and uses delocalized de Rham cohomology; this approach is powerful enough to construct all necessary ingredients in gauge theory such as products, push-forwards, an intersection pairing, and an $\mathbb{R}$-valued subfunctor, at the price of using broad classes of cocycles (geometric families of Dirac operators), some of which have no immediate physical interpretation in terms of Ramond–Ramond gauge theory.

The outline of the remainder of this paper, together with directions for complementary reading, is as follows.

In §2, we start from the Maxwell gauge theory of electromagnetism and generalize it to higher abelian gauge theories. We explain how Dirac charge quantization is naturally rooted in a description involving Cheeger–Simons cohomology; foundational aspects of this line of reasoning can be found in [26], while a more pedagogical introduction geared at physicists is [32]. We also describe the configuration space of abelian gauge fields in details, and how to properly incorporate currents. Gauge fields are modelled by groupoids, and equivalent groupoids are equally good for the purposes of defining the functional integral of the quantum gauge theory; we explore one such model which is contained in the seminal paper [37]. A complementary review can be found in [56], while more detailed mathematical aspects of differential cohomology are reviewed in [12].

In §3, we begin by briefly reviewing the relationship between D-branes and K-theory: We introduce the notion of D-brane, describe D-branes as K-cycles for geometric K-homology, explain the physical relevance of the Atiyah–Bott–Shapiro construction, and give the formula for D-brane charges. This leads us into the notion of Ramond–Ramond charges and how they are related to the semi-classical quantization of Ramond–Ramond fields in Type II superstring theory. We survey various models for differential K-theory from a physical perspective, describe the mathematical properties of flat Ramond–Ramond fields, and formulate the gauge theory of Ramond–Ramond fields, focusing in particular on their realization as self-dual fields and the use of differential K-theory in the description of their holonomies on D-branes. A review of certain aspects of D-branes

and K-theory from a mathematical perspective can be found in [53], while various aspects of differential K-theory in string theory is presented in [27]. A mathematical survey of differential K-theory is found in [16].

Finally, in §4 we begin with a quick overview of the mathematical formulation of functional integral quantization of generalized abelian gauge theories, together with many explicit examples. We then consider in some detail the hamiltonian quantization of self-dual generalized abelian gauge fields using the concept of Pontrjagin self-duality. We apply this formalism to the quantization of Ramond–Ramond fields, through the theory of Heisenberg groups and their representations; the seminal work on this approach to quantization is [31]. We work only with the simplest backgrounds that contain no H-flux or D-branes, and explicitly carry out the hamiltonian quantization of the Ramond–Ramond gauge theory by constructing the pertinent Heisenberg groups along the lines of [31]. By choosing the natural polarization on the configuration space of the self-dual Ramond–Ramond gauge theory, the Heisenberg group admits a unique irreducible unitary representation which is identified as the quantum Hilbert space of the gauge theory.

2 Abelian gauge theory and differential cohomology

2.1. Maxwell theory

In undergraduate physics courses on classical electromagnetic theory one learns about perhaps the most fundamental set of equations in physics, the Maxwell equations; these equations govern all classical electromagnetic phenomena and are responsible for much of modern technology. In this section we begin with a mathematical introduction to the classical Maxwell theory.

Let $M = \mathbb{R} \times N$ be a four-manifold with lorentzian signature metric $\mathrm{d}t \otimes \mathrm{d}t - g$, where $t \in \mathbb{R}$ parametrizes the "time" direction and (N, g) is a connected riemannian three-manifold which we will think of as "space". Classical electromagnetism takes place in *Minkowski spacetime* where N is taken to be the vector space $\mathbb{R}^3$.

Introduce a pair of differential forms

$$F \in \Omega^2(M) \qquad \text{and} \qquad j_e \in \Omega_c^3(M) \ ,$$

where the two-form F is called the gauge field strength or flux, while j_e is a differential form of compact spatial support called the electric current. Maxwell's equations then read

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \mathrm{d} \star F = j_e \ , \tag{2.1}$$

where $\star$ is the Hodge duality operator associated to the lorentzian metric on M. Consistency of the second equation in (2.1) requires the conservation law

$$\mathrm{d}j_e = 0 \ ,$$

which defines the electric charge

$$q_e := \left[j_e \big|_N \right] \ \in \ \mathrm{H}_c^3(N; \mathbb{R}) \cong \mathbb{R} \ .$$

We can generalize the first equation of (2.1) to include a magnetic current three-form j_m with

$$\mathrm{d}F = j_m \ ,$$

so that $\mathrm{d}F = 0$ outside the support of j_m. This defines the magnetic charge

$$q_m := \left[j_m \big|_N \right] \ \in \ \mathrm{H}_c^3(N; \mathbb{R}) \cong \mathbb{R} \ .$$

The magnetic current vanishes in the "classical theory", but the "quantum theory" allows for it; as we discuss below, this leads to the quantization of electric *and* magnetic charge. Since the two-form F is not required to have compact support, the charges q_e and q_m are non-zero generally; in fact, they live in the kernel of the natural forgetful map which forgets about the compact support condition,

$$q_e, q_m \; \in \; \ker \left(\mathrm{H}^3_c(N; \mathbb{R}) \; \longrightarrow \; \mathrm{H}^3(N; \mathbb{R}) \right) ,$$

since the currents j_e and j_m are trivialized by the flux F on the interior of the three-manifold N.

Absence of magnetic charge in the classical theory implies that the de Rham cohomology class of the flux is trivial,

$$[F]_{\mathrm{dR}} = 0 \qquad \text{in} \quad \mathrm{H}^2(M; \mathbb{R}) . \tag{2.2}$$

Hence there exists a one-form $A \in \Omega^1(M)$ such that

$$F = \mathrm{d}A . \tag{2.3}$$

The *gauge potential* A is only defined up to gauge transformations $A \mapsto A + \alpha$ by closed differential one-forms $\alpha \in \Omega^1_{\mathrm{cl}}(M)$. If M is contractible (in particular if $N = \mathbb{R}^3$), then the two conditions (2.2) and (2.3) are automatically satisfied and equivalent to each other. In general, the space of classical electromagnetic fields modulo gauge transformations is the infinite-dimensional abelian Lie group

$$\mathscr{F}_{\mathrm{class}}(M) = \Omega^1(M) \big/ \Omega^1_{\mathrm{cl}}(M) . \tag{2.4}$$

Alternatively, we can take A to be a connection on a principal $\mathbb{R}$-bundle over M (with the additive group structure on the real line $\mathbb{R}$); the space of such connections is an affine space based on $\Omega^1(M)/\mathrm{d}\Omega^0(M)$, and the quotient by gauge equivalence classes of flat connections is an affine space modelled on (2.4). This is the correct configuration space of fields for classical electromagnetism; as we discuss below, the story is rather different for the quantum theory.

It is also possible to derive these results from an action principle. Maxwell's equations (2.1) in this instance are the variational equations for the action functional

$$S_M[A] = \int_M \left(-\frac{1}{2} \mathrm{d}A \wedge \star \mathrm{d}A + A \wedge j_e \right) \tag{2.5}$$

with respect to compactly supported variations of A such that $\int_M \mathrm{d}(A \wedge \star F) = 0$. Then $S_M[A+\alpha] = S_M[A]$ for $\alpha \in \Omega^1_{\mathrm{cl}}(M)$ up to an exact term $-\int_M \mathrm{d}(\alpha \wedge \star F)$; in this sense the action functional (2.5) is classically well-defined on the quotient space (2.4).

2.2. *Semi-classical quantization*

If $\mathrm{H}^2(N; \mathbb{R})$ is non-trivial, we may well have $[F]_{\mathrm{dR}} \neq 0$ in $\mathrm{H}^2(M; \mathbb{R})$, e.g. outside the support of a magnetic current j_m in M. The *Dirac quantization condition* states that the de Rham cohomology class of the flux sits in a lattice

$$\tfrac{1}{2\pi} [F]_{\mathrm{dR}} \; \in \; \Lambda \subset \mathrm{H}^2(M; \mathbb{R}) ,$$

where

$$\Lambda = \mathrm{H}^2(M; \mathbb{Z}) \big/ \mathsf{Tor}\, \mathrm{H}^2(M; \mathbb{Z})$$

is the full lattice $\mathrm{H}^2(M; \mathbb{Z}) \hookrightarrow \mathrm{H}^2(M; \mathbb{R})$ induced in cohomology by the inclusion of abelian groups $\mathbb{Z} \hookrightarrow \mathbb{R}$, whose kernel consists of torsion classes in $\mathrm{H}^2(M; \mathbb{Z})$.

Let us pause to briefly explain how this is related to the usual notion of Dirac charge quantization in physics (see e.g. [26] for further details). Let us take $N = \mathbb{R}^3$, and the electric and magnetic currents to be of the form

$$j_e = q_e \, \delta_W \qquad \text{and} \qquad j_m = q_m \, \delta_{\mathbb{R}} \ ,$$

where $\delta_W = \mathsf{Pd}_M(W)$ is Poincaré dual to an oriented one-manifold $W \subset M$ (the "worldline" of a charged particle), while $\delta_{\mathbb{R}}$ is a distributional three-form on $M = \mathbb{R} \times \mathbb{R}^3$ dual to the one-manifold $\mathbb{R} \times 0 \subset M$. The global obstruction to the local representation (2.3) on $\mathbb{R} \times (\mathbb{R}^3 - 0)$ is then

$$\int_{S^2} F = \int_{B^3} \mathrm{d}F = \int_{B^3} q_m \, \delta_{\mathbb{R}} = q_m \ , \tag{2.6}$$

where $S^3 = \partial B^3$ is the unit sphere in $t \times (\mathbb{R}^3 - 0)$ for all $t \in \mathbb{R}$. This obstruction is due to the non-trivial cohomology $\mathrm{H}^2(\mathbb{R}^3 - 0; \mathbb{R}) \neq 0$ and it may be thought of as originating through the *Dirac string*, which is a semi-infinite solenoid represented by a ray from the origin $0 \in \mathbb{R}^3$; requiring the string to be physically invisible then yields the global obstruction (2.6). The Dirac quantization law then ensures that in the quantum theory the exponentiated charge coupling $\exp\left(\mathrm{i}\int_M A \wedge j_e\right) = \exp\left(\mathrm{i} q_e \int_W A\right)$ is well-defined. Since $\int_W A \in \mathbb{R}/q_m \mathbb{Z}$ by (2.6), the coupling is well-defined if

$$q_e \, q_m \ \in \ 2\pi \, \mathbb{Z} \ . \tag{2.7}$$

In the following we will interpret the quantization condition (2.7) geometrically. The wavefunction of a non-relativistic quantum mechanical particle of charge q_e on $\mathbb{R}^3 - 0$ is a section of a line bundle associated to the representation

$$\mathbb{R} \, / \, q_m \, \mathbb{Z} \ \longrightarrow \ \mathrm{U}(1) \ , \qquad x \ \longmapsto \ \mathrm{e}^{\,\mathrm{i}\, q_e\, x} \ ,$$

and is therefore well-defined if and only if (2.7) holds. In this sense the quantization of charge is a consequence of the compactness of the gauge group $\mathbb{R}/q_m \mathbb{Z}$.

There is an elegant model in differential geometry which combines locality of the gauge field F with global obstructions, including Dirac charge quantization. For this, we take F to be the curvature of a connection A on a principal $\mathbb{T}$-bundle $\pi : L \to M$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with first Chern class $c_1(L) \in \mathrm{H}^2(M; \mathbb{Z})$, i.e. A is a right-invariant one-form on L such that $\mathrm{d}A = \pi^*F$. Then the classical configuration space of fields (2.4) is replaced by the quantum groupoid of fields $\mathscr{F}_{\mathrm{qu}}(M)$. Recall that a *groupoid* is a small category in which all morphisms (viewed as arrows between objects) are invertible. The objects of the category $\mathscr{F}_{\mathrm{qu}}(M)$ are principal $\mathbb{T}$-bundles with connection, while its morphisms are connection-preserving bundle isomorphisms (gauge transformations). It has the structure of an infinite-dimensional abelian Lie group, with unit the trivial bundle, under tensor product of circle bundles with connection. The set of isomorphism classes $\pi_0 \mathscr{F}_{\mathrm{qu}}(M)$ is an infinite-dimensional abelian Lie group that fits into an exact sequence

$$0 \ \longrightarrow \ \mathrm{H}^1(M; \mathbb{T}) \ \longrightarrow \ \pi_0 \mathscr{F}_{\mathrm{qu}}(M) \ \overset{F}{\longrightarrow} \ \mathscr{F}_{\mathrm{class}}(M) \ \longrightarrow \ 0$$

which describes the quantum configuration space as an extension of the classical one (2.4) by gauge equivalence classes of flat connections; whence such connections are detectable quantum mechanically, but not classically. Symbolically,

$$\pi_0 \mathscr{F}_{\mathrm{qu}}(M) = \bigsqcup_{c_1 \in \mathrm{H}^2(M; \mathbb{Z})} \mathcal{A}(L_{c_1}) \big/ \mathcal{G} \ ,$$

where $\mathcal{A}(L_{c_1})$ is the affine space of smooth connections on a line bundle of first Chern class c_1 while $\mathcal{G} = \Omega^0(M; \mathbb{T})$ is the gauge group. The group $\mathrm{H}^1(M; \mathbb{T})$ of flat fields can be described as follows. The short exact sequence of abelian groups

$$0 \ \longrightarrow \ \mathbb{Z} \ \hookrightarrow \ \mathbb{R} \ \longrightarrow \ \mathbb{T} = \mathbb{R}/\mathbb{Z} \ \longrightarrow \ 0 \tag{2.8}$$

induces a short exact sequence in cohomology

$$0 \longrightarrow \mathrm{H}^1(M;\mathbb{Z}) \otimes \mathbb{T} \longrightarrow \mathrm{H}^1(M;\mathbb{T}) \xrightarrow{\beta} \mathsf{Tor}\,\mathrm{H}^2(M;\mathbb{Z}) \longrightarrow 0$$

where the group $\mathrm{H}^1(M;\mathbb{Z}) \otimes \mathbb{T}$ is the identity component of $\mathrm{H}^1(M;\mathbb{T})$, the torsion subgroup of $\mathrm{H}^2(M;\mathbb{Z})$ is the group of components of $\mathrm{H}^1(M;\mathbb{T})$, and β is the Bockstein homomorphism.

Another point of view of the space of quantum fields $\pi_0 \mathscr{F}_{\mathrm{qu}}(M)$ is as the group of holonomies

$$\chi_A \; : \; Z_1(M) \longrightarrow \mathrm{U}(1) \,, \qquad \chi_A(\gamma) = \exp\left(\mathrm{i} \oint_\gamma A\right),$$
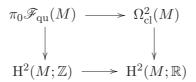
where $Z_p(M)$ is the group of smooth p-cycles on M. Such group homomorphisms are characterized by the feature that there is a unique integral two-form $F \in 2\pi\,\Omega_{\mathbb{Z}}^2(M)$ with

$$\chi_A(\partial D) = \exp\left(\mathrm{i} \int_D F\right) \qquad \text{for} \quad D \in C_2(M) \,,$$

where $\Omega_{\mathbb{Z}}^p(M)$ is the lattice of closed p-forms on M with integer periods, and $C_p(M)$ denotes the group of smooth p-chains on M. Then the group $\pi_0 \mathscr{F}_{\mathrm{qu}}(M)$ may be characterized by the short exact sequence

$$0 \longrightarrow \Omega^1(M)\,/\,\Omega_{\mathbb{Z}}^1(M) \longrightarrow \pi_0 \mathscr{F}_{\mathrm{qu}}(M) \xrightarrow{c_1} \mathrm{H}^2(M;\mathbb{Z}) \longrightarrow 0 \,, \qquad (2.9)$$

where the kernel of the characteristic class map c_1 consists of connections on the trivial line bundle over M modulo gauge equivalence; the quotient in (2.9) is the group of topologically trivial one-form fields on M. More generally, the space of quantum fields completes the pullback square

$$\begin{array}{ccc} \pi_0 \mathscr{F}_{\mathrm{qu}}(M) & \longrightarrow & \Omega_{\mathrm{cl}}^2(M) \\ \downarrow & & \downarrow \\ \mathrm{H}^2(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(M;\mathbb{R}) \end{array}$$

so that $\pi_0 \mathscr{F}_{\mathrm{qu}}(M) \subset \mathrm{H}^2(M;\mathbb{Z}) \times \Omega_{\mathrm{cl}}^2(M)$; thus a quantum field is a representative in $\Omega_{\mathrm{cl}}^2(M)$ of a first Chern class $c_1 \in \mathrm{H}^2(M;\mathbb{Z})$ in de Rham cohomology.

2.3.   Higher abelian gauge theory

There are various generalizations of Maxwell theory motivated from string theory which use higher degree differential form fields, for instance the two-form B-field of superstring theory, the three-form C-field of M-theory, and other supergravity fields. The higher Maxwell equations on a lorentzian $n+1$-manifold $M = \mathbb{R} \times N$, with N a riemannian n-manifold, read again

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \mathrm{d} \star F = j_e \,, \qquad (2.10)$$

but now generally $F \in \Omega^p(M)$ and $j_e \in \Omega_c^{n-p+2}(M)$ with $\mathrm{d}j_e = 0$. In the absence of electric current $j_e = 0$, the classical flux group is

$$\big([F]_{\mathrm{dR}}\,, [\star F]_{\mathrm{dR}}\big) \; \in \; \mathrm{H}^p(M;\mathbb{R}) \oplus \mathrm{H}^{n-p+1}(M;\mathbb{R}) \,,$$

which possesses "electric-magnetic duality" interchanging (magnetic) p-forms with (electric) $n - p + 1$-forms. On the other hand, the group of classical electric charges is

$$\big[j_e\big|_N\big] \; \in \; Q_e := \ker\big(\mathrm{H}_c^{n-p+2}(N;\mathbb{R}) \longrightarrow \mathrm{H}^{n-p+2}(N;\mathbb{R})\big) \,.$$

From the exact sequence

$$\mathrm{H}^{n-p+1}(M;\mathbb{R}) \xrightarrow{i^*} \mathrm{H}^{n-p+1}\left(M-\mathrm{supp}(j_e)\,;\mathbb{R}\right) \longrightarrow$$

$$\longrightarrow \mathrm{H}^{n-p+1}\left(M\,,\,M-\mathrm{supp}(j_e)\,;\mathbb{R}\right) \xrightarrow{\delta} \mathrm{H}^{n-p+2}(M;\mathbb{R})$$

one identifies the charge group as

$$Q_e \cong \mathrm{H}^{n-p+1}\left(M-\mathrm{supp}(j_e)\,;\mathbb{R}\right) \,/\, \mathrm{H}^{n-p+1}(M;\mathbb{R})\,.$$

We interpret this as the group of "charges measured by the flux at infinity", which are given by integrating the form $\star F$ over a gaussian sphere S_∞^{n-p+1}.

Again these equations can be obtained from a variational principle for the action functional

$$S_M[A] := \int_M \left(-\frac{1}{2}\,F \wedge \star F + A \wedge j_e \right)\,, \tag{2.11}$$

where $F = \mathrm{d}A$. Since the current form j_e is closed and compactly supported on M, by Poincaré duality there is a dual class in the real homology $[W_e] := \mathsf{Pd}_M(j_e) \in \mathrm{H}_{p-1}(M;\mathbb{R})$ which is represented by a compact oriented submanifold $W_e \subset M$ such that

$$\int_M a \wedge j_e = -\int_{W_e} a\big|_{W_e} \tag{2.12}$$

for any closed $p-1$-form a; if a is not closed, then the formula (2.12) still holds but j_e must be now regarded as a de Rham current, i.e. a distributional form supported on $W_e \subset M$. We think of the p_e+1-dimensional submanifold W_e as the worldvolume of an "electrically charged p_e-brane" where $p_e = p - 2$, with uniform charge density $q_e = \left[\star j_e\big|_{W_e}\right] \in \mathrm{H}^0(W_e;\mathbb{R})$ induced by the current which the electrically charged brane produces. Alternatively, given $q_e \in \mathrm{H}^0(W_e;\mathbb{R})$, the current $[j_e] = i_!(q_e) \in \mathrm{H}^{n-p+2}(M, M-W_e)$ is the pushforward of q_e induced by the embedding $i : W_e \hookrightarrow M$.

We can also introduce a magnetic current $j_m \in \Omega_c^{p+1}(M)$, which modifies the first equation of motion in (2.10) to $\mathrm{d}F = j_m$, and the corresponding magnetic p_m-brane $W_m \subset M$ with $p_m = n - p - 1$. Then the classical dyonic charge group is

$$\left([j_m]\,,\,[j_e]\right) \,\in\, \mathrm{H}^{p+1}(M, M-W_m;\mathbb{R}) \oplus \mathrm{H}^{n-p+2}(M, M-W_e;\mathbb{R})\,. \tag{2.13}$$

When p is even the lattice of charges is symplectic, while for p odd the lattice is symmetric.

Everything we said before concerning the semi-classical Maxwell theory has an analogue for these higher abelian gauge fields. In particular, Dirac quantization implies the quantization of classical charges and the quantum charge group is the real image of the lattice

$$\mathrm{H}^{p+1}(M, M-W_m;\mathbb{Z}) \,\oplus\, \mathrm{H}^{n-p+2}(M, M-W_e;\mathbb{Z})$$

in de Rham cohomology. From this one might expect that the quantum flux group is the real image of the lattice $\mathrm{H}^p(M;\mathbb{Z}) \oplus \mathrm{H}^{n-p+1}(M;\mathbb{Z})$ in de Rham cohomology, but we shall see that there are some subtleties with this naive guess. The proper geometric interpretation of the quantum theory of higher abelian gauge fields, which is a higher generalization of the description of electromagnetic fields in terms of line bundles with connection, is provided by studying isomorphism classes of fields in Cheeger–Simons differential cohomology [19, 8].

Definition 2.14. *The p-th Cheeger–Simons differential cohomology group of M is the subgroup*

$$\check{\mathrm{H}}^p(M) \subset \mathsf{Hom}_{\mathscr{A}b}\big(Z_{p-1}(M)\,,\,\mathrm{U}(1)\big)$$

in the category $\mathscr{A}b$ of abelian groups consisting of homomorphisms χ, called *differential characters, such that there exists a unique closed integral p-form $F_\chi \in 2\pi \, \Omega^p_{\mathbb{Z}}(M)$,* called the curvature of the *differential character χ, with*

$$\chi(\partial B) = \exp \left(\mathrm{i} \int_B F_\chi \right) \qquad for \quad B \in C_p(M) \ .$$

In the following we use a multiplicative notation for characters χ which are valued in the circle group $\mathrm{U}(1)$, and an additive notation for their classes $[\check{A}]$ which are valued in the abelian Lie algebra $\mathbb{T} = \mathbb{R}/\mathbb{Z}$; when we wish to utilize both descriptions simultaneously we will also write $\chi_{\check{A}}$. For reasons that will eventually become clear, the field theories we are interested in all fall into the following characterization.

Definition 2.15. *A* higher abelian gauge theory *is a field theory on a smooth manifold M whose (semi-classical) configuration space of gauge inequivalent field configurations is given by the differential cohomology group $\check{\mathrm{H}}^p(M)$ for some $p \in \mathbb{Z}$, and whose* charge group *is the integer cohomology $\mathrm{H}^p(M; \mathbb{Z})$.*

Properties

1. The map $M \mapsto \check{\mathrm{H}}^p(M)$ is a contravariant functor, with $\check{\mathrm{H}}^p(M)$ an infinite-dimensional abelian Lie group whose connected components are labelled by the charge group $\pi_0 \check{\mathrm{H}}^p(M) = \mathrm{H}^p(M; \mathbb{Z})$.

2. There is a *graded ring structure* $\check{\mathrm{H}}^{p_1}(M) \otimes \check{\mathrm{H}}^{p_2}(M) \to \check{\mathrm{H}}^{p_1+p_2}(M)$, denoted $\chi_1 \smile \chi_2$ for $\chi_1, \chi_2 \in \check{\mathrm{H}}^\bullet(M)$, and an *integration map* $\int^{\check{\mathrm{H}}}_M : \check{\mathrm{H}}^{n+2}(M) \to \check{\mathrm{H}}^1(\mathrm{pt}) \cong \mathbb{T}$ where pt denotes a one-point space; the existence of this integration requires a suitable notion of $\check{\mathrm{H}}$-orientation on M. More generally, given an $\check{\mathrm{H}}$-oriented bundle of manifolds $M \hookrightarrow X \to P$, there is an integration over the fibres $\int^{\check{\mathrm{H}}}_{X/P} : \check{\mathrm{H}}^s(X) \to \check{\mathrm{H}}^{s-n-1}(P)$.

3. There is a surjective *field strength map* defined by a natural transformation

$$F : \check{\mathrm{H}}^p(M) \longrightarrow \Omega^p_{\mathbb{Z}}(M) \ , \qquad F(\chi) := \tfrac{1}{2\pi} F_\chi$$

which is a graded ring homomorphism, i.e.

$$F(\chi_1 \smile \chi_2) = F_{\chi_1} \wedge F_{\chi_2} \ .$$

Then integration obeys a version of Stokes' theorem

$$\int^{\check{\mathrm{H}}}_{\partial N} [\check{A}] = \int^{\check{\mathrm{H}}}_N F([\check{A}])$$

for $[\check{A}] \in \check{\mathrm{H}}^p(N)$.

4. There is a surjective *characteristic class map* defined by a natural transformation

$$c : \check{\mathrm{H}}^p(M) \longrightarrow \mathrm{H}^p(M; \mathbb{Z})$$

which is a ring homomorphism, i.e.

$$c(\chi_1 \smile \chi_2) = c(\chi_1) \smile c(\chi_2) \ ,$$

9

and which is compatible with the field strength map, i.e.

$$\left[(c \otimes \mathbb{R})(\chi)\right] = \tfrac{1}{2\pi} \left[F_\chi\right]_{\mathrm{dR}} \; .$$

Together the maps c and F define the pullback square

$$
\begin{array}{ccc}
\check{\mathrm{H}}^p(M) & \longrightarrow & \Omega^p_{\mathrm{cl}}(M) \\
\downarrow & & \downarrow \\
\mathrm{H}^p(M;\mathbb{Z}) & \longrightarrow & \mathrm{H}^p(M;\mathbb{R})
\end{array}
$$

which leads to the exact sequence

$$0 \longrightarrow \mathrm{H}^{p-1}(M;\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow \check{\mathrm{H}}^p(M) \longrightarrow \Omega^p_\mathbb{Z}(M) \times_{[-]} \mathrm{H}^p(M;\mathbb{Z})$$
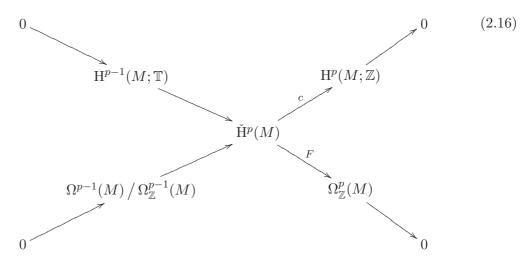
where $\Omega^p_\mathbb{Z}(M) \times_{[-]} \mathrm{H}^p(M;\mathbb{Z}) := \left\{(\omega,\xi) \mid [\omega]_{\mathrm{dR}} = \xi\right\}$.

5. Topologically trivial or flat fields $\chi_1 = \chi_{A_1}$ correspond to the class of a globally defined differential form A_1 on M; its product with any other character χ_2 is also topologically trivial and given by

$$\chi_{A_1} \smile \chi_2 = \chi_{A_1 \wedge F_{\chi_2}} \; .$$

More generally, the products of $\phi \in \ker F \subset \check{\mathrm{H}}^p(M)$ and $\xi \in \ker c \subset \check{\mathrm{H}}^p(M)$ with any character $\chi \in \check{\mathrm{H}}^l(M)$ correspond respectively to the classes $(-1)^l\, c(\chi) \smile \phi$ and $(-1)^l\, F(\chi) \smile \xi$.

6. The Cheeger–Simons groups are completely characterized by two short exact sequences, which can be summarised in the diagram

$$(2.16)$$

The sequence running from top to bottom is the field strength sequence, where, by Poincaré duality, $\mathrm{H}^{p-1}(M;\mathbb{T}) \cong \mathsf{Hom}_{\mathscr{A}b}(\mathrm{H}_{p-1}(M;\mathbb{Z}),\mathbb{T})$ is the group of *flat fields* χ with $F_\chi = 0$; it defines a torus $\mathbb{T}^p(M) \subset \check{\mathrm{H}}^p(M)$ with fundamental group

$$\pi_1 \check{\mathrm{H}}^p(M) \cong \mathrm{H}^{p-1}(M;\mathbb{Z}) \,/\, \mathsf{Tor}\; \mathrm{H}^{p-1}(M;\mathbb{Z})$$

based at the identity 0. The sequence from bottom to top is the characteristic class sequence, with $\Omega^{p-1}(M)/\Omega^{p-1}_\mathbb{Z}(M)$ the torus of *topologically trivial fields* whose classes $[A]$ have curvature $F([A]) = \mathrm{d}A$. One also has the exact sequence

$$0 \longrightarrow \mathrm{H}^{p-1}(M;\mathbb{R}) \,/\, \mathrm{H}^{p-1}(M;\mathbb{Z}) \longrightarrow \mathrm{H}^{p-1}(M;\mathbb{T}) \xrightarrow{\;\beta\;} \mathsf{Tor}\; \mathrm{H}^p(M;\mathbb{Z}) \longrightarrow 0 \;, \quad (2.17)$$

where the torsion subgroup is the group of *discrete Wilson lines* and β is the Bockstein homomorphism. This yields a geometric picture of $\check{\mathrm{H}}^p(M)$ as consisting of infinitely many connected components $\check{\mathrm{H}}^p_c(M)$ labelled by the charges $c \in \mathrm{H}^p(M; \mathbb{Z})$, with each topological sector a torus fibration over a vector space whose fibres are finite-dimensional tori $\Omega^{p-1}_{\mathrm{cl}}(M)/\Omega^{p-1}_{\mathbb{Z}}(M)$ represented by topologically trivial flat fields, called *Wilson lines*. In particular, there is a non-canonical splitting $\check{\mathrm{H}}^p(M) = \bigsqcup_{c \in \mathrm{H}^p(M;\mathbb{Z})} \check{\mathrm{H}}^p_c(M) \cong T \times \Gamma \times V$ where T is the torus of Wilson lines, Γ is the subgroup of topologically trivial flat fields, and $V \cong \mathrm{im}(\mathrm{d}^\dagger)$ is the vector space of *oscillator modes*; there is an isomorphism $\check{\mathrm{H}}^p_0(M) / \mathbb{T}^p(M) \cong \mathrm{d}\Omega^{p-1}(M)$ of vector spaces. Crucially, in contrast to ordinary cohomology groups, the differential cohomology contains information about both flat and topologically trivial fields on M, as they generally have non-zero classes in $\check{\mathrm{H}}^p(M)$.

7. The Cheeger–Simons groups satisfy *Pontrjagin–Poincaré duality*

$$\mathsf{Hom}_{\mathscr{A}b}\big(\check{\mathrm{H}}^p(M)\,,\,\mathbb{T}\big) \cong \check{\mathrm{H}}^{n+2-p}(M)\ .$$

This duality is a consequence of the fact that integration defines a perfect bilinear pairing

$$\check{\mathrm{H}}^p(M) \times \check{\mathrm{H}}^{n+2-p}(M)\ \longrightarrow\ \check{\mathrm{H}}^1(\mathrm{pt}) = \mathbb{T}$$

by

$$\langle \chi_1, \chi_2 \rangle := \int\!\!\!\!\!\!\!\int^{\check{\mathrm{H}}}_M \chi_1 \smile \chi_2\ .$$

On cocycles we also denote this pairing by $([\check{A}_1], [\check{A}_2]) \mapsto \int^{\check{\mathrm{H}}}_M [\check{A}_1] \smile [\check{A}_2] =: \langle [\check{A}_1], [\check{A}_2] \rangle$. If the characteristic class of $\check{A}_1$ is zero then the pairing is $\int_M A_1 \wedge F_{\check{A}_2} \bmod \mathbb{Z}$, and if also $c(\check{A}_2) = 0$ then this becomes $\int_M A_1 \wedge \mathrm{d}A_2 \bmod \mathbb{Z}$; note that both of these pairings are given by integrals of forms. On the other hand, if the curvature $F_{\check{A}_1} = 0$ with $[\check{A}_1] = \alpha_1 \in \mathrm{H}^{p-1}(M; \mathbb{T})$, then the pairing is $\int^{\mathrm{H}}_M \alpha_1 \smile c(\check{A}_2) \bmod \mathbb{Z}$.

8. A differential character $\chi \in \check{\mathrm{H}}^p(M)$ defines a *holonomy*

$$\mathsf{hol}_\Sigma(\chi) := \exp\Big(\mathrm{i} \oint_\Sigma A_\chi\Big)\ \in\ \mathrm{U}(1) \tag{2.18}$$

for any $p-1$-cycle $\Sigma \in Z_{p-1}(M)$, where the potential $A_\chi \in \Omega^{p-1}(\Sigma)$ is defined by $F_{\chi|_\Sigma} = \mathrm{d}A_\chi$ and we have used $\mathrm{H}^p(\Sigma; \mathbb{Z}) = 0$. For flat fields $F_\chi = 0$, the holonomy defines a class $\big[\mathsf{hol}(\chi)\big] \in \mathrm{H}^{p-1}(M; \mathrm{U}(1))$.

Examples

The groups $\check{\mathrm{H}}^p(M)$ vanish for all $p < 0$. For the first few non-vanishing groups we have the following identifications:

- For $p = 0$, $\check{\mathrm{H}}^0(M) \cong \mathrm{H}^0(M; \mathbb{Z})$ is identified via the characteristic class map c as the group of connected components of M; the field strength map F assigns an integer to each component.

- For $p = 1$, $\check{\mathrm{H}}^1(M) \cong \Omega^0(M; \mathrm{U}(1))$ is the space of differentiable circle-valued maps $f : M \to \mathrm{U}(1) \cong S^1$. The field strength and characteristic class maps are given in this case by

$$f \xmapsto{\ F\ } \tfrac{1}{2\pi\mathrm{i}}\, \mathrm{d}\log f \qquad \text{and} \qquad f \xmapsto{\ c\ } f^*[\mathrm{d}\theta]\ ,$$

where $[\mathrm{d}\theta]$ is the fundamental class of S^1; these maps describe how the function f acts on cohomology. The holonomy is the evaluation $\mathsf{hol}_x(f) = f(x)$ of f at $x \in M$.

- For $p = 2$, $\check{H}^2(M) \cong \mathsf{Pic}_\nabla(M)$ is the Picard group of gauge equivalence classes of line bundles with connection (L, ∇) on M and gauge group generated by $\check{H}^1(M)$. The field strength and characteristic class maps are given in this case by

$$(L, \nabla) \xmapsto{F} \tfrac{1}{2\pi\mathrm{i}} \nabla^2 \qquad \text{and} \qquad (L, \nabla) \xmapsto{c} c_1(L) \ .$$

  The connection ∇ determines a holonomy $\mathsf{hol}_\gamma(\nabla)$ for $\gamma \in Z_1(M)$ which coincides with the holonomy (2.18) of the corresponding differential character.

- For $p = 3$, $\check{H}^3(M)$ is isomorphic to the group of gauge equivalence classes of $\mathrm{U}(1)$ gerbes $\mathscr{G} \downarrow M$ with 2-connection (A, B) and gauge group generated by the differential cohomology $\check{H}^2(M)$. The field strength map gives the curvature $H = \mathrm{d}B$ of the B-field, while the characteristic class map returns the Dixmier–Douady class of $\mathscr{G}$. The 2-connection (A, B) determines a holonomy $\mathsf{hol}_\Sigma(B) \in \mathrm{U}(1)$ for $\Sigma \in Z_2(M)$.

- When M is a point, one can use the exact sequences in (2.16) to explicitly compute

$$\check{H}^p(\mathrm{pt}) = \begin{cases} \mathbb{Z} \ , & p = 0 \ , \\ \mathbb{R}/\mathbb{Z} \ , & p = 1 \ , \\ 0 \ , & p > 1 \ , \end{cases}$$

  where for $p = 0$ only the characteristic class contributes while for $p = 1$ only topologically trivial flat fields contribute. From the field strength exact sequence in (2.16) one also finds

$$\check{H}^{n+2}(M) \cong H^{n+1}(M; \mathbb{T}) \qquad \text{and} \qquad \check{H}^p(M) = 0 \quad \text{for} \quad p > n + 2 \ .$$

Deligne cohomology

An explicit cochain model for the Cheeger–Simons groups is provided by Deligne cohomology, see e.g. [8], which makes explicit the previous properties and examples together with their higher generalizations. The degree p smooth Deligne cohomology is the p-th Čech hypercohomology of the truncated sheaf complex

$$0 \longrightarrow \mathrm{U}(1)_M \xrightarrow{\mathrm{d}\log} \Omega^1_M \xrightarrow{\mathrm{d}} \Omega^2_M \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^p_M \ ,$$

where $\mathrm{U}(1)_M$ is the sheaf of smooth $\mathrm{U}(1)$-valued functions on M and Ω^p_M is the sheaf of differential p-forms on M. The degree p Deligne cohomology group $\mathrm{H}^p_{\mathfrak{D}}(M)$ can be calculated as the cohomology of the total complex of the double complex with respect to a good open cover $\mathfrak{M} = \{M_\alpha\}_{\alpha \in I}$ of M given by

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & & & \vdots \\
\delta \uparrow & & \delta \uparrow & & & & \delta \uparrow \\
C^2\big(\mathfrak{M};\, \mathrm{U}(1)_M\big) & \xrightarrow{\mathrm{d}\log} & C^2\big(\mathfrak{M};\, \Omega^1_M\big) & \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} & & & C^2\big(\mathfrak{M};\, \Omega^p_M\big) \\
\delta \uparrow & & \delta \uparrow & & & & \delta \uparrow \\
C^1\big(\mathfrak{M};\, \mathrm{U}(1)_M\big) & \xrightarrow{\mathrm{d}\log} & C^1\big(\mathfrak{M};\, \Omega^1_M\big) & \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} & & & C^1\big(\mathfrak{M};\, \Omega^p_M\big) \\
\delta \uparrow & & \delta \uparrow & & & & \delta \uparrow \\
C^0\big(\mathfrak{M};\, \mathrm{U}(1)_M\big) & \xrightarrow{\mathrm{d}\log} & C^0\big(\mathfrak{M};\, \Omega^1_M\big) & \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} & & & C^0\big(\mathfrak{M};\, \Omega^p_M\big)
\end{array}
$$

which is the quotient of the abelian group of Deligne p-cocycles by the subgroup of Deligne p-coboundaries; here δ is the usual Čech coboundary operator, and $C^p(\mathfrak{M}; \mathrm{U}(1)_M)$ and $C^p(\mathfrak{M}; \Omega_M^k)$ denote the Čech p-cochains. Below we describe these groups explicitly in low degree; contractible k-fold intersections of open sets of the cover $\mathfrak{M}$ are denoted by $M_{\alpha_1 \ldots \alpha_k} := M_{\alpha_1} \cap \cdots \cap M_{\alpha_k}$.

A degree 0 smooth Deligne class in $C^0(\mathfrak{M}; \mathrm{U}(1)_M)$ is just a smooth map $g : M \to \mathrm{U}(1)$. This case applies to the sigma-model on M of a scalar field g compactified on a circle; the charge group is the group of winding numbers $c \in \mathrm{H}^1(M; \mathbb{Z})$ of the field g around S^1. In particular, $\check{\mathrm{H}}^1(S^1)$ is the loop group $L\mathrm{U}(1)$ and if $\sigma \sim \sigma + 1$ is the coordinate on the circle $M = S^1$, then Fourier series expansion gives the explicit decomposition

$$\log g(\sigma) = 2\pi\,\mathrm{i}\,g_0 + 2\pi\,\mathrm{i}\,c\,\sigma + \sum_{k \neq 0} \frac{g_k}{k}\,\mathrm{e}^{\,2\pi\,\mathrm{i}\,k\,\sigma} \ \in \ \mathbb{T} \oplus \Gamma_c \oplus V \ .$$

A Deligne one-cocycle is a pair

$$(g_{\alpha\beta}, A_\alpha) \ \in \ C^1\big(\mathfrak{M}\,;\,\mathrm{U}(1)_M\big) \oplus C^0\big(\mathfrak{M}\,;\,\Omega_M^1\big)$$

satisfying the cocycle conditions

$$
\begin{aligned}
g_{\alpha\beta}\,g_{\beta\gamma}\,g_{\gamma\alpha} &= 1 &&\text{on}\quad M_{\alpha\beta\gamma}\ , \\
A_\alpha - A_\beta &= \mathrm{d}\log g_{\alpha\beta} &&\text{on}\quad M_{\alpha\beta}\ .
\end{aligned}
$$

A Deligne one-coboundary defines a gauge transformation and is a pair of the type

$$\big(h\,,\,\mathrm{d}\log h\big)$$

for a smooth map $h : M \to \mathrm{U}(1)$. The $\mathrm{U}(1)$ Čech one-cocycle $g_{\alpha\beta} : M_{\alpha\beta} \to \mathrm{U}(1)$ determines smooth transition functions on overlaps for a hermitian line bundle $L \to M$. This cocycle represents the first Chern class $c_1(L) = [g_{\alpha\beta}] \in \mathrm{H}^1(M; \mathrm{U}(1)) \cong \mathrm{H}^2(M; \mathbb{Z})$, where the canonical isomorphism follows from the exponential sequence

$$0 \ \longrightarrow \ \mathbb{Z} \ \longrightarrow \ \mathbb{R} \ \xrightarrow{\ \exp\ } \ \mathrm{U}(1) \ \longrightarrow \ 1 \ ;$$

it is the obstruction to triviality of the line bundle $L \to M$. The local one-forms $A_\alpha \in \Omega^1(M_\alpha)$ define a unitary connection $\nabla = \mathrm{d} + A$ on L. This is the case that arose in Maxwell theory.

A Deligne two-cocycle is a triple

$$(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha) \ \in \ C^2\big(\mathfrak{M}\,;\,\mathrm{U}(1)_M\big) \oplus C^1\big(\mathfrak{M}\,;\,\Omega_M^1\big) \oplus C^0\big(\mathfrak{M}\,;\,\Omega_M^2\big)$$

satisfying the cocycle conditions

$$
\begin{aligned}
g_{\alpha\beta\gamma}\,g_{\beta\gamma\delta}^{-1}\,g_{\gamma\delta\alpha}\,g_{\delta\alpha\beta}^{-1} &= 1 &&\text{on}\quad M_{\alpha\beta\gamma\delta}\ , \\
A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} &= \mathrm{d}\log g_{\alpha\beta\gamma} &&\text{on}\quad M_{\alpha\beta\gamma}\ , \\
B_\alpha - B_\beta &= \mathrm{d}A_{\alpha\beta} &&\text{on}\quad M_{\alpha\beta}\ . &&\text{(2.19)}
\end{aligned}
$$

A Deligne two-coboundary defines a gauge transformation and is a triple of the type

$$\big(h_{\alpha\beta}\,h_{\beta\gamma}\,h_{\gamma\alpha}\,,\,\mathrm{d}\log h_{\alpha\beta} + a_\alpha - a_\beta\,,\,\mathrm{d}a_\alpha\big)$$

for $(h_{\alpha\beta}, a_\alpha) \in C^1(\mathfrak{M}; \mathrm{U}(1)_M) \oplus C^0(\mathfrak{M}; \Omega_M^1)$. The $\mathrm{U}(1)$ Čech two-cocycle $g_{\alpha\beta\gamma} : M_{\alpha\beta\gamma} \to \mathrm{U}(1)$ specifies a hermitian "transition" line bundle $L_{\alpha\beta}$ over each overlap $M_{\alpha\beta}$, an isomorphism $L_{\alpha\beta} \cong$

$L_{\beta\alpha}^*$, and a trivialization of the line bundle $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ on each triple overlap $M_{\alpha\beta\gamma}$; the pair $\mathscr{G} = (L_{\alpha\beta}, g_{\alpha\beta\gamma})$ defines a gerbe on M. This cocycle represents the Dixmier–Douady class $dd(\mathscr{G}) = [g_{\alpha\beta\gamma}] \in \mathrm{H}^2(M; \mathrm{U}(1)) = \mathrm{H}^3(M; \mathbb{Z})$; it is the obstruction to triviality of the gerbe $\mathscr{G} \downarrow M$. The Čech one-cochain $A_{\alpha\beta}$ defines connection one-forms on each line bundle $L_{\alpha\beta} \to M_{\alpha\beta}$ such that the section $g_{\alpha\beta\gamma}$ is covariantly constant with respect to the induced connection on $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$; it defines a 0-connection (or connective structure) on the gerbe $\mathscr{G}$. The collection of two-forms $B_\alpha \in \Omega^2(M_\alpha)$ defines a 1-connection (or curving) on $\mathscr{G}$. The pair $(A_{\alpha\beta}, B_\alpha)$ defines a 2-connection on the gerbe $\mathscr{G} = (L_{\alpha\beta}, g_{\alpha\beta\gamma})$. The gauge group of the gerbe is generated by line bundles $\ell \to M$ with connection $\nabla = \mathrm{d} + a$ and curvature $f_\nabla = \mathrm{d}a$ with $F([\ell, \nabla]) = \frac{1}{2\pi \mathrm{i}} f_\nabla \in \Omega_{\mathbb{Z}}^2(M)$ through the gauge transformations

$$ L_{\alpha\beta} \longmapsto L_{\alpha\beta} \otimes \ell\big|_{M_{\alpha\beta}}, \qquad A_{\alpha\beta} \longmapsto A_{\alpha\beta} + a\big|_{M_{\alpha\beta}} \qquad \text{and} \qquad B_\alpha \longmapsto B_\alpha + f_\nabla. $$

The Deligne cohomology $\mathrm{H}_{\mathfrak{D}}^2(M)$ applies to the higher abelian gauge theory of the Kalb–Ramond B-field of superstring theory, with M a ten-dimensional manifold.

For $p = 3$, a Deligne class is represented by a quadruple

$$ (g_{\alpha\beta\gamma\delta}, A_{\alpha\beta\gamma}, B_{\alpha\beta}, C_\alpha) \in C^3\big(\mathfrak{M}; \mathrm{U}(1)_M\big) \oplus C^2\big(\mathfrak{M}; \Omega_M^1\big) \oplus C^1\big(\mathfrak{M}; \Omega_M^2\big) \oplus C^0\big(\mathfrak{M}; \Omega_M^3\big) $$

satisfying the equations

$$
\begin{aligned}
g_{\alpha\beta\gamma\delta} \, g_{\alpha\beta\delta\kappa} \, g_{\beta\gamma\delta\kappa} \, g_{\alpha\gamma\delta\kappa}^{-1} \, g_{\alpha\beta\gamma\kappa}^{-1} &= 1 && \text{on} \quad M_{\alpha\beta\gamma\delta\kappa}, \\
A_{\alpha\beta\gamma} + A_{\alpha\gamma\delta} - A_{\alpha\beta\delta} - A_{\beta\gamma\delta} &= \mathrm{d}\log g_{\alpha\beta\gamma\delta} && \text{on} \quad M_{\alpha\beta\gamma\delta}, \\
B_{\alpha\beta} + B_{\beta\gamma} + B_{\gamma\alpha} &= \mathrm{d}A_{\alpha\beta\gamma} && \text{on} \quad M_{\alpha\beta\gamma}, \\
C_\alpha - C_\beta &= \mathrm{d}B_{\alpha\beta} && \text{on} \quad M_{\alpha\beta}.
\end{aligned}
$$

Gauge transformations are generated by the Deligne three-coboundaries

$$ \big(h_{\alpha\beta\gamma} \, h_{\alpha\gamma\delta}^{-1} \, h_{\beta\gamma\delta} \, h_{\alpha\beta\delta}^{-1}, \ \mathrm{d}\log h_{\alpha\beta\gamma} + a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha}, \ \mathrm{d}a_{\alpha\beta} + b_\alpha - b_\beta, \ \mathrm{d}b_\alpha \big) $$

for $(h_{\alpha\beta\gamma}, a_{\alpha\beta}, b_\alpha) \in C^2(\mathfrak{M}; \mathrm{U}(1)_M) \oplus C^1(\mathfrak{M}; \Omega_M^1) \oplus C^0(\mathfrak{M}; \Omega_M^2)$. This cocycle represents a 2-gerbe $\underline{\mathscr{G}} = (\mathscr{G}_{\alpha\beta\gamma}, g_{\alpha\beta\gamma\delta})$ with 3-connection $(A_{\alpha\beta\gamma}, B_{\alpha\beta}, C_\alpha)$ [51, 38, 9]. The Deligne cohomology $\mathrm{H}_{\mathfrak{D}}^3(M)$ is the one appropriate to the abelian gauge theory of the three-form C-field of M-theory, with M an 11-dimensional manifold.

Holonomy and curvature

The construction of holonomy and curvature of a Deligne class defines an isomorphism [8]

$$ \mathrm{H}_{\mathfrak{D}}^p(M) \xrightarrow{\approx} \check{\mathrm{H}}^{p+1}(M). $$

Given a degree 1 smooth Deligne class $[(g_{\alpha\beta}, A_\alpha)]$ represented by a hermitian line bundle $L \to M$ with unitary connection $\nabla = \mathrm{d} + A$, the curvature is the globally defined two-form given by $F_\nabla = \mathrm{d}A_\alpha$ on M_α with $F([L, \nabla]) = \frac{1}{2\pi \mathrm{i}} F_\nabla \in \Omega_{\mathbb{Z}}^2(M)$. By Stokes' theorem, the holonomy of ∇ around any one-cycle $\gamma \subset M$ is then obtained from the product formula [35]

$$ \mathrm{hol}_\gamma(A) = \prod_{\alpha \in I} \exp\left(\mathrm{i} \int_{\gamma_\alpha} A_\alpha \right) \prod_{\alpha, \beta \in I} g_{\alpha\beta}(\gamma_{\alpha\beta}), $$

where $\gamma_\alpha \subset M_\alpha$ is a path in a subdivision of the loop γ into segments and $\gamma_{\alpha\beta} = \gamma_\alpha \cap \gamma_\beta$ is a point in $M_{\alpha\beta}$. This definition agrees with the general definition of holonomy (2.18) in terms of differential characters.

Given a degree 2 smooth Deligne class $[(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)]$ represented by a gerbe $\mathscr{G} \downarrow M$ with 2-connection (A, B), the curvature of the corresponding differential character is the globally defined closed three-form $H = H_{(A,B)}$ given by $H = \mathrm{d}B_\alpha$ on M_α with $F([\mathscr{G}, A, B]) = \frac{1}{2\pi\,\mathrm{i}}\, H \in \Omega^3_{\mathbb{Z}}(M)$, while its characteristic class is the Dixmier–Douady class $dd(\mathscr{G}) \in \mathrm{H}^3(M; \mathbb{Z})$ of the gerbe $\mathscr{G}$. Its holonomy around a two-cycle $\Sigma \subset M$ is obtained by choosing a triangulation $\{\Sigma_\alpha\}_{\alpha \in I}$ of Σ subordinate to the open cover $\Sigma \cap \mathfrak{M}$. Keeping careful track of orientations, by repeated application of Stokes' theorem one arrives at the product formula [8, 35]

$$\mathsf{hol}_\Sigma(B) = \prod_{\alpha \in I} \exp\left(\mathrm{i} \int_{\Sigma_\alpha} B_\alpha\right) \prod_{\alpha,\beta \in I} \exp\left(\mathrm{i} \int_{\Sigma_{\alpha\beta}} A_{\alpha\beta}\right) \prod_{\alpha,\beta,\gamma \in I} g_{\alpha\beta\gamma}(\Sigma_{\alpha\beta\gamma})$$

where $\Sigma_{\alpha\beta}$ is the common boundary edge of the surfaces Σ_α and Σ_β, and $\Sigma_{\alpha\beta\gamma} = \Sigma_{\alpha\beta} \cap \Sigma_{\beta\gamma} \cap \Sigma_{\gamma\alpha}$ are vertices of the triangulation of Σ. The coincidence between this expression and the general formula in terms of differential characters (2.18) is shown explicitly in [17]. This construction first appeared in the context of the Wess–Zumino–Witten model in [34].

For $p = 3$, the smooth Deligne class $[(g_{\alpha\beta\gamma\delta}, A_{\alpha\beta\gamma}, B_{\alpha\beta}, C_\alpha)]$ of a 2-gerbe $\mathscr{G} \downarrow\downarrow M$ with 3-connection (A, B, C) has curvature G given by $G = \mathrm{d}C_\alpha$ on M_α with $F([\mathscr{G}, A, B, C]) = \frac{1}{2\pi\,\mathrm{i}}\, G \in \Omega^4_{\mathbb{Z}}(M)$, and a degree 4 characteristic class $[g_{\alpha\beta\gamma\delta}] \in \mathrm{H}^4(M; \mathbb{Z})$. For a smooth three-cycle $\Sigma \subset M$, choose a triangulation subordinate to the open cover $\mathfrak{M}$ used to define the Deligne class; it consists of tetrahedra Σ_α, faces $\Sigma_{\alpha\beta}$, edges $\Sigma_{\alpha\beta\gamma}$, and vertices $\Sigma_{\alpha\beta\gamma\delta}$. Then the holonomy around Σ is given by [38]

$$\begin{aligned}
\mathsf{hol}_\Sigma(C) &= \prod_{\alpha \in I} \exp\left(\mathrm{i} \int_{\Sigma_\alpha} C_\alpha\right) \prod_{\alpha,\beta \in I} \exp\left(\mathrm{i} \int_{\Sigma_{\alpha\beta}} B_{\alpha\beta}\right) \prod_{\alpha,\beta,\gamma \in I} \exp\left(\mathrm{i} \int_{\Sigma_{\alpha\beta\gamma}} A_{\alpha\beta\gamma}\right) \\
&\times \prod_{\alpha,\beta,\gamma,\delta \in I} g_{\alpha\beta\gamma\delta}(\Sigma_{\alpha\beta\gamma\delta}) \, .
\end{aligned}$$

A general Deligne p-cocycle is represented by

$$\left(g_{\alpha_1 \dots \alpha_{p+1}}, A^1_{\alpha_1 \dots \alpha_p}, \dots, A^p_\alpha\right) \ \in \ C^p\big(\mathfrak{M}; \mathrm{U}(1)_M\big) \ \oplus \ \bigoplus_{k=1}^{p} C^{p-k}\big(\mathfrak{M}; \Omega^k_M\big) \, .$$

To compute its holonomy, triangulate a smooth p-cycle $\Sigma \subset M$ by a p-dimensional simplicial complex $\mathfrak{S}(\Sigma)$; the k-simplices of $\mathfrak{S}(\Sigma)$ are denoted σ^k for $k = 0, 1, \dots, p$. Let $\rho : \mathfrak{S}(\Sigma) \to I$ be the index map for the triangulation [35]. Then the holonomy around Σ is given by [38]

$$\mathsf{hol}_\Sigma(A^p) = \prod_{k=1}^{p} \prod_{\underline{\sigma}^k \in \mathfrak{S}(\Sigma)} \exp\left(\mathrm{i} \int_{\sigma^k} A^k_{\rho(\sigma^p) \dots \rho(\sigma^k)}\right) \prod_{\underline{\sigma}^0 \in \mathfrak{S}(\Sigma)} g_{\rho(\sigma^p) \dots \rho(\sigma^1)\rho(\sigma^0)}(\sigma^0) \, ,$$

where the products are taken over *flags* of simplices $\underline{\sigma}^k := \{(\sigma^k, \sigma^{k+1}, \dots, \sigma^p) \mid \sigma^k \subset \sigma^{k+1} \subset \dots \subset \sigma^p\}$ for $k = 0, 1, \dots, p$.

2.4. Configuration space and gauge transformations

As in Maxwell theory (or more generally in Yang–Mills theory), locality forces us to work with *gauge potentials* A, rather than with isomorphism classes of gauge fields F. The most convenient

mathematical framework for dealing with local quantum field theory is through categorification, wherein we work directly at the level of cochain complexes; this enables one to build a quantum field theory by "gluing" elementary constituents together. In particular, following the treatment of the configuration space of semi-classical Maxwell theory, we seek a suitable groupoid of higher abelian gauge fields which plays the role of the semi-classical configuration space. This framework also leads to a quantum definition of charge.

Quantum groupoid of fields

We seek a groupoid $\mathscr{H}^p(M)$ whose objects are gauge potentials $\check{A}$, whose morphisms are "gauge transformations", and whose set of isomorphism classes are gauge equivalence classes, so that

$$\pi_0 \mathscr{H}^p(M) \cong \check{\mathrm{H}}^p(M) \ .$$

In particular, every object has a group of automorphisms $\mathrm{H}^{p-2}(M; \mathbb{T})$. The particular sort of groupoid that we want is an example of an *action groupoid*, obtained by the action of a gauge group on a set of objects: Given an action $G \times X \to X$ of a group G on a set X, there is a groupoid $G \rightrightarrows X$ whose objects are the points $x \in X$ and whose morphisms are the group actions $x \xrightarrow{g} x' = g \triangleright x$ for $g \in G$ and $x \in X$. When X is e.g. a smooth manifold, this groupoid is sometimes denoted $[X/G]$ and called a *quotient stack*; it is naturally related to the orbifold X/G. The desired properties of the category $\mathscr{H}^p(M)$ can be described as follows.

Firstly, the gauge transformation law on corresponding characters $\chi_{\check{A}}$ is given on p-chains Σ with $\partial \Sigma \neq 0$ as

$$\chi_{\check{A}}(\Sigma) \ \longmapsto \ \check{\chi}(\partial \Sigma) \, \chi_{\check{A}}(\Sigma) \qquad \text{with} \quad \check{\chi}(\partial \Sigma) = \exp\left(\mathrm{i} \int_\Sigma F_{\check{\chi}} \right) \ .$$

Hence gauge transformations are given by differential characters $\check{\chi} \in \check{\mathrm{H}}^{p-1}(M)$; the action of $[\check{C}] \in \check{\mathrm{H}}^{p-1}(M)$ on a gauge potential $\check{A}$ is given by

$$g_{[\check{C}]} \triangleright \check{A} = \check{A} + F\big([\check{C}]\big) \ . \tag{2.20}$$

The groupoid of morphisms is thus the gauge group $\check{\mathrm{H}}^{p-1}(M)$.

Secondly, the connected components $\check{Z}_c^p(M)$ of the space of objects $\check{Z}^p(M)$ are labelled by the charge group $c \in \mathrm{H}^p(M; \mathbb{Z})$, each taken as a torsor for $\Omega^{p-1}(M)$. From (2.20) it follows that the flat characters in $\mathrm{H}^{p-2}(M; \mathbb{T})$ act trivially on the space of gauge fields $\mathscr{H}^p(M)$, and hence the group of automorphisms of any object $\check{A}$ is

$$\mathsf{Aut}_{\mathscr{H}^p(M)}(\check{A}) = \mathrm{H}^{p-2}(M; \mathbb{T}) \ .$$

On $M = \mathbb{R} \times N$, an automorphism $\alpha \in \mathrm{H}^{p-2}(N; \mathbb{T})$ acts on "wavefunctionals" $\psi(\check{A})$ in the quantum Hilbert space $\mathcal{H}(N)$ of the gauge theory via elements of the quantum electric charge group $Q \in \mathrm{H}^{n-p+2}(N; \mathbb{Z})$ as $\alpha \triangleright \psi(\check{A}) = \mathrm{e}^{2\pi \mathrm{i} \langle \alpha, Q \rangle} \psi(\check{A})$; the precise definition of the Hilbert space $\mathcal{H}(N)$ is the subject of §4.5. In each topological sector $c \in \mathrm{H}^p(M; \mathbb{Z})$ we choose a field strength F_c. Then an arbitrary field strength in this sector can be written as $F = F_c + \mathrm{d}a$ for $a \in \Omega^{p-1}(M)$; these are the oscillator modes. Gauge transformations act through $a \mapsto a + \omega$ with $\omega \in \Omega_{\mathbb{Z}}^{p-1}(M)$; if $\omega = \mathrm{d}\varepsilon$ is exact then this is called a *small gauge transformation* and there are also gauge transformations of the gauge transformations given by $\varepsilon \mapsto \varepsilon + \eta$ with $\eta \in \Omega_{\mathbb{Z}}^{p-2}(M)$.

The requisite quantum groupoid then has connected components given by the action groupoids $\mathscr{H}_c^p(M) = \big[\check{Z}_c^p(M) \,\big/\, \check{\mathrm{H}}^p(M) \big]$.

Category of differential cocycles

An explicit model for the category $\check{\mathscr{H}}^p(M)$ is described in [37]. In this formulation the space of objects $\check{Z}^p(M)$ of $\check{\mathscr{H}}^p(M)$ are *cocycles*; these are triples

$$(c, h, \omega) \ \in \ \check{C}^p(M) := C^p(M; \mathbb{Z}) \times C^{p-1}(M; \mathbb{R}) \times \Omega^p(M)$$

satisfying the cocycle condition

$$\mathrm{d}_{\check{\mathrm{H}}}(c, h, \omega) := (\delta c, \omega - c - \delta h, \mathrm{d}\omega) = (0, 0, 0)$$

with $\mathrm{d}_{\check{\mathrm{H}}}^2 = 0$. Here we think of c as the characteristic class $c(\chi)$ of a differential character χ, ω as its field strength F_χ, and h as the "monodromy" $\log \chi$. Note that the connected components of the space of objects of the category are indeed labelled by the charge group $\mathrm{H}^p(M; \mathbb{Z})$. Two objects are connected by a morphism $(c_1, h_1, \omega_1) \longrightarrow (c_2, h_2, \omega_2)$ if and only if

$$(c_1, h_1, \omega_1) = (c_2, h_2, \omega_2) + \mathrm{d}_{\check{\mathrm{H}}}(b, k, 0)$$

for some $(b, k) \in C^{p-1}(M; \mathbb{Z}) \times C^{p-2}(M; \mathbb{R})$ subject to the equivalence relation

$$(b, k, 0) \sim (b, k, 0) - \mathrm{d}_{\check{\mathrm{H}}}(b', k', 0) \qquad \text{for} \quad (b', k') \in C^{p-2}(M; \mathbb{Z}) \times C^{p-3}(M; \mathbb{R}) \ .$$

Then the set of isomorphism classes of objects in the category $\check{\mathscr{H}}^p(M)$ is isomorphic to the Cheeger–Simons differential cohomology group $\check{\mathrm{H}}^p(M)$. Moreover, the automorphism group of the trivial object is given by

$$\mathsf{Aut}_{\check{\mathscr{H}}^p(M)}(0, 0, 0) \cong \mathrm{H}^{p-2}(M; \mathbb{T}) \ ,$$

and whence the flat characters in $\mathrm{H}^{p-2}(M; \mathbb{T})$ act trivially on the configuration space of abelian gauge fields $\check{\mathscr{H}}^p(M)$. However, in this model there is no groupoid decomposition $\check{Z}^p(M) = \bigsqcup_{c \in \mathrm{H}^p(M; \mathbb{Z})} \check{Z}^p_c(M)$ into topological sectors.

For $p = 0$, $\check{\mathscr{H}}^0(M)$ is the category of maps $M \to \mathbb{Z}$ and identity morphisms.

For $p = 1$, $\check{\mathscr{H}}^1(M)$ is the category of smooth maps $M \to \mathrm{U}(1)$ and identity morphisms; to each object $(c, h, \omega) \in C^1(M; \mathbb{Z}) \times C^0(M; \mathbb{R}) \times \Omega^1(M)$ we associate its differential character.

For $p = 2$, $\check{\mathscr{H}}^2(M)$ is the monoidal category of $\mathrm{U}(1)$-bundles with connection, with groupoid structure of tensor product. An object $\check{A} = (c, h, \omega) \in \check{\mathscr{H}}^2(M)$ determines a $\mathrm{U}(1)$-bundle with connection in the following way: For each open set $U \subset M$, a principal homogeneous space $\Gamma(U)$ for the group of isomorphism classes in the group $\check{\mathscr{H}}^1(U)$ of smooth maps $U \to \mathrm{U}(1)$ is given by the set of trivialising "scalar potentials" $\check{\sigma} = (b, k, \theta) \in C^1(U; \mathbb{Z}) \times C^0(U; \mathbb{R}) \times \Omega^1(U)$ with $\mathrm{d}_{\check{\mathrm{H}}}\check{\sigma} = \check{A}$ modulo the equivalence relation $\check{\sigma} \sim \check{\sigma} + \mathrm{d}_{\check{\mathrm{H}}}\check{\tau}$ for $\check{\tau} \in C^1(U; \mathbb{Z}) \times \{0\} \times \Omega^0(U)$; any two potentials in $\Gamma(U)$ differ by an object of the category $\check{\mathscr{H}}^1(M)$. The connection $\nabla : \Gamma(U) \to \Omega^1(U)$ then sends $\check{\sigma} \mapsto \theta$.

Quantum 2-groupoid of fields

The model of differential cocycles illustrates that the construction of the category $\check{\mathscr{H}}^p(M)$ may be iterated to give higher categories as well, by the well-known process of categorification, which works for any cochain complex: We replace the equivalence relation on morphisms with 2-morphisms, and so on. These higher morphisms are necessary to obtain a fully local gauge theory with local gauge symmetries as well; they typically appear in the quantization of reducible gauge systems with large

symmetries, for example in Batalin–Vilkovisky quantization the off-shell higher gauge symmetries require introduction of both ghost fields and ghosts-for-ghosts. We recall here the pertinent category theory definitions; see e.g. [1] for further details and references.

Here we shall only consider the first member of this higher categorical hierarchy. A *2-category* consists of a set of objects $\check{A}$ (gauge potentials), a set of morphisms $g : \check{A} \longrightarrow \check{B}$ from a source potential $\check{A}$ to a target potential $\check{B}$ (gauge transformations), and a set of 2-morphisms $\alpha : g \Longrightarrow h$ from a source gauge transformation $g : \check{A} \longrightarrow \check{B}$ to a target gauge transformation $h : \check{A} \longrightarrow \check{B}$ (gauge-for-gauge transformations). The composition of gauge transformations $g : \check{B} \longrightarrow \check{C}$ and $h : \check{A} \longrightarrow \check{B}$ is written $g \circ h : \check{A} \longrightarrow \check{C}$; it defines an associative multiplication with unit the identity gauge transformation $\mathbb{1}_{\check{A}} : \check{A} \longrightarrow \check{A}$ for all gauge potentials $\check{A}$. For the gauge-for-gauge transformations there are two types of composition. The "vertical" composition of 2-morphisms $\alpha : g \Longrightarrow g'$ and $\alpha' : g' \Longrightarrow g''$ between gauge transformations $g, g', g'' : \check{A} \longrightarrow \check{B}$ with the same source and target is denoted $\alpha \cdot \alpha' : g \Longrightarrow g''$; it defines an associative multiplication with unit the identity vertical gauge-for-gauge transformation $\mathbb{1}_g : g \Longrightarrow g$ for all gauge transformations $g : \check{A} \longrightarrow \check{B}$. The "horizontal" composition of 2-morphisms $\alpha_1 : g_1 \Longrightarrow g_1'$ and $\alpha_2 : g_2 \Longrightarrow g_2'$ between gauge transformations $g_1, g_1' : \check{B} \longrightarrow \check{C}$ and $g_2, g_2' : \check{A} \longrightarrow \check{B}$ is written $\alpha_1 \circ \alpha_2 : g_1 \circ g_2 \Longrightarrow g_1' \circ g_2'$, where $g_1 \circ g_2, g_1' \circ g_2' : \check{A} \longrightarrow \check{C}$; again it gives an associative multiplication with unit the identity horizontal gauge-for-gauge transformation $\mathbb{1}_{\mathbb{1}_{\check{A}}} : \mathbb{1}_{\check{A}} \Longrightarrow \mathbb{1}_{\check{A}}$ for all gauge potentials $\check{A}$. The two compositions of 2-morphisms obey the interchange law

$$(\alpha_1' \cdot \alpha_1) \circ (\alpha_2' \cdot \alpha_2) = (\alpha_1' \circ \alpha_2') \cdot (\alpha_1 \circ \alpha_2) \ . \tag{2.21}$$

A *2-groupoid* structure on this 2-category is obtained by requiring that every gauge transformation $g : \check{A} \longrightarrow \check{B}$ has an inverse $g^{-1} : \check{B} \longrightarrow \check{A}$ such that $g^{-1} \circ g = \mathbb{1}_{\check{A}}$ and $g \circ g^{-1} = \mathbb{1}_{\check{B}}$, while every gauge-for-gauge transformation $\alpha : g \Longrightarrow h$ has a vertical inverse $\alpha_v^{-1} : h \Longrightarrow g$ such that $\alpha_v^{-1} \cdot \alpha = \mathbb{1}_g$ and $\alpha \cdot \alpha_v^{-1} = \mathbb{1}_h$, and a horizontal inverse $\alpha_h^{-1} : g^{-1} \Longrightarrow h^{-1}$ such that $\alpha_h^{-1} \circ \alpha = \mathbb{1}_{\mathbb{1}_{\check{A}}}$ and $\alpha \circ \alpha_h^{-1} = \mathbb{1}_{\mathbb{1}_{\check{B}}}$.

The corresponding set of 2-isomorphism classes now has the structure of a *2-group* $\mathscr{G}$, which is a 2-groupoid with one object; the morphisms g form a group G under composition whose unit element is the identity morphism, while the 2-morphisms $\alpha : g \Longrightarrow g'$ for $g, g' \in G$ form a group under horizontal composition and can be composed vertically with the two compositions tied together via the interchange law (2.21). Equivalently, a 2-group is a groupoid $\mathscr{G}$ equipped with a monoidal structure that obeys the usual group axioms; in many cases of interest we require that the axioms (and in general all associativity conditions on the category) hold only in a "weak" sense, i.e. up to natural isomorphism imposed e.g. by equivalence relations.

An important classification result states that 2-groups are the same things as *crossed modules* [1], which consist of pairs of groups G, H together with an action of G on H by automorphisms and a group homomorphism $\mathtt{t} : H \to G$ which is G-equivariant, i.e.

$$\mathtt{t}(g \triangleright h) = g \, \mathtt{t}(h) \, g^{-1}$$

for all $g \in G$ and $h \in H$, and which satisfies the Peiffer identity

$$\mathtt{t}(h) \triangleright h' = h \, h' \, h^{-1}$$

for all $h, h' \in H$. Given a 2-group $\mathscr{G}$, we construct a crossed module by taking G to be the group of morphisms of $\mathscr{G}$, H as the group of 2-morphisms of $\mathscr{G}$ whose source is the identity morphism, the homomorphism $\mathtt{t} : H \to G$ is defined by sending a 2-morphism to its target morphism, and the G-action on H is defined by $g \triangleright h = \mathbb{1}_g \circ h \circ \mathbb{1}_{g^{-1}}$. In this classification the 2-morphisms of $\mathscr{G}$ form the crossed-product group $G \ltimes H$ under horizontal composition: 2-morphisms $g \Longrightarrow g'$ are equivalent

to pairs $(g, h) \in G \times H$ with $g' = \mathtt{t}(h)\, g$. Moreover, the vertical composition of $g \Longrightarrow \tilde{g} = \mathtt{t}(h)\, g$ and $g' \Longrightarrow \tilde{g}' = \mathtt{t}(h')\, g'$ is given by

$$(g, h) \cdot (g', h') = (g, h'\, h)$$

for composable 2-morphisms, i.e. when $g' = \mathtt{t}(h)\, g$ so that $\tilde{g}' = \mathtt{t}(h')\, \mathtt{t}(h)\, g = \mathtt{t}(h'\, h)\, g$, while horizontal composition can be represented as

$$(g, h) \circ (g', h') = \big(g\, g',\; h\, (g \triangleright h')\big)\, .$$

In the "weak" case one replaces the homomorphism $\mathtt{t} : H \to G$ with an element $[a]$ of the group cohomology $\mathrm{H}^3(G, H)$ which comes from the associator isomorphism a on $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ [1].

For example, define the *shifted* group $\mathrm{bU}(1)$ as the Lie 2-group where $G = \mathbb{1}$ is the trivial group, $H = \mathrm{U}(1)$, and $\mathtt{t}$ is the trivial homomorphism. Then a principal $\mathrm{bU}(1)$-2-bundle with 2-connection, i.e. an object in the 2-category of $\mathrm{U}(1)$-bundles with connection, is a gerbe with 2-connection [2]. On a trivial gerbe, a 2-connection is a globally defined two-form $B \in \Omega^2(M)$; in this case the holonomies of paths, regarded as morphisms between points $x \in M$ in the path 2-groupoid of the smooth manifold M, are all trivial while the 2-holonomies over two-chains $\Sigma \subset M$ with $\partial \Sigma \neq 0$, regarded as 2-morphisms in the path 2-groupoid, give elements $\exp\big(\mathrm{i} \int_\Sigma B\big) \in \mathrm{U}(1)$.

Currents

In the non-vacuum theory, i.e. when there are currents present, the description of the configuration space changes somewhat. Consider a differential cocycle

$$\check{j} = (c, h, \omega) \;\in\; \check{\mathscr{H}}^p(M)\, ,$$

so that $c = c(\check{j})$, $j := \omega = F(\check{j})$, and h is the "holonomy" of $\check{j}$. A *trivialization* of $\check{j}$ is a *differential cochain*

$$\check{A} \;\in\; \check{C}^{p-1}(M) = C^{p-1}(M; \mathbb{Z}) \times C^{p-2}(M; \mathbb{R}) \times \Omega^{p-1}(M) \qquad \text{with} \quad \mathrm{d}_{\check{\mathrm{H}}} \check{A} = \check{j}\, .$$

Associated to $\check{A}$ is a differential form $A \in \Omega^{p-1}(M)$ such that $\mathrm{d}A = F(\check{j}) = j \in \Omega^p_{\mathrm{cl}}(M)$.

Starting from a current j, we consider a *refinement* of j to a differential cocycle $\check{j}$ such that $F(\check{j}) = j$, and regard the configuration space of higher abelian gauge fields as the set of trivializations of $[\check{j}] \in \check{\mathrm{H}}^p(M)$. The set of all such trivializations is a torsor for the group $\check{\mathrm{H}}^p(M)$, and the group of gauge transformations is isomorphic to $\check{\mathrm{H}}^{p-1}(M)$.

2.5. Generalized abelian gauge theory

A *generalized differential cohomology theory* $\check{\mathrm{E}}^\bullet$ is a geometric refinement of a generalized cohomology theory $\mathrm{E}^\bullet$ on the category of smooth manifolds. For any manifold M, it completes the pullback square

$$
\begin{array}{ccc}
\check{\mathrm{E}}^\bullet(M) & \longrightarrow & \Omega_{\mathrm{cl}}(M; E^\bullet)^\bullet \\
\downarrow & & \downarrow {\scriptstyle [-]_{\mathrm{dR}}} \\
\mathrm{E}^\bullet(M) & \underset{\varphi}{\longrightarrow} & \mathrm{H}(M; E^\bullet)^\bullet
\end{array}
\qquad (2.22)
$$

where $E^\bullet := \mathrm{E}^\bullet(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a graded real vector space, the image of the morphism φ is a full lattice while its kernel is the torsion subgroup $\ker \varphi = \mathsf{Tor}\, \mathrm{E}^\bullet(M)$ (in particular $\varphi \otimes \mathbb{R}$ is a group isomorphism), and the grading on ordinary cohomology is given by the total degree

$$\mathrm{H}(M; E^\bullet)^d := \bigoplus_{p+q=d} \mathrm{H}^p(M; E^q)\, .$$

Theorem 2.23. *Differential cohomology theories exist for any generalized cohomology theory.*

The proof of Thm. 2.23 can be found in [37] and it consists in replacing differential cocycles by certain maps to the corresponding classifying spaces $B\,\mathrm{E}$, the collection of which are called "differential function spaces": One now considers triples (c, h, ω) where $c : M \to B\,\mathrm{E}$ is a smooth map, $h \in C^\bullet(M)$ is a cochain on M, and $\omega \in \Omega^\bullet_{\mathrm{cl}}(M)$ is a closed differential form on M such that $\delta h = \omega - c^* i$ for some cocycle $i \in Z^\bullet(B\,\mathrm{E})$, together with homotopy equivalence relations. Existence and uniqueness of generalized differential cohomology theories is also investigated by [13, 14]. The groups $\check{\mathrm{E}}^d(M)$ have analogous properties to those of the Cheeger–Simons groups, including:

(i) The connected components of $\check{\mathrm{E}}^d(M)$ are labelled by the charge group $\pi_0 \check{\mathrm{E}}^d(M) = \mathrm{E}^d(M)$.

(ii) There is a field strength map $F : \check{\mathrm{E}}^d(M) \to \Omega_{\mathrm{cl}}(M; E^\bullet)^d$.

(iii) There is a torus $\mathbb{T}^d_{\mathrm{E}}(M) = \mathrm{E}^{d-1}(M; \mathbb{T})$ of flat fields in $\check{\mathrm{E}}^d(M)$, where E-cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$ is constructed using stable homotopy theory in terms of a spectrum in such a way that it fits into natural long exact sequences

$$\cdots \longrightarrow \mathrm{E}^{d-1}(M) \longrightarrow \mathrm{E}^{d-1}(M) \otimes \mathbb{R} \longrightarrow \mathrm{E}^{d-1}(M; \mathbb{T}) \longrightarrow \mathrm{E}^d(M) \longrightarrow \cdots .$$

See [14] and [16, §2.3] for conditions under which there is an isomorphism $\mathbb{T}^d_{\mathrm{E}}(M) \cong \ker(F)$ of cohomology theories, and [37] for an explicit realization of the isomorphism.

Most of what we have said before concerning (higher) abelian gauge theories have analogs in this more general setting. A *generalized abelian gauge theory* is determined by a multiplicative generalized cohomology theory $\mathrm{E}^\bullet$ with invertible closed differential forms

$$\omega_M = \sqrt{F\big([\mathtt{or}_{\mathrm{E}}(M)]\big)} \tag{2.24}$$

depending functorially and locally on M that normalize the morphism φ in (2.22) such that the intersection pairing $\mathrm{E}^\bullet(M) \otimes \mathrm{E}^\bullet(M) \to \mathbb{Z}$ becomes compatible with the integration of curvatures. Here $[\mathtt{or}_{\mathrm{E}}(M)] \in \check{\mathrm{E}}^\bullet(M)$ is a smooth E-orientation on M, and we use invertibility to regard cup products with the cohomology class of $F\big([\mathtt{or}_{\mathrm{E}}(M)]\big)$ as an invertible linear operator on $\mathrm{H}(M, E^\bullet)^\bullet$; if the cohomology is finitely-generated, then the square root (2.24) can be defined using the usual Jordan normal form. A gauge potential $\check{A} \in \check{C}^d(M)$ is a non-flat trivialization of a current $\check{j} \in \check{Z}^{d+1}(M)$ for some $d \in \mathbb{Z}$. If $\check{j} = 0$, then the potential has a class $[\check{A}] \in \check{\mathrm{E}}^d(M)$; we will mostly consider this source-free case in this article, as the gauge theory in this instance involves only free fields and so its quantization can be carried out in a rigorous way, e.g. along the lines discussed in [31, 32]. The gauge field $F = F([\check{A}])$ is a differential form of total degree d on spacetime, i.e. an element

$$F \in \Omega(M; E^\bullet)^d = \bigoplus_{k \in \mathbb{Z}} \Omega^k(M; E^{d-k}) .$$

In this paper we are primarily interested in two particular physical applications of this general construction.

- When $\mathrm{E}^\bullet = \mathrm{H}^\bullet$ is ordinary cohomology, this construction gives the differential cohomology $\check{\mathrm{H}}^\bullet$ considered in this section. In this case $H^\bullet = \mathbb{R}$ and φ is the map $\mathrm{H}^\bullet(M; \mathbb{Z}) \hookrightarrow \mathrm{H}^\bullet(M; \mathbb{R})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ of abelian groups with $F\big([\mathtt{or}_{\mathrm{H}}(M)]\big) = 1$. This is the model that is appropriate to higher abelian gauge theory with extended $p-1$-brane charges, e.g. the fluxes of supergravity.

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- When $E^\bullet = R^{-1}$ is the version of connective KO-theory defined by its truncated Postnikov spectrum of real vector spaces at degree four, which sits inside an exact sequence

$$0 \longrightarrow H^3(M;\mathbb{Z}) \longrightarrow R^{-1}(M) \longrightarrow \begin{array}{c} H^0(M;\mathbb{Z}_2) \\ \oplus \\ H^1(M;\mathbb{Z}_2) \end{array} \longrightarrow 0 \,,$$

  the corresponding smooth refinement $\check{R}^{-1}$ can be modelled by a 2-groupoid consisting of invertible elements of a symmetric monoidal 2-category whose objects are $\mathbb{C}$-algebras, morphisms are $\mathbb{Z}_2$-graded algebra bimodules, 2-morphisms are intertwiners between bimodules, and monoidal structure provided by tensor product of $\mathbb{C}$-algebras. This model consistently reconciles the target space and worldsheet field theories of the B-field; the three-form H-flux in Type II superstring theory is Dirac quantized by a "differential B-field" which is an object of $\check{R}^{-1}(M)$, whose charge group classifies the twistings of the complex K-theory of orbifolds. See [23, 24] for details.

- When $E^\bullet = K^\bullet$ is complex K-theory, this construction gives the differential K-theory $\check{K}^\bullet$ of vector bundles with connection on M. Here $K^\bullet = \mathbb{R}(u)$ with u^{-1} the Bott generator of degree $\deg(u^{-1}) = -2$, and φ is the modified Chern character from K-theory to real cohomology with normalising differential form $F([\mathsf{or}_K(M)]) = \widehat{A}(M)$ the Atiyah–Hirzebruch class of the tangent bundle of M. This is the model that is appropriate to Ramond–Ramond gauge theory with D-brane charges in Type II superstring theory. In this case the p-forms F of §2.3. correspond to Ramond–Ramond fields and the submanifolds W_e to the worldvolumes of D-branes; now the charges q_e are classes in the complex K-theory group $K^0(W_e)$ and their pushforwards under the embedding $W_e \hookrightarrow M$ gives the Ramond–Ramond charge $[j_e]$ of a D-brane wrapping W_e. These considerations are the subject of the next section.

3   Ramond–Ramond fields and differential K-theory

3.1.   D-branes and K-cycles

We begin with a brief mathematical introduction to D-branes in Type II superstring theory; see [53] for further details and references. A D-brane is a suitable boundary condition for the Euler–Lagrange equations in a two-dimensional superconformal field theory on an open oriented Riemann surface Σ. It is realized as a submanifold $W \subset M$ of spacetime onto which "open strings attach". Topologically, open strings are the relative maps

$$(\Sigma, \partial\Sigma) \longrightarrow (M, W) \,.$$

Compatibility with superconformal invariance constrains the allowed "worldvolumes" W, e.g. in the absence of H-flux, Freed–Witten anomaly cancellation requires that W be a spin^c manifold (and hence K-oriented).

D-branes actually have more structure: The worldvolume W carries a complex "Chan–Paton vector bundle" E with connection ∇; here the rank of E is the "number of D-branes wrapping" $W \subset M$. Hence the "charge" of a D-brane may be naturally considered as the complex K-theory class $[E] \in K^0(W)$.

However, in analogy with the charges in higher abelian gauge theory, a homology theory is more natural in the description of D-brane charges (see e.g. [52]); here we shall use the Baum–Douglas geometric formulation of K-homology [3], which encodes important physical aspects of D-branes in Type I and Type II string theory [48, 53, 49]. Most importantly, D-branes are dynamical

objects that "interact with" or "couple to" the Ramond–Ramond fields of supergravity; this requires considerations from differential K-theory which we explain later on. For the moment, we briefly highlight a few of the salient features of this topological description.

Let M be a ten-dimensional spin manifold; a choice of spin structure enables us to introduce fermions. Sometimes the manifold M can also refer only to a subspace of spacetime, for example when one considers string compactifications; in that case $m = \dim(M) < 10$. We suppose that there is no background H-flux, which means that we can use untwisted cohomology theories.

Definition 3.1. *A* D-brane *in* M *is a Baum–Douglas K-cycle* (W, E, f)*, where* $f : W \hookrightarrow M$ *is a closed spin^c submanifold called the* worldvolume*, and* $E \to W$ *is a complex vector bundle with connection called the* Chan–Paton gauge bundle*.*

Note that the Chan–Paton bundle defines a *stable* element of $\mathrm{K}^0(W)$. The collection of D-branes described in this way forms an additive category under disjoint union of K-cycles. The quotient by Baum–Douglas "gauge equivalence" is isomorphic to the analytic K-homology of M; the isomorphism classes of K-cycles generate the geometric K-homology $\mathrm{K}_\bullet(M)$. In this way D-branes naturally provide K-homology classes on M which are dual to K-theory classes $f_![E] \in \mathrm{K}^d(M)$, where $f_!$ is the Gysin map in K-theory and d is the codimension of W in M. There is a natural $\mathbb{Z}_2$-grading on K-cycles given by the parity of the dimension of the worldvolume; the K-cycle (W, E, f) is odd in Type IIA string theory and even in Type IIB string theory.

Here we have used the usual physical intuition of a D-brane as an embedded submanifold $S \subset M$ of spacetime. However, not all D-branes (regarded as consistent boundary conditions in the underlying boundary conformal field theory) admit such a geometric description. When they do we say that the D-brane is "representable". The description of D-branes in terms of Baum–Douglas K-cycles for K-homology naturally requires non-representable branes [48] with arbitrary maps $f : W \to M$. A D-brane (W, E, f) is said to *wrap* $S \subset M$ if $f(W) \subset S$, and to *fill* S if $f(W) = S$.

As a consequence of the gauge equivalence relation of vector bundle modification, together with the K-theory Thom isomorphism, the class of any D-brane on W may be expressed in terms of virtual K-cycles on M through

$$[M, \mathcal{S}_E^+, \mathrm{id}_M] - [M, \mathcal{S}_E^-, \mathrm{id}_M] = \pm\, [W, E, f] \,,$$

where $\mathcal{S}_E^\pm \to M$ are twisted spinor bundles. This is the K-homology version of the *Atiyah–Bott–Shapiro construction*; the reduction of classes from the left-hand side to the right-hand side of this equation represents the standard Sen–Witten construction of D-branes through *tachyon condensation* on a brane-antibrane system wrapping the whole spacetime M [58, 46]. More generally, for a given worldvolume S the Sen–Witten construction naturally assigns D-brane charges to classes of wrapped branes in the K-homology $\mathrm{K}_\bullet(S)$ [48].

The cohomological formula for the charge of a D-brane arises physically through "anomaly cancellation" arguments. Mathematically, it arises very naturally through the *modified Chern character* $\mathsf{ch} \mapsto \mathsf{ch} \smile \sqrt{\widehat{A}(M)}$ which, by the Atiyah–Singer index theorem, is an isometric isomorphism between K-theory and cohomology groups over $\mathbb{R}$ with respect to their natural bilinear pairings. On K-theory this pairing is given by the index of the twisted Dirac operator, which coincides with the natural intersection form on boundary states and computes the chiral fermion anomaly on D-branes; on cohomology it is given by evaluation on the fundamental class. Then the *charge* of a D-brane (W, E, f) is given by

$$Q(W, E, f) = \mathsf{ch}\big(f_![E]\big) \smile \sqrt{\widehat{A}(M)} \ \in \ \mathrm{H}^\bullet(M; \mathbb{R}) \,. \tag{3.2}$$

This form of the charge vector is called the *Minasian–Moore formula* [43]. It respects the Baum–Douglas gauge equivalence relations.

3.2. Ramond–Ramond gauge theory

Let us now look at D-branes and their charges as currents in a suitable generalized abelian gauge theory. As before, let M be a ten-dimensional spin manifold. The generalized abelian gauge theory of Ramond–Ramond fields arises as a low-energy limit of Type II superstring theory from the quantum Hilbert space of states of closed superstrings on the manifold M; we are interested in the corresponding equations of motion in Type II supergravity. We begin with a description of topologically trivial Ramond–Ramond fields as elements of the differential complex $\Omega^\bullet(M)$.

For this, let $K^\bullet$ be the $\mathbb{Z}$-graded real vector space of Laurent polynomials $\mathbb{R}(u)$, where u^{-1} is the Bott generator of $\mathrm{K}^{-2}(\mathrm{pt}) \cong \mathbb{Z}[u^{-1}]$ with $\deg(u) = 2$. Let $\Omega(M; K^\bullet)^j$ denote the vector space of differential forms on M of *total* degree j. In this article we are interested mostly in the cases $j = 0, -1$; an element $F \in \Omega(M; K^\bullet)^j$ then admits an expansion in even degree forms

$$F = \sum_{k=0}^{5} u^{-k} \otimes F_{2k} \qquad \text{for} \quad j = 0 \tag{3.3}$$

and in odd degree forms

$$F = \sum_{k=1}^{5} u^{-k} \otimes F_{2k-1} \qquad \text{for} \quad j = -1 \, , \tag{3.4}$$

where $F_p \in \Omega^p(M)$. We call an element $F \in \Omega(M; K^\bullet)^j$ the total *Ramond–Ramond field strength*, where $j = 0$ for the Type IIA string theory while $j = -1$ for the Type IIB string theory. In the IIA theory, the standard supergravity Bianchi identity is

$$\mathrm{d}F = 0 \, .$$

The space of forms $\Omega(M; K^\bullet)^j$ is a symplectic vector space with symplectic form given by

$$\omega_j = \left[\frac{1}{2} \int_M \delta F \wedge \Psi_j^{-1}(\delta F) \right]_{u^0} \, ,$$

where the operation $[-]_{u^0} := u^{-\deg/2}\,(-)$ projects out the constant coefficient in $\Omega^j(M)$ of a Laurent polynomial in $\mathbb{R}(u)$, and

$$\Psi_j^{-1} \, : \, \Omega(M; K^\bullet)^j \, \longrightarrow \, \Omega(M; K^\bullet)^{10-j}$$

is the map which essentially sends F to its complex conjugate $\overline{F}$, where $\overline{u} := -u$; on decomposable elements $F = u^k \otimes f$ it is given by

$$\Psi_j^{-1}(u^k \otimes f) := (-1)^{j\,(j-1)/2}\, u^{5-j}\,(-u)^k \otimes f \, .$$

In the IIA/IIB theories, one has the respective expansions

$$\omega_0(F, G) = \int_M \sum_{k=0}^{5} (-1)^{k+1}\, F_{2k} \wedge G_{10-2k}$$

and

$$\omega_{-1}(F, G) = \int_M \sum_{k=1}^{5} (-1)^{k+1}\, F_{2k-1} \wedge G_{11-2k} \, .$$

Now we let (M, g) be a ten-dimensional lorentzian spin manifold. Then there is a metric of indefinite signature on the vector space $\Omega(M; K^\bullet)^j$ given by

$$g_j(F, G) := \Big[\int_M F \wedge \check{\iota}(G) \Big]_{u^0} ,$$

where the map

$$\check{\iota} : \Omega(M; K^\bullet)^j \longrightarrow \Omega\big(M; (K^\bullet)^*\big)^{10-j}$$

is essentially the Hodge duality operator $\star$ associated to the lorentzian metric g on M, with the convention $\star : u \mapsto u^{-1}$; on decomposable elements $F = u^k \otimes f$ it is given by

$$\check{\iota}(u^k \otimes f) = u^{-k} \otimes \star f .$$

In the IIA/IIB theories, one has the respective expansions

$$g_0(F, G) = \int_M \sum_{k=0}^{5} F_{2k} \wedge \star G_{2k} \qquad \text{and} \qquad g_{-1}(F, G) = \int_M \sum_{k=1}^{5} F_{2k-1} \wedge \star G_{2k-1} .$$

The metric and symplectic form together define an involution

$$I_j : \Omega(M; K^\bullet)^j \longrightarrow \Omega(M; K^\bullet)^j , \qquad I_j(F) := -\big((\Psi_j^{-1})^{-1} \circ \check{\iota}\big)(F) \tag{3.5}$$

which has the property

$$g_j(F, G) = \omega_j\big(I_j(F), G\big) .$$

In the IIA/IIB theories, one has the respective expansions

$$I_0(F) = \sum_{k=0}^{5} (-1)^{k+1}\, u^{-k} \otimes \star F_{10-2k} \qquad \text{and} \qquad I_{-1}(F) = \sum_{k=1}^{5} (-1)^{k}\, u^{-k} \otimes \star F_{11-2k} . \tag{3.6}$$

This involution defines a $\mathbb{Z}_2$-grading on the vector space of differential forms through the decomposition into ± 1 eigenspaces

$$\Omega(M; K^\bullet)^j = \Omega_+(M; K^\bullet)^j \oplus \Omega_-(M; K^\bullet)^j$$

with I_j acting as multiplication by ± 1 on $\Omega_\pm(M; K^\bullet)^j$.

Forms $F^+ \in \Omega_+(M; K^\bullet)^j$, i.e. forms F^+ which satisfy

$$I_j(F^+) = + F^+ , \tag{3.7}$$

are called *self-dual forms*. An alternative formulation of the self-duality equation (3.7) can be given using Clifford algebras [5]. On any lorentzian manifold (M, g) of dimension $4k+2$, $k \in \mathbb{N}$, the associated volume form vol_g defines a Clifford algebra involution $\Gamma := \mathsf{c}(\mathrm{vol}_g)$, where $\mathsf{c}(\omega)$ denotes Clifford multiplication by the form ω. We may then define the involution I_j on $\Omega(M; K^\bullet)^j$ by

$$\mathsf{c}\big(I_j(F)\big) := \Gamma\, \mathsf{c}(F) .$$

This definition agrees with (3.6) by using the property

$$\Gamma\, \mathsf{c}(F_p) = (-1)^{p\,(p+1)/2}\, \mathsf{c}(\star F_p)$$

of Clifford multiplication for $F_p \in \Omega^p(M)$. Note that if we work in euclidean signature instead, then I_j would define a complex structure which is compatible with the euclidean metric g_j and the symplectic form ω_j; see [5] for further details.

All the supergravity equations of motion for the Ramond–Ramond fields in Type II string theory can be rewritten using only the self-dual form F^+. In particular, the equation

$$\mathrm{d}F^+ = 0$$

contains both the Bianchi identity and the Ramond–Ramond equation of motion

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \mathrm{d} \star F = 0 \ .$$

These equations are identical to the equations of motion that we encountered in §2 for abelian gauge theory, like the Maxwell theory. This suggests that the Ramond–Ramond gauge theory should be modelled on some sort of generalized differential cohomology theory; we shall elaborate on this model soon.

For the Type IIA theory, the self-duality equations are

$$F_6 = - \star F_4 \ , \qquad F_8 = \star F_2 \qquad \text{and} \qquad F_{10} = - \star F_0 \ ,$$

and hence from (3.3) the Type IIA self-dual Ramond–Ramond field has an expansion

$$F^+ = u^0 \otimes F_1 + u^{-1} \otimes F_2 + u^{-2} \otimes F_4 + u^{-3} \otimes \star F_4 - u^{-4} \otimes \star F_2 + u^{-5} \otimes \star F_0$$

with the equations of motion

$$\mathrm{d}F_0 = \mathrm{d}F_2 = \mathrm{d}F_4 = 0 = \mathrm{d} \star F_4 = \mathrm{d} \star F_2 \ .$$

Similarly, from (3.4) the Type IIB self-dual Ramond–Ramond field has an expansion

$$F^+ = u^{-1} \otimes F_1 + u^{-2} \otimes F_3 + u^{-3} \otimes F_5^+ + u^{-4} \otimes \star F_3 - u^{-5} \otimes \star F_1 \ ,$$

with $\star F_5^+ = -F_5^+$ and the equations of motion

$$\mathrm{d}F_1 = \mathrm{d}F_3 = 0 = \mathrm{d} \star F_3 = \mathrm{d} \star F_1 \qquad \text{and} \qquad \mathrm{d}F_5^+ = 0 \ .$$

Note that these expressions are just particular parametrizations of the self-dual field F^+: It can always be written as

$$F^+ = F + I_j(F) \qquad \text{with} \quad F \in \Omega(M; K^\bullet)^j \ ,$$

but the form F is *not* unique. The self-duality condition on the total Ramond–Ramond field strength is a feature that we will have to deal with carefully later on. Together with the constraint on the parity of differential form degree, it is the requirement of the GSO projection in the underlying closed superstring theory on the manifold M. Note also that, generally, a necessary condition for the existence of solutions to the self-duality equations on a lorentzian manifold (M, g) is that $\dim(M) = 4k + 2$ with $k \in \mathbb{N}$; only under these conditions is the map (3.5) an involution.

<u>Ramond–Ramond currents</u>

At the level of topologically trivial fields, the Ramond–Ramond fields are *sourced* by D-branes on M, just like the electromagnetic fields of Maxwell theory are *sourced* by electrically charged particles. "Anomaly cancellation" arguments in the Ramond–Ramond abelian gauge theory requires that, in the presence of a D-brane (W, E, f), the Ramond–Ramond fields obey the equations of motion

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \mathrm{d} \star F = j(W, E, f) \ , \tag{3.8}$$

where $j(W, E, f)$ is the "Ramond–Ramond current" whose cohomology class is given by the D-brane charge vector (3.2) and as before we use a distributional representative of the Poincaré dual class $\mathsf{Pd}_M(f(W))$. Note the formal equivalence of (3.8) to the equations of motion (2.10) of (higher) abelian gauge theories. In particular, the Ramond–Ramond field is a trivialization of the Ramond–Ramond current. However, these equations are incompatible with the self-duality of F.

3.3. Semi-classical quantization

Since the Ramond–Ramond charges of D-branes are classified by the complex K-theory $\mathrm{K}_c^j(M)$ of spacetime with compact support (with $j = 0/-1$ in Type IIB/IIA theory), it is natural to expect that the Ramond–Ramond fields are also classified in some way via K-theory. This statement about the K-theory classification of Ramond–Ramond fields on a locally compact spacetime M follows from the relation between D-brane charges and the group of Ramond–Ramond fluxes "measured at infinity".

For this, let M be a non-compact manifold, e.g. $M = \mathbb{R} \times N$ with N non-compact. If the worldvolume W is compact inside the K-cycles (W, E, f), then the current $j(W, E, f)$ is supported in the interior $\mathring{M} \subset M$. Let M_∞ be the "boundary of M at infinity", e.g. $(\mathbb{R}^n)_\infty = S^{n-1}$. Since $j(W, E, f)$ is trivialized by F in $\mathring{M}$, the D-brane charge lives in the kernel of the natural forgetful homomorphism

$$\mathfrak{q}^\bullet \ : \ \mathrm{K}_c^\bullet(M) \ \longrightarrow \ \mathrm{K}^\bullet(M)$$

induced by the inclusions

$$(M, \emptyset) \ \hookrightarrow \ (M, M_\infty) \qquad \text{and} \qquad i \ : \ M_\infty \ \hookrightarrow \ M$$

of (relative pairs of) locally compact spaces; this morphism forgets about the compact support condition, with $\mathrm{K}_c^\bullet(M) \cong \mathrm{K}^\bullet(M, M_\infty)$ given by relative K-theory. By Bott periodicity, the long exact sequence for the pair (M, M_∞) in K-theory truncates to the six-term exact sequence

$$
\begin{array}{ccccc}
\mathrm{K}^{-1}(M_\infty) & \longrightarrow & \mathrm{K}^0(M, M_\infty) & \xrightarrow{\ \mathfrak{q}^0\ } & \mathrm{K}^0(M) \\
{\scriptstyle i^*} \uparrow & & & & \downarrow {\scriptstyle i^*} \\
\mathrm{K}^{-1}(M) & \xleftarrow[\ \mathfrak{q}^{-1}\ ]{} & \mathrm{K}^{-1}(M, M_\infty) & \longleftarrow & \mathrm{K}^0(M_\infty)
\end{array}
$$

It follows that the D-brane charge groups are given by

$$\ker(\mathfrak{q}^0) \cong \mathrm{K}^{-1}(M_\infty) \big/ i^*\big(\mathrm{K}^{-1}(M)\big) \qquad \text{and} \qquad \ker(\mathfrak{q}^{-1}) \cong \mathrm{K}^0(M_\infty) \big/ i^*\big(\mathrm{K}^0(M)\big) \ .$$

We interpret these formulas in the following way. D-brane charge, regarded as the total charge of a Ramond–Ramond current $j(W, E, f)$, can only be detected by classes of fields at infinity M_∞ which are *not* the restrictions of fields defined on $\mathring{M}$, i.e. which cannot be extended to all of spacetime

M. Moreover, the group $\mathrm{K}^j(M)$ for $j = 0, -1$ classifies fields F which do not contribute to the D-brane charge, i.e. $\mathrm{K}^j(M)$ topologically classifies gauge equivalence classes of Ramond–Ramond fields in the absence of D-branes, where $j = 0, -1$ for Type IIA/IIB string theory. In the following we will *assume* that this relation between Ramond–Ramond fields and K-theory holds for arbitrary spacetime manifolds M.

Let us now make explicit the relation between the Ramond–Ramond fields and cohomology, i.e. the de Rham cohomology class $[F(\xi)]_{\mathrm{dR}}$ associated to an element $\xi \in \mathrm{K}^j(M)$ that determines (the class of) the Ramond–Ramond field F. In [45] it is argued using the Ramond–Ramond equation of motion from (3.8) that

$$[F]_{\mathrm{dR}} \ \in \ \Lambda_{\mathrm{K}^j} \subset \mathrm{H}(M; K^\bullet)^j \ ,$$

where the full lattice Λ_{K^j} is the image of the modified Chern character map

$$\mathsf{ch} \wedge \sqrt{\widehat{A}(M)} \ : \ \mathrm{K}^j(M) \ \longrightarrow \ \mathrm{H}(M; K^\bullet)^j$$

which is a group isomorphism over $\mathbb{R}$. This means that the cohomology class of the Ramond–Ramond field F associated to the K-theory class $\xi \in \mathrm{K}^j(M)$ is

$$\left[F(\xi)\right] = \mathsf{ch}(\xi) \smile \sqrt{\widehat{A}(M)} \ . \tag{3.9}$$

This formula is interpreted as the Dirac quantization condition for Ramond–Ramond fields in Type II string theory. It follows that the Ramond–Ramond field is given by a representative for an element of *differential K-theory*, which we now proceed to describe in some detail.
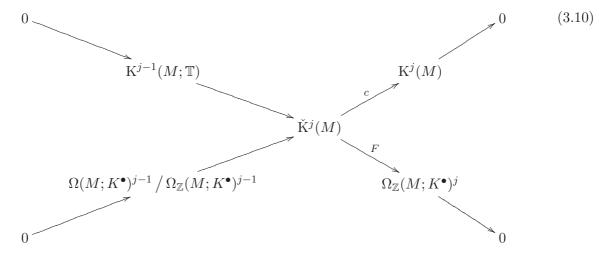
3.4. Differential K-theory

The set of gauge inequivalent Ramond–Ramond fields (or equivalently gauge equivalence classes of Ramond–Ramond currents) lives inside an infinite-dimensional abelian Lie group $\check{\mathrm{K}}^j(M)$, the differential K-theory group of spacetime M; its connected components are labelled by the complex K-theory $\mathrm{K}^j(M)$, the group of D-brane charges. The group $\check{\mathrm{K}}^j(M)$ extends the setwise fibre product

$$\Omega_{\mathbb{Z}}(M; K^\bullet)^j \times_{[-]} \mathrm{K}^j(M) := \left\{ (F, \xi) \ \Big| \ \mathsf{ch}(\xi) \wedge \sqrt{\widehat{A}(M)} = [F]_{\mathrm{dR}} \right\}$$

by the torus of topologically trivial flat Ramond–Ramond fields, i.e. there is an exact sequence

$$0 \ \longrightarrow \ \mathrm{K}^{j-1}(M) \otimes \mathbb{T} \ \longrightarrow \ \check{\mathrm{K}}^j(M) \ \longrightarrow \ \Omega_{\mathbb{Z}}(M; K^\bullet)^j \times_{[-]} \mathrm{K}^j(M) \ \longrightarrow \ 0 \ .$$

In general, a Ramond–Ramond *potential* is a representative for a class $[\check{C}]$ in the differential K-theory $\check{\mathrm{K}}^j(M)$. As before, this group can be characterised by two short exact sequences which are summarised in the diagram

$$\tag{3.10}$$

where the kernel of the field strength map F is the group of flat Ramond–Ramond fields on M, while the kernel of the characteristic class map c is the torus of topologically trivial Ramond–Ramond gauge potentials. Let us pause to briefly describe the structures of each of these classes of fields, as they will play a prominent role in our applications.

Flat Ramond–Ramond fields

The group $\mathrm{K}^j(M;\mathbb{T})$ of flat fields can be described as explained in §2.5., by using the short exact sequence of abelian groups (2.8) to write the corresponding long exact sequence in complex K-theory

$$\cdots \longrightarrow \mathrm{K}^j(M) \longrightarrow \mathrm{K}^j(M) \otimes \mathbb{R} \longrightarrow \mathrm{K}^j(M;\mathbb{T}) \xrightarrow{\delta} \mathrm{K}^{j+1}(M) \longrightarrow \cdots . \qquad (3.11)$$

Using the Chern character homomorphism, the flat Ramond–Ramond fields are thus described by the short exact sequence

$$0 \longrightarrow \mathrm{H}^j(X;\mathbb{R})\big/\Lambda_{\mathrm{K}^j} \longrightarrow \mathrm{K}^j(M;\mathbb{T}) \xrightarrow{\beta} \mathsf{Tor}\,\mathrm{K}^{j+1}(M) \longrightarrow 0 ,$$

where the Bockstein homomorphism β is induced from the connecting homomorphism δ in the long exact sequence (3.11). If the K-theory group $\mathrm{K}^{j+1}(M)$ is pure torsion, then the flat torsion Ramond–Ramond fields can be represented in terms of virtual flat vector bundles over M as

$$\mathrm{K}^j(M;\mathbb{T}) \cong \mathsf{Tor}\,\mathrm{K}^{j+1}(M) . \qquad (3.12)$$

By Pontrjagin duality of K-theory, we have

$$\mathrm{K}^j(M;\mathbb{T}) \cong \mathsf{Hom}_{\mathscr{A}b}\big(\mathrm{K}_j(M)\,,\,\mathbb{T}\big) , \qquad (3.13)$$

where $\mathrm{K}_\bullet$ is geometric K-homology. The isomorphism (3.13) follows by using the fact that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a divisible group along with the universal coefficient theorem for K-theory to find

$$\mathsf{Ext}_{\mathscr{A}b}\big(\mathrm{K}_j(M)\,,\,\mathbb{T}\big) = 0$$

for all $j \in \mathbb{Z}$; this implies that the contravariant functor $G \mapsto \mathsf{Hom}_{\mathscr{A}b}(G,\mathbb{T})$ from the category of abelian groups into itself takes exact sequences into exact sequences.

The Ramond–Ramond flux couplings implied by (3.12)–(3.13) can be made explicitly to "background D-branes", which are K-chains $\big(\widetilde{W}\,,\,\widetilde{E}\,,\,\widetilde{f}\,\big)$ whose boundary

$$\partial_{\mathrm{K}}\big(\widetilde{W}\,,\,\widetilde{E}\,,\,\widetilde{f}\,\big) := \big(\partial\widetilde{W}\,,\,\widetilde{E}\,\big|_{\partial\widetilde{W}}\,,\,\widetilde{f}\,\big|_{\partial\widetilde{W}}\big) = (W,E,f) \qquad (3.14)$$

is a Baum–Douglas K-cycle. The *holonomy* over such a D-brane background with flat Ramond–Ramond flux given by

$$\xi = [E_0] - [E_1] \ \in\ \mathrm{K}^{-1}(M;\mathbb{T}) \cong \mathsf{Hom}_{\mathscr{A}b}\big(\mathrm{K}_{\mathrm{odd}}(M)\,,\,\mathbb{T}\big) ,$$

where E_0, E_1 are complex vector bundles on M of equal rank, is then defined by the virtual K-chain

$$\big(\widetilde{W}\,,\,\widetilde{f^*E_0}\,,\,\widetilde{f}\,\big) - \big(\widetilde{W}\,,\,\widetilde{f^*E_1}\,,\,\widetilde{f}\,\big) . \qquad (3.15)$$

Unlike the couplings to D-branes, these couplings do not define spin^c bordism invariants.

Topologically trivial Ramond–Ramond fields

The torus of topologically trivial fields $\Omega(M; K^\bullet)^j / \Omega_{\mathbb{Z}}(M; K^\bullet)^j$ consists of globally defined differential forms on M, i.e. $[\check{C}] = [C]$ with $C \in \Omega(M; K^\bullet)^j$; these are the Ramond–Ramond potentials that are normally considered in the physics literature. The field strength of such a potential is

$$F\big([C]\big) = \mathrm{d}C$$

and the gauge invariance is

$$C \;\longmapsto\; C + \mathrm{d}\xi \qquad \text{with} \quad \xi \in \Omega(M; K^\bullet)^{j-1} \big/ \Omega_{\mathbb{Z}}(M; K^\bullet)^{j-1} \;.$$

Properties

1. The differential K-theory is 2-periodic:

$$\check{\mathrm{K}}^{j+2}(M) \cong \check{\mathrm{K}}^j(M) \;.$$

2. There is a graded-commutative ring multiplication

$$\smile \,:\, \check{\mathrm{K}}^j(M) \otimes \check{\mathrm{K}}^{j'}(M) \;\longrightarrow\; \check{\mathrm{K}}^{j+j'}(M)$$

which acts on topologically trivial fields C as

$$[C] \otimes [\check{C}'] \;\longmapsto\; \big[C \wedge F\big([\check{C}']\big) \big] \;.$$

3. A $\check{\mathrm{K}}$-*orientation* or *smooth K-orientation* on a manifold M is a choice of spin^c structure together with a riemannian structure and a compatible smooth connection on the spin^c bundle. For M $\check{\mathrm{K}}$-oriented there is an integration map

$$\int\!\!\!\!\!\int_M^{\check{\mathrm{K}}} \,:\, \check{\mathrm{K}}^j(M) \;\longrightarrow\; \check{\mathrm{K}}^{j-m}(\mathrm{pt})$$

where $m = \dim(M) = 10$. For topologically trivial fields C it is given by

$$\int\!\!\!\!\!\int_M^{\check{\mathrm{K}}} [C] = u^{\lfloor m/2 \rfloor} \Big[\int_M C \wedge \widehat{A}(M) \Big]_{u^0} \quad \mathrm{mod}\ \mathbb{Z} \;.$$

The integration map commutes with the field strength map, if the integration of forms is defined appropriately using the orientation curvature, in the sense that the curvature of $\int^{\check{\mathrm{K}}}_M [\check{C}]$ is $\int_M \widehat{A}(M) \wedge F([\check{C}])$ by the Riemann–Roch theorem.

Examples

Using the exact sequences from (3.10) one can work out differential K-theory groups explicitly in a number of basic examples.

- When M is a point one has

$$\check{\mathrm{K}}^0(\mathrm{pt}) = \mathbb{Z} \qquad \text{and} \qquad \check{\mathrm{K}}^{-1}(\mathrm{pt}) = \mathbb{T} \;.$$

- Let R be a linear vector space over $\mathbb{C}$. Then R is contractible, so one has $\mathrm{H}(R; K^\bullet)^{-1} = 0 = \mathrm{K}^{-1}(R)$ and $\mathrm{K}^0(R) = \mathbb{Z}$. Whence the group of Type IIA Ramond–Ramond potentials on R is given by

$$\check{\mathrm{K}}^0(R) = \Omega_{\mathbb{Z}}(R; K^\bullet)^0 \longrightarrow \mathbb{Z} \oplus \Omega(R; K^\bullet)^{-1} / \Omega_{\mathbb{Z}}(R; K^\bullet)^{-1} \ .$$

  It naturally contains those fields which trivialize the Ramond–Ramond currents sourced by the stable D0-branes of the Type IIA theory, corresponding to characteristic classes $[c] \in \mathrm{K}^0(R) = \mathbb{Z}$. Since R is contractible, the potential is determined in positive degree by a globally-defined differential form C of curvature $F([C]) = \mathrm{d}C$, with the gauge invariance $C \mapsto C + \mathrm{d}\xi$. The arrow is then the natural map which associates to the field strength F the corresponding globally well-defined Ramond–Ramond potential C. The group of Type IIB Ramond–Ramond potentials on R is on the other hand given by

$$\check{\mathrm{K}}^{-1}(R) = \Omega(R; K^\bullet)^0 / \Omega_{\mathbb{Z}}(R; K^\bullet)^0 \ .$$

  In this case there is no extension as the Type IIB theory has no stable D0-branes, and hence the Ramond–Ramond fields are determined entirely by the potentials C which are globally defined differential forms of even degree.

3.5. Models for differential K-theory

Let us now look at some explicit models for the differential K-theory groups, stressing the virtues and drawbacks of each approach from a physical perspective. For brevity we consider only the degree zero groups $\check{\mathrm{K}}^0(M)$ pertinent to the Type IIA string theory.

Differential function spaces

The foundational construction of differential K-theory is found in [37]; this approach is based on classifying maps for complex K-theory. Let $\mathtt{Fred}$ denote the algebra of Fredholm operators on a separable Hilbert space. Taking the index bundle of a homotopy class of maps $c : M \to \mathtt{Fred}$ determines an isomorphism

$$[M, \mathtt{Fred}] \xrightarrow{\ \mathsf{Index}\ } \mathrm{K}^0(M) \ . \tag{3.16}$$

The cocycles for $\check{\mathrm{K}}^0(M)$ are triples (c, h, ω), where $c : M \to \mathtt{Fred}$ represents a K-theory class via the isomorphism (3.16), $h \in C(M; K^\bullet)^{-1}$ is a cochain on M, and $\omega \in \Omega_{\mathrm{cl}}(M; K^\bullet)^0$ is a closed differential form on M of even degree such that $c^*\mathtt{u} - \delta h = \omega$ where the cocycle $\mathtt{u} \in Z(\mathtt{Fred}; K^\bullet)^0$ represents the Chern character of the universal vector bundle. One then defines an equivalence relation by declaring two cocycles to be equivalent "up to homotopy". Although powerful because of its generality, this approach is not very useful for constructing the extra geometrical ingredients required in gauge theory; a graded-commutative product structure in this model is described in [55].

Chan–Paton gauge fields

A more geometric approach is given in [26], which may be thought of as equipping the Chan–Paton vector bundles of background D-branes that wrap the whole spacetime M with connections, and then constructing Ramond–Ramond potentials in an analogous way to that described in §3.1. In this model one represents classes in $\check{\mathrm{K}}^0(M)$ by pairs (E, ∇), where $E \to M$ is a hermitian vector bundle and ∇ is a unitary connection on E. The cocycles are then triples (f, η, ω), where $f : M \to BU$

is a classifying map for E, $\omega = \mathsf{ch}(\nabla)$ is the Chern–Weil representative of the Chern characteristic class $\mathsf{ch}([E])$, and η is a Chern–Simons form with the property

$$\mathrm{d}\eta = f^*\omega_{BU} - \omega \ ,$$

with $\omega_{BU} = \mathsf{ch}(\nabla_{BU})$ the Chern characteristic class of the universal bundle over BU with universal connection ∇_{BU}. A refinement of the index theorem in this model can be found in [29].

Configuration space and gauge transformations of Ramond–Ramond fields

As we have explained, locality of quantum field theory requires the use of cocycles rather than isomorphism classes. We will regard the Ramond–Ramond gauge fields as objects in a certain monoidal category $\check{\mathscr{K}}_M$ whose morphisms are gauge transformations. A construction of a category $\check{\mathscr{K}}_M$ of differential K-cocycles is sketched in [32] whose isomorphism classes are precisely the differential K-theory classes in the group $\check{\mathrm{K}}^0(M)$, and which is a *groupoid*; below we elaborate on this description. We identify gauge fields up to isomorphism of cocycles. The monoidal structure on $\check{\mathscr{K}}_M$ is generated by the sum of cocycles.

The objects of the category $\check{\mathscr{K}}_M$ are triples $\check{C} = (E, \nabla, C)$, where $E \to M$ is a $\mathbb{Z}_2$-graded hermitian vector bundle, ∇ is a unitary connection on E, and $C \in \Omega(M; K^\bullet)^{-1}$ is a differential form on M of odd degree. The field strength of such an object is given by [29]

$$F(E, \nabla, C) = \mathsf{ch}(\nabla) + \mathrm{d}C \ ,$$

where

$$\mathsf{ch}(\nabla) = \mathrm{Tr}\,\exp\big(-\tfrac{1}{2\pi\,\mathrm{i}}\,u^{-1} \otimes \nabla^2\big) \ \in\ \Omega_{\mathrm{cl}}(M; K^\bullet)^0$$

is the corresponding Chern character form which gives the curvature contribution from background D-branes. In the topologically trivial case we take $E = \emptyset$ to be the empty vector bundle and the differential form C is what is normally called the Ramond–Ramond field (or more precisely potential) of Type IIA string theory. The monoidal structure on objects is given by

$$\hat{C} + \hat{C}' = (E, \nabla, C) + (E', \nabla', C') = (E \oplus E', \nabla \oplus \nabla', C + C') \ ,$$

and a zero object $\hat{C} = 0$ is represented by $(E \oplus E^{\mathrm{op}}, \nabla \oplus \nabla, 0)$ for a $\mathbb{Z}_2$-graded vector bundle E, where E^{op} denotes the bundle E with the opposite grading.

A pre-morphism $g : \check{C}_0 \longrightarrow \check{C}_1$ between two objects $\check{C}_0 = (E_0, \nabla_0, C_0)$ and $\check{C}_1 = (E_1, \nabla_1, C_1)$ is a triple $\big(\widetilde{G}, \widetilde{\nabla}, \lambda\big)$, where $\big(\widetilde{G}, \widetilde{\nabla}\big)$ is a $\mathbb{Z}_2$-graded vector bundle with connection on $M \times I$, $I = [0,1]$ such that

$$\big(\widetilde{G}, \widetilde{\nabla}\big)\big|_{M \times 0} = (E_0, \nabla_0) \qquad \text{and} \qquad \big(\widetilde{G}, \widetilde{\nabla}\big)\big|_{M \times 1} = (E_1, \nabla_1) \ ,$$

with $\widetilde{F} = F\big(\widetilde{G}, \widetilde{\nabla}, \lambda\big)$ constant along $t \in [0,1]$, i.e. $i_{\partial/\partial t}\widetilde{F} = 0$, and $\lambda \in \Omega(M; K^\bullet)^0$ such that

$$C_1 = C_0 + \mathsf{CS}(\nabla_0, \nabla_1) + \mathrm{d}\lambda$$

is given by a transgression form obtained from a canonically defined Chern–Simons class [42]

$$\mathsf{CS}(\nabla_0, \nabla_1) = \int_0^1 \mathsf{ch}\big(\widetilde{\nabla}\big) \ \in\ \Omega(M; K^\bullet)^{-1}\big/\mathrm{im}(\mathrm{d})$$

with the property

$$\mathrm{dCS}(\nabla_0, \nabla_1) = \mathrm{ch}(\nabla_0) - \mathrm{ch}(\nabla_1) \; .$$

The relations are $E_2 = E_1 + E_3$ whenever there is a short exact sequence of vector bundles

$$0 \; \longrightarrow \; E_1 \; \xrightarrow{i} \; E_2 \; \xrightarrow{j} \; E_3 \; \longrightarrow \; 0 \; .$$

Choosing a splitting $s : E_3 \to E_2$ gives an isomorphism $i \oplus s : E_1 \oplus E_3 \to E_2$. If ∇^{E_i} is a connection on $E_i \to M$, then the corresponding relative Chern–Simons form

$$\mathsf{CS}\big(\nabla^{E_1}, \, \nabla^{E_2}, \, \nabla^{E_3}\big) := \mathsf{CS}\big((i \oplus s)^* \nabla^{E_2}, \, \nabla^{E_1} \oplus \nabla^{E_3}\big) \; \in \; \Omega(M; K^\bullet)^{-1} \big/ \mathrm{im}(\mathrm{d})$$

is independent of the choice of splitting morphism s and satisfies

$$\mathrm{dCS}\big(\nabla^{E_1}, \, \nabla^{E_2}, \, \nabla^{E_3}\big) = \mathsf{ch}\big(\nabla^{E_2}\big) - \mathsf{ch}\big(\nabla^{E_1}\big) - \mathsf{ch}\big(\nabla^{E_3}\big) \; .$$

We then set $C_2 = C_1 + C_3 - \mathsf{CS}\big(\nabla^{E_1}, \, \nabla^{E_2}, \, \nabla^{E_3}\big)$.

Two pre-morphisms

$$(E_0, \nabla_0, C_0) \xrightarrow{(\widetilde{G}_0, \widetilde{\nabla}_0, \lambda_0)} (E_1, \nabla_1, C_1) \qquad \text{and} \qquad (E_0, \nabla_0, C_0) \xrightarrow{(\widetilde{G}_1, \widetilde{\nabla}_1, \lambda_1)} (E_1, \nabla_1, C_1)$$

are said to be equivalent if there exists a triple $(\boldsymbol{G}, \boldsymbol{\nabla}, \boldsymbol{\lambda})$, where $(\boldsymbol{G}, \boldsymbol{\nabla})$ is a $\mathbb{Z}_2$-graded vector bundle with connection on $M \times I \times I$ such that

$$(\boldsymbol{G}, \boldsymbol{\nabla})\big|_{M \times 0 \times I} = \big(\widetilde{G}_0, \, \widetilde{\nabla}_0\big) \qquad \text{and} \qquad (\boldsymbol{G}, \boldsymbol{\nabla})\big|_{M \times I \times 0} = (E_0 \times I, \nabla_0 \times 1)$$

while

$$(\boldsymbol{G}, \boldsymbol{\nabla})\big|_{M \times 1 \times I} = \big(\widetilde{G}_1, \, \widetilde{\nabla}_1\big) \qquad \text{and} \qquad (\boldsymbol{G}, \boldsymbol{\nabla})\big|_{M \times I \times 1} = (E_1 \times I, \nabla_1 \times 1) \; ,$$

with $\boldsymbol{\lambda} \in \Omega(M; K^\bullet)^{-1}/\mathrm{im}(d)$ such that

$$\lambda_1 = \lambda_0 + \int_0^1 \mathsf{ch}(\boldsymbol{\nabla}) + \mathrm{d}\boldsymbol{\lambda} \; .$$

This generates an equivalence relation. Note that because of the geometric product structure of $(\boldsymbol{G}, \boldsymbol{\nabla})$, one has $\mathrm{dCS}\big(\widetilde{\nabla}_0, \, \widetilde{\nabla}_1\big) = \mathsf{ch}\big(\widetilde{\nabla}_0\big) - \mathsf{ch}\big(\widetilde{\nabla}_1\big)$ and hence

$$\int_0^1 \mathsf{ch}\big(\widetilde{\nabla}_0\big) + \mathrm{d}\lambda_0 = \int_0^1 \mathsf{ch}\big(\widetilde{\nabla}_1\big) + \mathrm{d}\lambda_1 \; .$$

A *gauge transformation* from $\check{C}_0$ to $\check{C}_1$ is an equivalence class of pre-morphisms $\big(\widetilde{G}, \, \widetilde{\nabla}, \, \lambda\big)$. We refer to the shift of C_0 by $\mathrm{d}\lambda$ as a *small* gauge transformation, while the shift by the transgression form is called a *large* gauge transformation. Gauge transformations leave the corresponding Ramond–Ramond field strengths invariant, $F([\check{C}_0]) = F([\check{C}_1])$.

Next we construct the composition of gauge transformations. The composition of pre-morphisms

$$\check{C}_0 = (E_0, \nabla_0, C_0) \xrightarrow{(\widetilde{G}_1, \widetilde{\nabla}_1, \lambda_1)} \check{C}_1 = (E_1, \nabla_1, C_1)$$

and

$$\check{C}_1 = (E_1, \nabla_1, C_1) \xrightarrow{(\widetilde{G}_2, \widetilde{\nabla}_2, \lambda_2)} \check{C}_2 = (E_2, \nabla_2, C_2)$$

is the pre-morphism $\check{C}_0 \xrightarrow{(\widetilde{G},\widetilde{\nabla},\lambda)} \check{C}_2$ defined as follows. There is a graded vector bundle with connection on $M \times I$ defined by

$$\big(\widetilde{G}' , \widetilde{\nabla}' \big) := \big(\widetilde{G}_1 \oplus \widetilde{G}_2 , \widetilde{\nabla}_1 \oplus \widetilde{\nabla}_2 \big)$$

with

$$C_2 = C_0 + \int_0^1 \mathsf{ch}\big(\widetilde{\nabla}' \big) + \mathrm{d}(\lambda_1 + \lambda_2) \ .$$

Via concatenation of paths, we now set $\big(\widetilde{G} , \widetilde{\nabla} \big)\big|_{M \times t}$ equal to $\big(\widetilde{G}_1 , \widetilde{\nabla}_1 \big)\big|_{M \times (2t)}$ for $0 \leq t \leq \frac{1}{2}$ and to $\big(\widetilde{G}_2 , \widetilde{\nabla}_2 \big)\big|_{M \times (2t-1)}$ for $\frac{1}{2} \leq t \leq 1$. This is compatible with the required identity

$$\mathsf{CS}(\nabla_0, \nabla_2) = \int_0^1 \mathsf{ch}\big(\widetilde{\nabla} \big) = \mathsf{CS}(\nabla_0, \nabla_1) + \mathsf{CS}(\nabla_1, \nabla_2) \ .$$

Likewise, we use concatenation of paths to define the graded vector bundle with connection $(\boldsymbol{E}, \blacktriangledown)$ on $M \times I \times I$ having the three boundary faces

$$\big((E_0, \nabla_0), (E_1, \nabla_1)\big) \ , \qquad \big((E_1, \nabla_1), (E_2, \nabla_2)\big) \qquad \text{and} \qquad \big((E_2, \nabla_2), (E_0, \nabla_0)\big) \ .$$

One shows that $\mathrm{d} \int_0^1 \mathsf{ch}(\blacktriangledown) = \int_0^1 \big(\mathsf{ch}(\widetilde{\nabla}_1) + \mathsf{ch}(\widetilde{\nabla}_2) - \mathsf{ch}(\widetilde{\nabla}') \big)$. We set

$$\lambda = \lambda_1 + \lambda_2 + \int_0^1 \mathsf{ch}(\blacktriangledown) + \mathrm{d}\kappa$$

for some $\kappa \in \Omega(M; K^\bullet)^{-1}/\mathrm{im}(\mathrm{d})$.

Lemma 3.17. *The composition $\big(\widetilde{G}_1 , \widetilde{\nabla}_1 , \lambda_1 \big) \circ \big(\widetilde{G}_2 , \widetilde{\nabla}_2 , \lambda_2 \big) := \big(\widetilde{G} , \widetilde{\nabla} , \lambda \big)$ in $\mathscr{K}_M$ is well-defined and associative.*

Proof. We show that the composition is well-defined. Suppose, for example, that the gauge transformation $g : \check{C}_0 \longrightarrow \check{C}_1$ is represented by equivalent pre-morphisms $\big(\widetilde{G}_1 , \widetilde{\nabla}_1 , \lambda_1 \big)$ and $\big(\widetilde{G}_1' , \widetilde{\nabla}_1' , \lambda_1' \big)$, implemented by a triple $(\boldsymbol{G}, \boldsymbol{\nabla}, \boldsymbol{\lambda})$. Set $\big(\widetilde{G}' , \widetilde{\nabla}' \big) = \big(\widetilde{G}_1' , \widetilde{\nabla}_1' \big) \circ \big(\widetilde{G}_2 , \widetilde{\nabla}_2 \big)$ and $\lambda' = \lambda_1' + \lambda_2 + \int_0^1 \mathsf{ch}(\blacktriangledown') + \mathrm{d}(\kappa - \boldsymbol{\lambda})$, where the vector bundle with connection $(\boldsymbol{E}', \blacktriangledown')$ on $M \times I \times I$ is obtained from $(\boldsymbol{E}, \blacktriangledown)$ by replacing $\widetilde{G}_1$ with $\widetilde{G}_1'$. Then the condition $\lambda_1' = \lambda_1 + \int_0^1 \mathsf{ch}(\boldsymbol{\nabla}) + \mathrm{d}\boldsymbol{\lambda}$ immediately implies that $\big(\widetilde{G} , \widetilde{\nabla} , \lambda \big)$ and $\big(\widetilde{G}' , \widetilde{\nabla}' , \lambda' \big)$ determine equivalent gauge transformations. We leave the check of associativity to the interested reader. $\square$

The composition law above defines a group structure on the set of all gauge transformations on $\mathscr{K}_M$. The identity morphism is given by $\mathbb{1}_{(E,\nabla,C)} : (E, \nabla, C) \xrightarrow{(E \times I, \nabla \times 1, 0)} (E, \nabla, C)$, with $\mathsf{CS}(\nabla, \nabla) = \int_0^1 \mathsf{ch}(\nabla \times 1) = 0$. The inverse of a morphism $(E, \nabla, C) \xrightarrow{(\widetilde{G},\widetilde{\nabla},\lambda)} (E', \nabla', C')$ is given by $(E', \nabla', C') \xrightarrow{(\widetilde{G}^{\mathrm{op}},\widetilde{\nabla},-\lambda)} (E, \nabla, C)$. To see this, we must show that the composition $\big(\widetilde{G}' , \widetilde{\nabla}' , \lambda' \big)$ of these two morphisms is equivalent to the identity $\mathbb{1}_{(E,\nabla,C)}$. Consider the morphism $(E', \nabla', C') \xrightarrow{(\widetilde{G},\widetilde{\nabla},\lambda)^{-1}} (E, \nabla, C)$ defined via path inversion

$$\big(\widetilde{G} , \widetilde{\nabla} , \widetilde{\lambda} \big)^{-1}\Big|_{M \times t} := \big(\widetilde{G} , \widetilde{\nabla} , \widetilde{\lambda} \big)\Big|_{M \times (1-t)}$$

for all $t \in I$. Note that $\big(\widetilde{G}\,', \widetilde{\nabla}\,', \widetilde{\lambda}\,' \big)\big|_{M \times t}$ is equal to $\big(\widetilde{G}, \widetilde{\nabla}, \widetilde{\lambda} \,\big)\big|_{M \times (2t)}$ for $0 \leq t \leq \frac{1}{2}$ and to $\big(\widetilde{G}, \widetilde{\nabla}, \widetilde{\lambda} \,\big)\big|_{M \times 2(1-t)}$ for $\frac{1}{2} \leq t \leq 1$, so that

$$\int_0^1 \mathsf{ch}\big(\widetilde{\nabla}\,' \big) = 0 \ .$$

Furthermore, in this case we have

$$\mathrm{d}\int_0^1 \mathsf{ch}(\blacktriangledown) = \int_0^1 \big(\mathsf{ch}(\widetilde{\nabla}) - \mathsf{ch}(\widetilde{\nabla}) - \mathsf{ch}(\nabla \times 1) \big) = 0$$

and

$$\lambda - \lambda + \int_0^1 \mathsf{ch}(\blacktriangledown) = \int_0^1 \mathsf{ch}(\blacktriangledown) \ .$$

This shows that every morphism is invertible.

This construction defines the *category $\check{\mathscr{K}}_M$ of Ramond–Ramond fields* on the manifold M, which is a strictly symmetric monoidal category; it defines an action groupoid and has a physical interpretation in boundary string field theory, generalized to incorporate superconnections [32]. The objects are the gauge fields $\check{C} = (E, \nabla, C)$, while the morphisms are the gauge transformations $\big(\widetilde{G}, \widetilde{\nabla}, \lambda \big)$ together with the group law described above. The set of gauge equivalence classes is the group of isomorphism classes which coincides with the differential K-theory $\check{\mathrm{K}}^0(M)$. Moreover, we see that gauge transformations generate the differential K-theory group $\check{\mathrm{K}}^1(M)$; this follows from the description of $\check{\mathrm{K}}^1(M)$ given in [32] by integrating classes in $\check{\mathrm{K}}^0(M \times S^1)$, which are trivial at each point of S^1, over S^1.

Every object of the cocycle category of fields $\check{\mathscr{K}}_M$ is invertible: The inverse of the gauge field $\check{C} = (E, \nabla, C)$ is $\check{C}^{-1} = (E^{\mathrm{op}}, \nabla, -C)$. Thus $\check{\mathscr{K}}_M$ is a Picard category. The zero object (of the monoidal structure) has non-trivial automorphisms; from the above construction it follows that the corresponding automorphism group coincides with the group of flat fields

$$\mathsf{Aut}_{\check{\mathscr{K}}_M}(0) \cong \mathrm{K}^0(M; \mathbb{T}) \ .$$

The categorical structure of the gauge theory configuration space may be iterated to define higher categories $\check{\mathscr{K}}_M^k$ in a similar manner. Let us describe explicitly the next member $\check{\mathscr{K}}_M^2$ in the multi-categorical hierarchy. In the same way that we went from the definition of the differential K-theory to the category $\check{\mathscr{K}}_M = \check{\mathscr{K}}_M^1$, we now replace the equivalence relation on (1-)morphisms by 2-morphisms. Then we lose the strict notion of composition of 1-morphisms, as explained in §2.4. Since every gauge transformation has an inverse, we can consider the composition as a certain subset $\mathsf{Comp}^1 \subset \mathsf{Hom}_{\check{\mathscr{K}}_M^1}(\check{C}_0, \check{C}_1) \times \mathsf{Hom}_{\check{\mathscr{K}}_M^1}(\check{C}_1, \check{C}_2) \times \mathsf{Hom}_{\check{\mathscr{K}}_M^1}(\check{C}_2, \check{C}_0)$ such that $(g_1, g_2, g_3) \in \mathsf{Comp}^1$ if and only if $g_1 \circ g_2 \circ g_3 = \mathbb{1}_{\check{C}_0}$. On the other hand, it is no longer true in general that g_1 and g_2 determine g_3 uniquely for the subset $(g_1, g_2, g_3) \in \mathsf{Comp}^2 \subset \mathsf{Hom}_{\check{\mathscr{K}}_M^2}(\check{C}_0, \check{C}_1) \times \mathsf{Hom}_{\check{\mathscr{K}}_M^2}(\check{C}_1, \check{C}_2) \times \mathsf{Hom}_{\check{\mathscr{K}}_M^2}(\check{C}_2, \check{C}_0)$.

We now explain some details of the construction. The objects of the category $\check{\mathscr{K}}_M^2$ are the same as those for $\check{\mathscr{K}}_M^1$. Every object $\check{C}$ has a canonical inverse $\check{C}^{-1}$. As in $\check{\mathscr{K}}_M^1$, a morphism $\check{C}_0 \longmapsto \check{C}_1$ is the same thing as a morphism $\check{C}_0^{-1} + \check{C}_1 \longmapsto 0$. Therefore, we need only describe morphisms to the zero object. A morphism $g : (E, \nabla, C) \longmapsto 0$ is given by a triple $\big(\widetilde{G}, \widetilde{\nabla}, \lambda \big)$, with the pair $\big(\widetilde{G}, \widetilde{\nabla} \big)$ on $M \times I$ such that $\big(\widetilde{G}, \widetilde{\nabla} \big)\big|_{M \times 0} = (E, \nabla)$ and $\big(\widetilde{G}, \widetilde{\nabla} \big)\big|_{M \times 1} = 0$, while $-C + \mathrm{d}\lambda = \int_0^1 \mathsf{ch}(\widetilde{\nabla})$. Let $g' : (E, \nabla, C) \xrightarrow{(\widetilde{G}\,', \widetilde{\nabla}\,', \lambda')} 0$ be another morphism. A 2-morphism $\alpha : g \Longrightarrow g'$ is given by a triple $(\boldsymbol{G}, \boldsymbol{\nabla}, \boldsymbol{\lambda})$, where $(\boldsymbol{G}, \boldsymbol{\nabla})$ is graded vector bundle with connection on $M \times I \times I$ such that

$(\boldsymbol{G}, \boldsymbol{\nabla})|_{M \times 0 \times I} = (\widetilde{G}, \widetilde{\nabla})$, $(\boldsymbol{G}, \boldsymbol{\nabla}, \boldsymbol{\lambda})|_{M \times I \times 0} = (E \times I, \nabla \times 1)$ and $(\boldsymbol{G}, \boldsymbol{\nabla})|_{M \times 1 \times I} = (\widetilde{G}', \widetilde{\nabla}')$, $(\boldsymbol{G}, \boldsymbol{\nabla})|_{M \times I \times 1} = 0$, and $\boldsymbol{\lambda} \in \Omega(M; K^\bullet)^{-1}/\operatorname{im}(d)$ such that $\lambda' = \lambda + \int_0^1 \operatorname{ch}(\boldsymbol{\nabla}) + \mathrm{d}\boldsymbol{\lambda}$.

We declare a pair of 2-morphisms $\alpha_0, \alpha_1 : g \implies g'$ to be equivalent if there is a triple $(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}}, \widetilde{\boldsymbol{\lambda}})$, where $(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})$ is a $\mathbb{Z}_2$-graded vector bundle with connection on $M \times I \times I \times I$ having the boundary values

$$(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})|_{M \times 0 \times I \times I} = (\boldsymbol{G}_0, \boldsymbol{\nabla}_0) \qquad \text{and} \qquad (\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})|_{M \times 1 \times I \times I} = (\boldsymbol{G}_1, \boldsymbol{\nabla}_1),$$

$$(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})|_{M \times I \times 0 \times I} = (\widetilde{G}_0 \times I, \widetilde{\nabla}_0 \times 1) \qquad \text{and} \qquad (\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})|_{M \times I \times 1 \times I} = (\widetilde{G}_1 \times I, \widetilde{\nabla}_1 \times 1),$$

$$(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})|_{M \times I \times I \times 0} = (E \times I \times I, \nabla \times 1 \times 1) \qquad \text{and} \qquad (\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{\nabla}})|_{M \times I \times I \times 1} = 0,$$

and $\widetilde{\boldsymbol{\lambda}} \in \Omega(M; K^\bullet)^0/\operatorname{im}(d)$ such that $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_0 - \int_0^1 \operatorname{ch}(\widetilde{\boldsymbol{\nabla}}) = \mathrm{d}\widetilde{\boldsymbol{\lambda}}$.

Composition of 2-morphisms is now defined in the same manner as composition for 1-morphisms in the preceeding level of the hierarchy. The set $\mathsf{Comp}^2 \subset \mathsf{Hom}_{\check{\mathscr{K}}_M^2}(\check{C}_0 + \check{C}_1^{-1}, 0) \times \mathsf{Hom}_{\check{\mathscr{K}}_M^2}(\check{C}_1 + \check{C}_2^{-1}, 0) \times \mathsf{Hom}_{\check{\mathscr{K}}_M^2}(\check{C}_2 + \check{C}_0^{-1}, 0)$ now consists of all triples (g_1, g_2, g_3) such that there exists a 2-morphism $g_1 \circ g_2 \circ g_3 \implies \mathsf{Symm}$, where Symm represents the symmetric 2-morphisms. One can show that all 2-morphisms are isomorphisms. The set of isomorphism classes in $\check{\mathscr{K}}_M^2$ coincides with the set of morphisms in $\check{\mathscr{K}}_M^1$. The composition of 2-morphisms is associative. The composition of 1-morphisms is associative up to 2-morphisms. One can also replace cylinders in the construction of $\check{\mathscr{K}}_M^k$ by general bordisms.

Geometric cocycles

The index theory model of [13] readily permits all constructions required in gauge theory, at the price of introducing a very large configuration space, containing broad classes of fields, some of which have no interpretation in terms of D-branes wrapping cycles. Let $\pi : E \to M$ be a proper submersion with closed fibres and even-dimensional vertical bundle $T^v\pi := \ker(\mathrm{d}\pi)$. Choose a fibrewise riemannian metric on $T^v\pi$, and a complement $T^h\pi \subset TE$ which defines a horizontal distribution. We pick an orientation on $T^v\pi$, and also a family of Dirac bundles over E, i.e. a $\mathbb{Z}_2$-graded hermitian vector bundle with connection (V, ∇) on E and Clifford multiplication $\mathsf{c} : T^v\pi \otimes V \to V$. In [10, 13] this collection of data was subsumed into the notion of a geometric family $\mathcal{E}$. A cocycle for a differential K-theory class in $\check{\mathrm{K}}^0(M)$ is a pair $(\mathcal{E}, \xi)$, where $\mathcal{E}$ is a geometric family and $\xi \in \Omega(M; K^\bullet)^{-1}/\operatorname{im}(d)$ is a class of differential forms on M of odd degree; the equivalence relations on cocycles can be found in [13]. These classes are extensions of those defined above in terms of vector bundles with connection, to which they reduce when $\pi = \mathrm{id}_M$ has fibre consisting of just a point; within this model lie the well-developed and very powerful techniques of local index theory, whose properties can be used as a "black box" to efficiently carry out all constructions. The index of the family of Dirac operators $\not{D}(\mathcal{E})$ on a geometric family $\mathcal{E}$ over M can be naturally considered as an element of K-theory $\mathsf{Index}(\mathcal{E}) \in \mathrm{K}^0(M)$; it defines the characteristic class map $c : \check{\mathrm{K}}^0(M) \to \mathrm{K}^0(M)$ as

$$c\big([\mathcal{E}, \xi]\big) := \mathsf{Index}(\mathcal{E}).$$

For a geometric family $\mathcal{E}$, the local index form $\Omega(\mathcal{E}) \in \Omega(M; K^\bullet)^0$ [10, 13] is the adiabatic limit of local traces of the heat kernel of a Bismut superconnection on the associated Hilbert bundle $H(\mathcal{E}) \to M$ with fibres $H_x := \mathrm{L}^2(E_x; V|_{E_x})$ for $x \in M$; it provides a canonical and explicit de Rham representative for the Chern character of the index of $\mathcal{E}$ through the index theorem for families [7] which reads $\operatorname{ch}\big(\mathsf{Index}(\mathcal{E})\big) = \big[\Omega(\mathcal{E})\big]$. The field strength map $F : \check{\mathrm{K}}^0(M) \to \Omega_{\mathbb{Z}}(M; K^\bullet)^0$ is then given by

$$F\big([\mathcal{E}, \xi]\big) := \Omega(\mathcal{E}) - \mathrm{d}\xi.$$

A refinement of the Chern character homomorphism between differential K-theory and differential cohomology in this model can be found in [13].

We will now explain how topologically non-trivial quantized Ramond–Ramond fields naturally fit into this framework as cocycles in the absence of D-brane sources, i.e. as induced solely by the closed string background (M, g). For this, we will give a physical interpretation of this description in terms of brane-antibrane annihilation in boundary string field theory by presenting a special class of cocycles which exhibit the salient features of the construction of the Ramond–Ramond field associated to a K-theory class in (3.9). In this setting a Ramond–Ramond field on the manifold M is taken to be a pair $\check{C} = (\mathcal{E}, -C)$, where $\mathcal{E}$ is a geometric family over M and $C \in \Omega(M; K^\bullet)^{-1}$ (not taken modulo im(d)). In the topologically trivial case one sets $\mathcal{E} = \emptyset$ and the differential form C is what is usually called the Ramond–Ramond field of the Type IIA theory.

Consider first the Ramond–Ramond field $\check{C} = (\mathcal{V}, 0)$ for the geometric family $\mathcal{V}$ with underlying fibre bundle $\pi = \mathrm{id}_M : M \to M$ having zero-dimensional vertical bundle. The $\mathbb{Z}_2$-graded bundle $V = V^+ \oplus V^- \to M$ represents $\mathsf{Index}(\mathcal{V}) = [V] := [V^+] - [V^-] \in \mathrm{K}^0(M)$. The geometric K-homology class

$$[M, V^+, \mathrm{id}_M] - [M, V^-, \mathrm{id}_M]$$

represents a brane-antibrane pair filling M with Chan–Paton bundles (V^+, V^-) [48]. The connection ∇ and hermitian structure of the family of Dirac bundles $\mathcal{V}$ is the extra dynamical information on the D-branes encoded in the boundary string field theory, which naturally defines an element of differential K-theory. Every class in $\mathrm{K}^0(M)$ (and hence every D-brane on M) can be realized via this construction as the index of a geometric family [13]. Using the explicit expression for the local index form $\Omega(\mathcal{V})$ in this case [10], the corresponding Ramond–Ramond field strength is given by

$$F(V) = \mathsf{ch}(V) \wedge \sqrt{\widehat{A}(M)} \ .$$

This construction thus reproduces the Moore–Witten derivation [45] for the Ramond–Ramond field strength (3.9) associated to the K-theory element $[V] \in \mathrm{K}^0(M)$ classifying the background brane-antibrane system wrapping M.

As in the case of geometric K-homology [48], the inclusion of more general families $\check{C} = (\mathcal{E}, -C)$ extends this description to include non-representable brane-antibrane pairs filling M which are represented by the K-homology classes

$$\left[E \, , \, \mathsf{Index}(\mathcal{E}) \, , \, \pi \right] \ ,$$

where $\pi : E \to M$ is the underlying fibre bundle of $\mathcal{E}$ and $\mathsf{Index}(\mathcal{E}) \in \mathrm{K}^0(M)$. This modifies the associated Ramond–Ramond field strength to

$$F\big(\mathsf{Index}(\mathcal{E}) \big) = \Big(\int_{E/M} \mathsf{ch}(W) \wedge \widehat{A}(T^v \pi) + \mathrm{d}C \Big) \wedge \sqrt{\widehat{A}(M)} \ ,$$

where we have decomposed the family of Dirac bundles into the spinor bundle of $T^v \pi$ as $V = \mathcal{S}(T^v \pi) \otimes W$ for a twisting bundle $W \to E$ with metric and compatible connection. Here the fibrewise integral is the curvature generated by the (non-representable) background D-branes given by the local index form $\Omega(\mathcal{E}) \in \Omega(M; K^\bullet)^0$, which depends only on the geometric family $\mathcal{E}$, while $\mathrm{d}C$ is the contribution of a topologically trivial Ramond–Ramond field, which as such is not sourced by any D-branes.

Holonomy on D-branes

In analogy to the model of differential cohomology provided by Cheeger–Simons differential characters, which are U(1)-valued homomorphisms on the group of smooth cycles in M, one can define the

differential K-theory $\check{K}^\bullet(M)$ as a group of U(1)-valued homomorphisms on the set of Baum–Douglas K-cycles for geometric K-homology; these maps are called differential characters for K-theory in [6] and are interpreted as holonomies on D-branes in [49]. They are characterized by their restrictions to boundaries of K-chains (3.14), which are given by pairing a certain differential form $\omega|_{\widetilde{W}}$ with the index density $\mathsf{ch}\big(\widetilde{E}\big) \wedge \widehat{A}\big(\widetilde{W}\big)$.

As an explicit example, define the reduced eta-invariant of a K-chain with boundary (3.14) by

$$\Xi\big(\widetilde{W}\,,\,\widetilde{E}\,,\,\widetilde{f}\,\big) = \tfrac{1}{2}\left(\dim\big(\mathcal{H}^W_E\big) + \eta\big(\slashed{D}^W_E\big)\right) \;\in\; \mathbb{R}/\mathbb{Z}\;,$$

where $\mathcal{H}^W_E$ is the space of harmonic E-valued spinors on W, and $\eta\big(\slashed{D}^W_E\big)$ is the spectral asymmetry of the E-twisted Dirac operator $\slashed{D}^W_E$ on W which is the meromorphic continuation at $s = 0$ of the absolutely convergent series

$$\eta\big(s, \slashed{D}^M_E\big) = \sum_{\lambda \in \mathrm{spec}^0(\slashed{D}^M_E) - 0} \frac{\lambda}{|\lambda|^{s+1}}$$

for $s \in \mathbb{C}$ with $\mathfrak{Re}(s) \gg 0$, with the sum taken over the spectrum of the closure of $\slashed{D}^M_E$ which is the bounded Fredholm operator $\slashed{D}^M_E \left(1 + (\slashed{D}^M_E)^2\right)^{-1/2}$. The map Ξ respects disjoint union, direct sum and Baum–Douglas vector bundle modification of K-chains, but *not* spin^c bordism [6]. Then the *holonomy* of the flat D-brane background defined by (3.15) is given by

$$\Omega\big(\widetilde{W}\,,\,\widetilde{\xi}\,,\,\widetilde{f}\,\big) = \exp\left(2\pi\,\mathrm{i}\,\big(\Xi(\widetilde{W}, \widetilde{f^*E_0}, \widetilde{f}\,) - \Xi(\widetilde{W}, \widetilde{f^*E_1}, \widetilde{f}\,)\big)\right) \;\in\; \mathrm{U}(1)\;.$$

3.6. Self-dual field theories

We will now formulate the self-duality property of Ramond–Ramond fields more precisely. We do this first in the more general setting of §2.5.

Definition 3.18. *A generalized cohomology theory* $\mathrm{E}^\bullet$ *is* Pontrjagin self-dual *if there exists a "shift"* $s \in \mathbb{Z}$ *such that*

$$\mathrm{E}^\bullet(M) \cong \mathsf{Hom}_{\mathbb{Z}}\big(\mathrm{E}_{\bullet+s}(M)\,,\,\mathbb{Z}\big)$$

for all spaces M.

In Def. 3.18, $\mathrm{E}_\bullet$ is the homology theory obtained from $\mathrm{E}^\bullet$ through its *spectrum* $\{\mathcal{E}_k\}$ such that $\mathrm{E}^k(M)$ is the set of homotopy classes of maps $M \to \mathcal{E}_k$; there is a homotopy equivalence between $\mathcal{E}_k$ and the based loop space $\Omega\mathcal{E}_{k+1}$. The E-homology is given by the directed limit

$$\mathrm{E}_k(M) = \varinjlim_{r} \pi_{k+r}\big(M_+ \wedge \mathcal{E}_k\big)\;,$$

where $M_+ = M \sqcup m_0$ is the one-point compactification of M by a fixed base point $m_0 \in M$, and $X \wedge Y = X \times Y/(X \times y_0 \sqcup x_0 \times Y)$ is the smash product of locally compact spaces. Def. 3.18 is equivalent to the statement that for each $k \in \mathbb{Z}$ the natural pairing of real vector spaces

$$\check{\iota}\,:\, E^{k-s} \otimes E^{-k} \;\longrightarrow\; \mathbb{R} \tag{3.19}$$

is non-degenerate, where $E^k := \mathrm{E}^k(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 3.20. *A* self-dual *generalized abelian gauge theory on a compact n-dimensional smoothly* E-*oriented manifold N consists of a Pontrjagin self-dual multiplicative cohomology theory* $E^\bullet$ *with shift* $s \in \mathbb{Z}$, *and its associated configuration space of gauge fields* $\check{E}^d(N)$ *for some* $d \in \mathbb{Z}$, *together with a natural isomorphism*

$$\Psi^{-1} : E^d(N) \xrightarrow{\approx} E^{n-s+1-d}(N) . \tag{3.21}$$

Let N be an E-oriented riemannian manifold of dimension n, and $M = \mathbb{R} \times N$ the associated lorentzian spacetime of signature $(1, n)$. The *self-duality equations* for the gauge field $F \in \Omega(M; E^\bullet)^d$ read

$$\mathrm{d}F = 0 \qquad \text{and} \qquad \Psi^{-1}(F) = \check{\imath}(\star F) \in \Omega(M; E^\bullet)^{n-s+1-d} , \tag{3.22}$$

where the first equation is the Bianchi identity while the second equation is the self-duality condition. Here

$$\Psi^{-1} : \Omega^k(M; E^{d-k}) \longrightarrow \Omega^k(M; E^{n-s+1-d-k})$$

is induced by the isomorphism (3.21), the map

$$\star : \Omega^{n+1-k}(M; E^{d+k-n-1}) \longrightarrow \Omega^k\big(M; (E^{d+k-n-1})^*\big)$$

is the lorentzian Hodge duality operator, and

$$\check{\imath} : \Omega^k\big(M; (E^{d+k-n-1})^*\big) \longrightarrow \Omega^k(M; E^{n-s+1-d-k})$$

is induced by the pairing (3.19). The equations (3.22) define a first order linear hyperbolic differential equation, so a solution is determined at any fixed time $t \in \mathbb{R}$; whence the space of solutions is isomorphic to the real vector space $\Omega_{\mathrm{cl}}(N; E^\bullet)^d$.

Note that the *classical flux* $[F]_{\mathrm{dR}}$ defines a map $\Omega_{\mathrm{cl}}(N; E^\bullet)^d \to H(N; E^\bullet)^d$. In the semi-classical theory with Dirac charge quantization, the gauge field is a geometric representative of a class in $\check{E}^d(M)$; the space of classical solutions on M is then the differential cohomology group $\check{E}^d(N)$.

Incorporating sources into a self-dual gauge theory further requires an isomorphism between electric and magnetic currents j_e and j_m, as well as a quadratic refinement of the bilinear pairing $\check{E}^d(N) \otimes \check{E}^{n-s+1-d}(N) \to \mathbb{T}$ between the corresponding differential cocycles $\check{\jmath}_e$ and $\check{\jmath}_m$.

Type II Ramond–Ramond fields

In our main application, we take $n = 9$ and $E^\bullet = K^\bullet$ to be complex K-theory, so that $s = 0$. A Ramond–Ramond field on a compact riemannian spin manifold N has a gauge equivalence in the differential K-theory $\check{K}^j(N)$, where $j = 0$ for Type IIA string theory and $j = -1$ for Type IIB. The K-theory of a point is given by the Laurent polynomial ring $K^\bullet(\mathrm{pt}) \cong \mathbb{Z}(u)$, where u has degree two and the dual involution maps $u \mapsto u^* := u^{-1}$. The automorphism Ψ^{-1} is the Adams operation on K-theory which acts as complex conjugation, with $u \mapsto -u$. The lift of the Adams operation $\check{\Psi}^{-1}$ to differential K-theory is then given by

$$\check{\Psi}^{-1}\big([\check{C}]\big) = u^\ell \, \overline{[\check{C}]} , \tag{3.23}$$
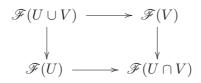
where $\ell = 5$ for Type IIA and $\ell = 6$ for Type IIB, and if $[\check{C}] \in \check{K}^j(N)$ is represented by a complex vector bundle $E \to N$ with connection ∇, then the class $\overline{[\check{C}]}$ is represented by the complex conjugate bundle $\overline{E} \to N$ with conjugate connection $\overline{\nabla}$; a model independent construction of all Adams operations $\check{\Psi}^k$, $k \in \mathbb{Z}$, is given in [11]. The self-duality equations for the field strengths $F \in \Omega\big(\mathbb{R} \times N, K^\bullet\big)^j$ are then as described in §3.2.

4 Quantization of generalized abelian gauge fields

4.1. Quantum actions and partition functions

We begin with some general remarks about the approach to the quantization of abelian gauge theories that we shall pursue. Recall that a generalized abelian gauge field on a manifold M is an object of a suitable groupoid $\check{\mathscr{E}}^d(M)$ whose isomorphism class sits in the generalized differential cohomology group $\check{\mathrm{E}}^d(M)$; this semi-classical quantization of the gauge theory leads to integrality of coupling constants and secondary invariants, and also to Dirac charge quantization. Once we have identified the configuration space $\check{\mathscr{E}}^d(M)$ of a generalized abelian gauge theory, in the functional integral approach to quantization we "integrate" over the isomorphism classes $\check{\mathrm{E}}^d(M)$ using a suitable translation invariant measure; such a Haar-like measure exists at least formally for gaussian fields and is induced by the riemannian metric on M. Let us briefly explain the meaning of such an integration, illustrated through several explicit examples.

To set up the path integral of the gauge theory, we regard the set of local fields $\mathscr{F}$ as a certain covariant functor from the (opposite) category of smooth manifolds with suitable morphisms to the category of sets. Locality of the fields is implemented by the requirement that the functor $\mathscr{F}$ satisfies a Mayer–Vietoris sheaf property, i.e. there is a pullback square

$$
\begin{array}{ccc}
\mathscr{F}(U \cup V) & \longrightarrow & \mathscr{F}(V) \\
\downarrow & & \downarrow \\
\mathscr{F}(U) & \longrightarrow & \mathscr{F}(U \cap V)
\end{array}
$$

for any pair of open charts U, V. In most of our examples we take $\mathscr{F} = \Omega^q$, with $q = 0$ corresponding to scalar fields, $q = 1$ to gauge fields, and so on. In our gauge theory examples we can generalize this to require that the fields be valued in the suitable configuration space, which requires replacing the category of sets such that one considers sheaves of groupoids, higher groupoids, or even ∞-groupoids; such is the case for e.g. double covers whose target is the category of simplicial sets.

Let $\mathscr{B}ord_m$ be the bordism category of smooth m-manifolds; an object of $\mathscr{B}ord_m$ is a closed $m-1$-manifold N, while a morphism from N_0 to N_1 is an m-manifold M with boundary $\partial M = N_0 \sqcup N_1$ and composition defined by gluing. Given a collection of fields $\mathscr{F}$, the bordism category $\mathscr{B}ord_m(\mathscr{F})$ enriched by $\mathscr{F}$ has the same objects, but its morphisms are extended to pairs (M, Φ) where $\Phi \in \mathscr{F}(M)$. Let $\mathscr{V}ect_\mathbb{C}$ be the category of complex vector spaces with linear transformations. The partition function of our gauge theory is a monoidal functor

$$
\mathscr{Z}_\mathscr{F} : \mathscr{B}ord_m(\mathscr{F}) \longrightarrow \mathscr{V}ect_\mathbb{C} , \tag{4.1}
$$

which sends disjoint unions to tensor products. Semi-classical quantization corresponds to restricting $\mathscr{Z}_\mathscr{F}$ so that it takes values in invertible objects of $\mathscr{V}ect_\mathbb{C}$; if M is closed then $\mathscr{Z}_\mathscr{F}(M, \Phi) \in \mathbb{C}^\times$ and we write $\mathscr{Z}_\mathscr{F}(M, \Phi) =: \exp \mathrm{i}\, S_M[\Phi]$. See [33] for further details and constructions.

Let us look at a simple example of a higher abelian gauge theory to demonstrate the need for using cocycles as objects in a suitable category in order to formulate the path integral of the quantum theory. In the setting of §2.3., the field content of a generic higher abelian gauge theory is $\Phi = (g, \check{j}_e, \check{j}_m, F)$ where g is the metric of M, the differential cocycles $\check{j}_e$ and $\check{j}_m$ are smooth refinements of electric and magnetic current forms $j_e = \mathrm{d} \star F \in \Omega^{n-p+2}(M)$ and $j_m = \mathrm{d}F \in \Omega^{p+1}(M)$, and $F \in \Omega^p(M)$. We take $p = 1$, and set $j_e = \sum_{x \in W_e} q_e(x)\, \mathsf{Pd}_M(x)$ where $q_e(x) \in \mathbb{R}$ are electric charges inserted at a collection of points $x \in W_e \subset M$. If $j_m = 0$, then $\mathrm{d}F = 0$ and F can be refined to a differential cohomology class $\lambda \in \check{\mathrm{H}}^1(M) = \Omega^0(M; \mathrm{U}(1))$, i.e. a smooth

map $\lambda : M \rightarrow S^1$. Then $F = \mathrm{d}\log\lambda$. In the quantum gauge theory, exponentiation of the action functional (2.11) gives

$$\exp \mathrm{i}\, S_M[\lambda] = \exp\left(-\frac{\mathrm{i}}{2}\int_M \frac{\mathrm{d}\lambda \wedge \star \mathrm{d}\lambda}{\lambda^2}\right) \prod_{x \in W_e} \lambda(x)^{q_e(x)}\,, \qquad (4.2)$$

which is well-defined and $\mathbb{C}$-valued provided that electric charge is quantized, $q_e(x) \in \mathbb{Z}$. Suppose now that $j_m \neq 0$. Since $j_m = \mathrm{d}F$ is trivialised, we can refine it to a class in the differential cohomology $\check{\mathrm{H}}^2(M)$, represented by a hermitian line bundle with connection (L, ∇) on M. We now take $\lambda \in \Omega^0(M; L)$ and set $F = \mathrm{d}_\nabla \log \lambda$ so that $j_m = \nabla^2$ is the curvature of ∇. Then the product in (4.2) lies in the fibres $\bigotimes_{x \in W_e}(L_x)^{\otimes q_e(x)}$, and the quantum action (4.2) takes values in a line bundle (rather than in $\mathbb{C}$). The line bundle is an obstruction to defining the path integral and it represents an *anomaly*. "Anomaly cancellation" corresponds to a trivialization of this line bundle; see [26] for further details, and [26, 28] for an extension to Green–Schwarz anomaly cancellation in Type I superstring theory.

More generally, the product in (4.2) is replaced with

$$\chi(W) = \exp\left(\mathrm{i}\int_W q_e\, A\big|_W\right)$$

for a p-brane W. Locality requires that $\chi \in \mathsf{Hom}_{\mathscr{A}b}(Z_{p-1}(M), \mathrm{U}(1))$. The equations of motion imply that

$$\chi(W') = \chi(W)\, \exp\left(\mathrm{i}\int_B q_e\, F\right)$$

if B is a bordism between W and W'. This means that χ is a Cheeger–Simons differential character (Def. 2.14), and $q_e\, F \in \mathrm{im}(\delta_1)$ where $\delta_1 : \check{\mathrm{H}}^p(M) \rightarrow \Omega^p_{\mathbb{Z}}(M)$ is the field strength map $\delta_1(\chi) = F_\chi$. When $p = 0$ and the field strength F is produced by a magnetic brane as above, we immediately arrive at Dirac quantization of charge $q_e\, q_m \in 2\pi\,\mathbb{Z}$, as argued from a different perspective in §2.2. More generally, the charges live in the lattice obtained from the image of integral cohomology in (2.13). In dimensions $m = n + 1 = 4s + 3$ with $p = 2s + 2$, one can also add a Chern–Simons term

$$\exp\left(\mathrm{i}\,\langle \check{A}, \check{A}\rangle\right)\,.$$

For $s = 0$ this term is well-defined by picking a spin structure on the three-manifold M.

One can also have charges in images of generalised cohomology theories. For example, in Type II superstring theory the D-brane (W, E, f) carries a vector bundle with connection (E, ∇). If C is a globally-defined Ramond–Ramond gauge potential, then the product in (4.2) is replaced with

$$\exp\left(\mathrm{i}\int_W Q(W, E, f) \wedge C\big|_W\right)$$

where $Q(W, E, f)$ is the charge vector (3.2).

Let us finally consider an example from M-theory. The field content $\Phi = (\sigma, g, A, \psi, C)$ of supergravity on an 11-dimensional spin manifold M consists of a spin structure σ on M, a riemannian metric $g \in \Omega^0(M; T^*M \otimes T^*M)$, a connection one-form A on a principal bundle over M, a twisted spinor field $\psi \in \Omega^0(M; T^*M \otimes \mathcal{S}_M)$ where $\mathcal{S}_M$ is the twisted spin bundle of M, and an abelian gauge potential $C \in \Omega^3(M)$ with field strength $G = \mathrm{d}C \in \Omega^4(M)$. The relevant terms in the quantum supergravity action are

$$\exp\left(\mathrm{i}\int_M G \wedge \star G + 2\,\mathrm{i}\int_M \left(C \wedge G \wedge G - C \wedge I_8(g)\right)\right) \qquad (4.3)$$

where $I_8(g) = \frac{1}{48}\big(4p_2(M) - p_1(M)^{\wedge 3}\big)$ and $p_k(M) \in \mathrm{H}^{4k}(M;\mathbb{Z})$ are the Pontrjagin classes of the tangent bundle of M. For topologically non-trivial fields, we refine the three-form C to a class $\check{C} \in \check{\mathrm{H}}^4(M)$. Then the topological terms in (4.3) refine to $\exp\big(\frac{\mathrm{i}}{6}\int_M^{\check{\mathrm{H}}} \check{C} \smile \check{C} \smile \check{C}\big)$; making this term well-defined requires a cubic refinement of the trilinear form $\check{\mathrm{H}}^4(M) \times \check{\mathrm{H}}^4(M) \times \check{\mathrm{H}}^4(M) \to \mathbb{T}$ defined by it. Since $K(\mathbb{Z},4) = B\,\mathrm{E}_8$ the charge $c \in \mathrm{H}^4(M;\mathbb{Z})$ is an isomorphism class of a principal E_8-bundle over M [21] (up to approximation on the skeleton of M). The groupoid of fields $\mathscr{H}^4(M)$ consists of cocycles $\check{C} = (P, \nabla, C) \in \check{Z}^4(M)$, where $P \to M$ is an E_8-bundle with connection ∇ and $C \in \Omega^3(M)$; gauge transformations connect cocycles (P, ∇, C) and (P', ∇', C') with $C' - C = \mathsf{CS}(\nabla, \nabla') + F_\chi$ for some $\chi \in \check{\mathrm{H}}^3(M)$. This implies that $G = F(\check{C}) = \mathrm{Tr}\big(F_\nabla \wedge F_\nabla\big) - \frac{1}{2}\,\mathrm{Tr}\big(R(g) \wedge R(g)\big) + \mathrm{d}C$ is gauge-invariant, where $F_\nabla = \nabla^2$ is the curvature of the connection ∇ and $R(g)$ is the curvature two-form of the metric g with $\mathrm{Tr}\big(R(g) \wedge R(g)\big) = \frac{1}{4}\,p_1(M)$; see [57, 22, 30] for further details.

The functor (4.1) with values in an invertible quantum field theory can be formally gotten by performing the functional integral over the configuration groupoid $\mathscr{H}^p(M)$, with $\pi_0 \mathscr{H}^p(M) = \check{\mathrm{H}}^p(M)$, of free higher abelian gauge theory, which is studied in [40] using techniques of covariant quantization on compact closed manifolds: For $\Phi = (g, \check{A})$ with g a riemannian metric on M and $[\check{A}] \in \check{\mathrm{H}}^p(M)$, by [40, Thm. 4.5] the partition function is rigorously defined by the formula

$$\mathscr{Z}_p(M) = \prod_{j=0}^{p-1} \left(\frac{\mathrm{det}'\big(\mathrm{d}^\dagger \mathrm{d}\big|_{\Omega^j(M) \cap \mathrm{im}(\mathrm{d}^\dagger)}\big)}{\mathrm{vol}\big(\mathtt{harm}^j(M)\,/\,\mathtt{harm}^j_{\mathbb{Z}}(M)\big)^2} \right)^{\frac{1}{2}(-1)^{p-j}} \Theta_p(M)\,\big|\mathsf{Tor}\,\mathrm{H}^p(M;\mathbb{Z})\big| \qquad (4.4)$$

where $\Theta_p(M) = \sum_{f \in \mathtt{harm}^p_{\mathbb{Z}}(M)} \exp\big(-\frac{1}{2}\int_M f \wedge \star f\big)$ is a Riemann theta-function on the lattice of harmonic p-forms f on M, i.e. $\mathrm{d}f = 0 = \mathrm{d} \star f$, with integer periods; it can be interpreted as a section of a line bundle over $\check{\mathrm{H}}^p(M)$, of the type that arises in Chern–Simons theory. The product in (4.4) arises from a formal gaussian integral over oscillator modes $F_0 + \mathrm{d}a$, with $a \in \Omega^{p-1}(M)$ modulo the small gauge invariances $a \mapsto a + \mathrm{d}\varepsilon$, $\varepsilon \mapsto \varepsilon + \mathrm{d}\eta$, and so on; it can be interpreted as a sort of analytic torsion [4], i.e. as a Quillen norm of a section of some determinant line bundle over the space of metrics on M. In [40] it is shown that the partition function (4.4) exhibits an electric-magnetic duality relation wherein $\mathscr{Z}_p(M)$ and $\mathscr{Z}_{n+1-p}(M)$ are proportional to each other, which follows from Hodge theory and the Poisson resummation formula. A functional integral approach to the quantization of self-dual higher abelian gauge fields is similarly described in [4] using higher-dimensional Chern–Simons theories, and further elucidated in [44]; an analogous path integral quantization of Ramond–Ramond gauge theory is carried out in [5].

4.2. Hamiltonian quantization

Another approach to constructing the functor (4.1) into an invertible quantum field theory is to categorify the partition function to the Hilbert space of the quantum field theory; as the partition function is generally valued in a line bundle, the Hilbert space is thus valued in a gerbe. In the remainder of this article we will explain how to construct this Hilbert space; we set $\check{j}_e = 0 = \check{j}_m$ for the rest of the section.

Consider again the spacetime manifold $M = \mathbb{R} \times N$ where N is a compact E-oriented n-dimensional riemannian manifold. As we demonstrate below, the configuration space of a free generalized abelian gauge theory on M is the generalized differential cohomology group $\check{\mathrm{E}}^d(N)$. Heuristically, the general principles of hamiltonian quantization suggest that the *Hilbert space* of the quantum field theory on which the fields act as operators is the space $\mathcal{H} = \mathrm{L}^2\big(\check{\mathrm{E}}^d(N)\big)$ of square integrable functions on the manifold $\check{\mathrm{E}}^d(N)$ with respect to a suitable measure. The problem, however, is that the differential cohomology $\check{\mathrm{E}}^d(N)$ is an infinite-dimensional vector space, so it

is tricky to define measures on it. Instead, we will approach the problem of quantization from a group theory perspective, and appeal to the representation theory of the Heisenberg group. This will identify the quantum Hilbert space as a representation of a certain Heisenberg extension of $\check{\mathrm{E}}^d(N)$ [31, 32].

Recall the classical definition of a Heisenberg group $\mathtt{Heis}(V, \omega)$ associated to a symplectic vector space (V, ω): It is a central extension of the translation group V by the circle group $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. Topologically $\mathtt{Heis}(V, \omega) \cong V \times \mathrm{U}(1)$ with a twisted multiplication

$$(v_1, z_1) \cdot (v_2, z_2) = \left(v_1 + v_2, \ \mathrm{e}^{\pi \, \mathrm{i} \, \omega(v_1, v_2)} \, z_1 \, z_2\right) .$$

The idea behind the relevance of the Heisenberg group in quantization is as follows. Let $\mathcal{G}$ be a topological abelian Lie group with Haar measure, and $\widehat{\mathcal{G}}$ the Pontrjagin dual group of characters $\chi : \mathcal{G} \to \mathrm{U}(1)$. The groups $\mathcal{G}$ and $\widehat{\mathcal{G}}$ both act on the Hilbert space $\mathcal{H} := \mathrm{L}^2(\mathcal{G})$, respectively as the translation or "momentum" operators

$$(T_h \psi)(g) = \psi(g + h)$$

and as the multiplication or "position" operators

$$(M_\chi \psi)(g) = \chi(g) \, \psi(g)$$

for $\psi \in \mathcal{H}$, $h, g \in \mathcal{G}$, and $\chi \in \widehat{\mathcal{G}}$. The Hilbert space $\mathcal{H}$ is not a representation of $\widetilde{\mathcal{G}} = \mathcal{G} \times \widehat{\mathcal{G}}$, since

$$T_h \circ M_\chi = \chi(h) \, M_\chi \circ T_h . \tag{4.5}$$

But the commutation relations (4.5) can be thought of as originating from a suitable cocycle, and $\mathcal{H}$ *is* a representation of the Heisenberg group $\mathtt{Heis}(\widetilde{\mathcal{G}})$ associated to $\widetilde{\mathcal{G}}$, which is a certain central extension of $\widetilde{\mathcal{G}}$ by $\mathrm{U}(1)$; specifically, $\mathcal{H}$ is the unique Stone–von Neumann representation of $\mathtt{Heis}(\widetilde{\mathcal{G}})$. We will now proceed to define these concepts precisely.

4.3. Heisenberg groups and their representations

We will begin by collecting some general results concerning central extensions of Lie groups and their representations, following [31].

Definition 4.6. *Let $\mathcal{G}$ be an abelian Lie group. A* generalized Heisenberg group *is a Lie group $\mathtt{Heis}(\mathcal{G})$ which sits inside the exact sequence*

$$1 \ \longrightarrow \ \mathrm{U}(1) \ \longrightarrow \ \mathtt{Heis}(\mathcal{G}) \ \longrightarrow \ \mathcal{G} \ \longrightarrow \ 0 \, ,$$

with the circle group $\mathrm{U}(1)$ contained in the center $Z_{\mathtt{Heis}(\mathcal{G})}$. We will further require that the group manifold of $\mathtt{Heis}(\mathcal{G})$ is a smooth, locally trivial circle bundle over $\mathcal{G}$. This is guaranteed by assuming the group $\mathcal{G}$ fits inside the exact sequence [31]

$$1 \ \longrightarrow \ \pi_1(\mathcal{G}) \ \longrightarrow \ \mathfrak{g} \ \overset{\exp}{\longrightarrow} \ \mathcal{G} \ \longrightarrow \ \pi_0(\mathcal{G}) \ \longrightarrow \ 0$$

where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$, and $\exp$ is the exponential map. A generalized Heisenberg group is said to be maximally noncommutative *if $Z_{\mathtt{Heis}(\mathcal{G})} = \mathrm{U}(1)$. A maximally noncommutative generalized Heisenberg group is simply called a* Heisenberg group.

Any smooth map $c : \mathcal{G} \times \mathcal{G} \to \mathrm{U}(1)$ satisfying the cocycle condition

$$c(g_1, g_2)\, c(g_1 + g_2, g_3) = c(g_1, g_2 + g_3)\, c(g_2, g_3)$$

for all $g_1, g_2, g_3 \in \mathcal{G}$ defines a generalized Heisenberg group denoted $\mathtt{Heis}(\mathcal{G}; c)$. As a manifold $\mathtt{Heis}(\mathcal{G}; c)$ is topologically the product $\mathcal{G} \times \mathrm{U}(1)$, while the multiplication is defined by

$$(g_1, z_1) \cdot (g_2, z_2) := \big(g_1 + g_2 \, , \, c(g_1, g_2)\, z_1\, z_2 \big) \tag{4.7}$$

for all $g_1, g_2 \in \mathcal{G}$ and $z_1, z_2 \in \mathrm{U}(1)$. From the group cocycle c we construct the *commutator map*

$$s \, : \, \mathcal{G} \times \mathcal{G} \; \longrightarrow \; \mathrm{U}(1) \tag{4.8}$$

defined by

$$s(g_1, g_2) := \frac{c(g_1, g_2)}{c(g_2, g_1)} \tag{4.9}$$

for all $g_1, g_2 \in \mathcal{G}$. The commutator map s enjoys the following properties:

1. From the definition (4.7) of the group operation in $\mathtt{Heis}(\mathcal{G}; c)$ and (4.9) it follows that the group commutator is given by

$$\big[(g_1, z_1) \, , \, (g_2, z_2) \big] = \big(0 \, , \, s(g_1, g_2) \big) \, . \tag{4.10}$$

2. s is alternating:

$$s(g, g) = 1 \, .$$

3. s is bimultiplicative:

$$s(g_1 + g_2, h) = s(g_1, h)\, s(g_2, h) \qquad \text{and} \qquad s(g, h_1 + h_2) = s(g, h_1)\, s(g, h_2) \, .$$

4. s is skew-symmetric:

$$s(g, h) = s(h, g)^{-1} \, .$$

Given a smooth map $f : \mathcal{G} \to \mathrm{U}(1)$, consider the cocycle $\tilde{c}$ for the group $\mathcal{G}$ defined by

$$\tilde{c}(g_1, g_2) := \frac{f(g_1\, g_2)}{f(g_1)\, f(g_2)}\, c(g_1, g_2) \, .$$

We say that $\tilde{c}$ and c differ by a *coboundary*. The map

$$(g, z) \; \longmapsto \; \big(g \, , \, f(g)\, z \big)$$

induces an isomorphism

$$\mathtt{Heis}(\mathcal{G}; c) \; \xrightarrow{\;\approx\;} \; \mathtt{Heis}(\mathcal{G}; \tilde{c}) \, .$$

It follows easily that

$$\tilde{s}(g_1, g_2) = s(g_1, g_2)$$

for all $g_1, g_2 \in \mathcal{G}$. A complete characterization of generalized Heisenberg groups is given in [31].

Proposition 4.11. *Given an abelian Lie group $\mathcal{G}$, any generalized Heisenberg group $\mathtt{Heis}(\mathcal{G})$ is of the form $\mathtt{Heis}(\mathcal{G}; c)$ for some cocycle c, and $\mathtt{Heis}(\mathcal{G})$ is uniquely determined up to isomorphism by its commutator map (4.8). Conversely, every alternating and bimultiplicative map $s : \mathcal{G} \times \mathcal{G} \to \mathrm{U}(1)$ uniquely determines a Heisenberg group $\mathtt{Heis}(\mathcal{G})$ up to isomorphism.*

Denote by $\mathscr{E}_{\mathcal{G}}$ the category of central extensions of $\mathcal{G}$ by the circle group $\mathrm{U}(1)$ with the usual morphisms. Let $\mathscr{C}_{\mathcal{G}}$ be the category whose objects are bimultiplicative maps $\psi : \mathcal{G} \times \mathcal{G} \to \mathrm{U}(1)$ (and hence automatically satisfy the cocycle condition), and whose morphisms are *quadratic* maps $f : \mathcal{G} \to \mathrm{U}(1)$, sending ψ to the map $\psi\,\psi_f$ where

$$\psi_f(g_1, g_2) = \frac{f(g_1\,g_2)}{f(g_1)\,f(g_2)} \ .$$

The bimultiplicativity of the coboundary ψ_f is precisely what is meant by f being quadratic. Then there is a functor

$$\mathscr{C}_{\mathcal{G}} \longrightarrow \mathscr{E}_{\mathcal{G}}$$

which assigns a central extension to a cocycle. By Prop. 4.11, this defines a natural equivalence of categories.

Suppose that we weaken the definition of the commutator map s so that it is skew-symmetric but not alternating. This implies

$$s(g, g)^2 = 1$$

for all $g \in \mathcal{G}$, and so the group $\mathcal{G}$ acquires a natural $\mathbb{Z}_2$-grading given by the homomorphism $g \mapsto s(g, g) \in \mathbb{Z}_2$. A *graded (generalized) Heisenberg group* is a central extension $\mathtt{Heis}(\mathcal{G})$ of $\mathcal{G}$ by $\mathrm{U}(1)$ which is at the same time a $\mathbb{Z}_2$-graded group; the maps in the central extension are $\mathbb{Z}_2$-graded homomorphisms, with $\mathrm{U}(1)$ regarded as trivially graded. A graded central extension naturally determines a commutator map which is skew-symmetric but not alternating [31]. Note that to every such skew-symmetric bimultiplicative map s one can assign an alternating bimultiplicative map $\widetilde{s}$ by defining

$$\widetilde{s}(g_1, g_2) := s(g_1, g_2) \ \exp\big(-\pi\,\mathrm{i}\,\varepsilon(g_1)\,\varepsilon(g_2) \big) \tag{4.12}$$

where $\varepsilon(g)$ is defined (modulo 2) through $s(g, g) = \exp\big(\pi\,\mathrm{i}\,\varepsilon(g)\big)$.

Proposition 4.13. *Every graded central extension of an abelian Lie group $\mathcal{G}$ by $\mathrm{U}(1)$ is determined uniquely up to isomorphism by its graded commutator map (4.8). Conversely, every skew-symmetric and bimultiplicative map $s : \mathcal{G} \times \mathcal{G} \to \mathrm{U}(1)$ uniquely determines such a graded central extension up to isomorphism.*

Generalized Stone–von Neumann theorem

One of the main results in the theory of generalized Heisenberg groups is the fact that the irreducible unitary representations are uniquely determined. This is essentially an extension of the Stone–von Neumann theorem. Consider the group

$$Z_{\mathcal{G};c} := \big\{ g \in \mathcal{G} \,\big|\, s(g, h) = 1 \quad \forall h \in \mathcal{G} \big\} \ . \tag{4.14}$$

The center $Z_{\mathtt{Heis}(\mathcal{G};c)}$ of $\mathtt{Heis}(\mathcal{G}; c)$ sits in the exact sequence

$$1 \ \longrightarrow \ \mathrm{U}(1) \ \stackrel{i}{\longrightarrow} \ Z_{\mathtt{Heis}(\mathcal{G};c)} \ \longrightarrow \ Z_{\mathcal{G};c} \ \longrightarrow \ 0$$

where i is the inclusion. In any irreducible unitary representation ρ of $\mathtt{Heis}(\mathcal{G}; c)$, by Schur's lemma the center acts by scalar multiplication as elements of the circle group $\mathrm{U}(1)$. Since the representations we are considering satisfy $(\rho \circ i)(z) = z\,\mathrm{id}$ for all $z \in \mathrm{U}(1)$, it follows that this sequence splits non-canonically via a homomorphism $\chi : Z_{\mathtt{Heis}(\mathcal{G};c)} \to \mathrm{U}(1)$.

Proposition 4.15. *Any irreducible unitary representation of a generalized Heisenberg group of finite dimension $\mathtt{Heis}(\mathcal{G}; c)$ for which $\mathrm{U}(1) \subseteq Z_{\mathtt{Heis}(\mathcal{G};c)}$ acts by the identity character is uniquely determined up to isomorphism by a splitting homomorphism $\chi : Z_{\mathtt{Heis}(\mathcal{G};c)} \to \mathrm{U}(1)$. Conversely, any such homomorphism χ gives rise to such an irreducible unitary representation of $\mathtt{Heis}(\mathcal{G}; c)$.*

Corollary 4.16. *If the commutator map s is* non-degenerate, *i.e. the group (4.14) is the trivial group $Z_{\mathcal{G};c} = 0$, then up to isomorphism there is a unique irreducible unitary representation of* $\mathrm{Heis}(\mathcal{G};c)$ *for which the center $Z_{\mathrm{Heis}(\mathcal{G};c)} = \{(0, z) \mid z \in \mathrm{U}(1)\}$ acts by scalar multiplication.*

Examples

- Let us return to the example $\widetilde{\mathcal{G}} = \mathcal{G} \times \widehat{\mathcal{G}}$ and $\mathcal{H} = \mathrm{L}^2(\mathcal{G})$ from §4.2. The Heisenberg group extending $\widetilde{\mathcal{G}}$ has a representation

$$\mathrm{Heis}(\widetilde{\mathcal{G}}) \longrightarrow \mathrm{GL}(\mathcal{H}) , \qquad \big((g, \chi), z\big) \longmapsto z\, T_g \circ M_\chi .$$

  The cocycle in this case is given by

$$c\big((g_1, \chi_1), (g_2, \chi_2)\big) = \frac{1}{\chi_1(g_2)} ,$$

  and its antisymmetrization gives the commutator map

$$s\big((g_1, \chi_1), (g_2, \chi_2)\big) = \frac{\chi_2(g_1)}{\chi_1(g_2)} .$$

- Let $\mathcal{G} = \mathbb{R}$, which we regard as parametrizing "coordinate" operators $\mathrm{e}^{\,\mathrm{i}\,p\,\hat{x}}$. The Pontrjagin dual $\widehat{\mathcal{G}} = \widehat{\widehat{\mathbb{R}}} \cong \mathbb{R}$ may then be regarded as parametrizing "momentum" operators $\mathrm{e}^{\,\mathrm{i}\,x\,\hat{p}}$, so that $\widetilde{\mathcal{G}} = \mathbb{R} \times \mathbb{R}$ is "phase space". More precisely, any character on $\mathcal{G}$ in this case is of the form $\chi(p) = \mathrm{e}^{\,\mathrm{i}\,x\,p}$ for some $x \in \mathbb{R}$. Then the commutator map s gives the canonical symplectic pairing on phase space, and the uniqueness result of Prop. 4.15 is the usual Stone–von Neumann theorem of quantum mechanics expressing uniqueness of the irreducible Schrödinger representation of the Heisenberg commutation relations.

Polarization

Prop. 4.15 can be generalized to infinite-dimensional abelian groups which are *polarized* [31].

Definition 4.17. *A* polarization *of an abelian Lie group $\mathcal{G}$ is an action of the real line $\mathbb{R}$ on the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ via operators $\{u_t\}_{t \in \mathbb{R}}$ which preserve the Lie bracket and decompose the complexification $\mathfrak{g}_{\mathbb{C}}$ into a countable direct sum of finite-dimensional subspaces $\mathfrak{g}_\lambda$, $\lambda \in \mathbb{R}$, such that u_t for each t acts on $\mathfrak{g}_\lambda$ as multiplication by $\mathrm{e}^{\,\mathrm{i}\,\lambda\,t}$. If $\mathcal{G}$ is a polarized group, then a unitary representation of the Heisenberg group $\mathrm{Heis}(\mathcal{G})$ on a Hilbert space $\mathcal{H}$ is said to be of* positive energy *if there is a unitary action of the real line on $\mathcal{H}$ by operators $U_t = \exp(\mathrm{i}\,t\,H)$, $t \in \mathbb{R}$, which intertwine with the action of $\mathrm{Heis}(\mathcal{G})$ on $\mathcal{H}$ such that the operator H has discrete non-negative spectrum.*

In physical applications, the one-parameter family $\{u_t\}$ is typically given by hamiltonian flow on phase space while $\{U_t\}$ determines the time evolution of the associated quantum theory, hence the positive energy condition.

Proposition 4.18. *For a polarized generalized Heisenberg group $\mathrm{Heis}(\mathcal{G};c)$, any irreducible unitary representation of positive energy for which $\mathrm{U}(1) \subseteq Z_{\mathrm{Heis}(\mathcal{G};c)}$ acts by the identity character is uniquely determined up to isomorphism by a splitting homomorphism $\chi : Z_{\mathrm{Heis}(\mathcal{G};c)} \to \mathrm{U}(1)$. Conversely, any such homomorphism χ gives rise to such an irreducible unitary representation of $\mathrm{Heis}(\mathcal{G};c)$ of positive energy.*

If $\mathrm{Heis}(\mathcal{G})$ is a $\mathbb{Z}_2$-graded generalized Heisenberg group, then the quantum Hilbert space $\mathcal{H}$ automatically acquires a $\mathbb{Z}_2$-grading as well.

4.4. Quantization of free generalized abelian gauge theory

We will now explain the setting for the quantization of generalized abelian gauge theories following [31, 32]. We consider hamiltonian quantization of an abelian gauge theory with semi-classical configuration space $\check{\mathrm{E}}^d(M)$ given by a smooth refinement $\check{\mathrm{E}}^\bullet$ of a generalized cohomology theory $\mathrm{E}^\bullet$ on the spacetime $M = \mathbb{R} \times N$ with metric of indefinite signature, where N is a compact oriented riemannian manifold. For $[\check{A}] \in \check{\mathrm{E}}^d(M)$, the classical equations of motion for the gauge theory are

$$\mathrm{d}F([\check{A}]) = 0 \qquad \text{and} \qquad \mathrm{d} \star F([\check{A}]) = 0 \tag{4.19}$$

where $F : \check{\mathrm{E}}^d(M) \to \Omega_{\mathbb{Z}}(M; E^\bullet)^d$ is the field strength map (curvature), with $E^\bullet = \mathrm{E}^\bullet(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\star$ denotes the Hodge duality operator on M. Of course, the first equation is automatically satisfied. We are interested in the *space of classical solutions*, i.e. the subspace $\mathcal{M} \subset \check{\mathrm{E}}^d(M)$ of gauge fields $[\check{A}]$ which solve the equations (4.19).

For this, we decompose $F([\check{A}])$ on M as

$$F([\check{A}]) = B(t) - E(t) \wedge \mathrm{d}t \tag{4.20}$$

where t is the time coordinate on $\mathbb{R}$, and $B(t)$ and $E(t)$ for each $t \in \mathbb{R}$ are d- and $d-1$-forms on N, respectively. (The notation stems from the fact that when $\mathrm{E}^\bullet = \mathrm{H}^\bullet$ is ordinary cohomology and $d = 2$, i.e. in Maxwell theory, the forms $B(t)$ and $E(t)$ are the magnetic and electric fields, respectively.) Then (4.19) can be rewritten as

$$\frac{\partial}{\partial t} B = -\tilde{\mathrm{d}}E \qquad \text{and} \qquad \frac{\partial}{\partial t} \tilde{\star}E = \tilde{\mathrm{d}}\tilde{\star}B \ , \tag{4.21}$$

where $\tilde{\star}$ denotes the Hodge duality operator and $\tilde{\mathrm{d}}$ the exterior derivative on N. The Cauchy data for these first order linear elliptic differential equations are the values of $B(t)$ and $E(t)$ at a given initial time $t = t_0$; the corresponding solutions uniquely determine $F([\check{A}])$ on M through (4.20).

In particular, we can identify the solution space $\mathcal{M}$ with the tangent bundle $T\check{\mathrm{E}}^d(N)$ in the following way. First, notice that $T\check{\mathrm{E}}^d(N)$ can be trivialized as $\check{\mathrm{E}}^d(N) \times \Omega(N; E^\bullet)^{d-1}/\mathrm{im}(\mathrm{d})$. Consider the map

$$i_{t_0} : N \longrightarrow \mathbb{R} \times N \ , \qquad i_{t_0}(x) = (t_0, x) \ .$$

It induces a map from $\mathcal{M}$ to $T\check{\mathrm{E}}^d(N)$ by assigning to $[\check{A}]$ the pair $(i_{t_0}^*[\check{A}], E(t_0))$, where $E(t)$ is determined by the decomposition (4.20). The inverse map is obtained by assigning to the pair $([\check{B}], E)$ the unique element $[\check{A}] \in \check{\mathrm{E}}^d(M)$ such that $i_{t_0}^*[\check{A}] = [\check{B}]$ and $F([\check{A}]) = B(t) - E(t) \wedge \mathrm{d}t$, where $B(t)$ and $E(t)$ are obtained from (4.21) with initial conditions given by $B(t_0) = B$ and $\tilde{\star}E(t_0) = \tilde{\star}E$. The uniqueness of $[\check{A}]$ is assured by the fact that the map i_{t_0} induces the isomorphism $\mathrm{E}^{d-1}(M; \mathbb{T}) \cong \mathrm{E}^{d-1}(N; \mathbb{T})$ of cohomology classes (flat fields) in the kernel of the field strength transformation.

With the identification of the space of classical solutions as $\mathcal{M} = T\check{\mathrm{E}}^d(N)$, the standard hamiltonian quantization scheme suggests that the quantum Hilbert space of the generalized abelian gauge theory is given heuristically by the "space of L^2-functions on $\mathcal{G} = \check{\mathrm{E}}^d(N)$". A more precise definition is given in [31, 32], where it is proposed that the quantum (projective) Hilbert space $\mathcal{H}$ of a generalized abelian gauge theory with configuration space a group $\mathcal{G}$ is an irreducible representation of the generalized Heisenberg group

$$\mathtt{Heis}\big(\mathcal{G} \times \widehat{\mathcal{G}}\big) \ ,$$

where $\widehat{\mathcal{G}} = \mathsf{Hom}_{\mathscr{A}b}(\mathcal{G}, \mathrm{U}(1))$ is the group of characters of $\mathcal{G}$ in the category $\mathscr{A}b$ of abelian groups. The case of self-dual gauge theories is somewhat simpler, as then the phase space $\mathcal{G} \times \widehat{\mathcal{G}}$ degenerates

to the configuration space; in this case, due to the self-duality equations (3.22), the space of classical solutions on M may be identified with the diagonal subgroup $\mathcal{G} \cong \widehat{\mathcal{G}}$ and the quantization is carried out using the Heisenberg group $\mathtt{Heis}(\mathcal{G})$ itself. This technique can be applied to any abelian group $\mathcal{G}$ based on a smooth refinement of a Pontrjagin self-dual generalized cohomology theory $\mathrm{E}^{\bullet}$ [31, App. B]; one quantizes the Poisson manifold $\mathcal{G}$ in this case.

Isomorphism classes of Heisenberg group extensions are determined by maps $s : \widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}} \to \mathrm{U}(1)$ which are skew-symmetric, alternating and bimultiplicative, where $\widetilde{\mathcal{G}} = \mathcal{G} \times \widehat{\mathcal{G}}$. For any generalized differential cohomology theory one can define a pairing

$$\check{\mathrm{E}}^d(N) \otimes \check{\mathrm{E}}^{n-s+1-d}(N) \longrightarrow \mathbb{T} , \qquad ([\check{A}], [\check{A}']) \longmapsto \int\!\!\!\!\!\int_N^{\check{\mathrm{E}}} [\check{A}] \smile [\check{A}'] . \qquad (4.22)$$

The perfectness of the pairing (4.22) is a feature of any generalized differential cohomology theory $\check{\mathrm{E}}^{\bullet}$, defined as explained in §2.5., for which $\mathrm{E}^{\bullet}$ is a Pontrjagin self-dual generalized cohomology theory, defined as in §3.6. The proof makes use of the fact that for such theories there is a perfect pairing

$$\mathrm{E}^d(N) \otimes \mathrm{E}^{n-s-d}(N; \mathbb{T}) \longrightarrow \mathbb{T} ,$$

and that the $\mathbb{R}/\mathbb{Z}$ cohomology $\mathrm{E}^{\bullet}(N; \mathbb{T})$ appears as the kernel of the field strength map; see [31, App. B] for details. The commutator map s may then be constructed by exponentiating the pairing (4.22).

The equations (4.19) can be obtained as the variational equations for the action functional

$$S([\check{A}]) := -\frac{1}{2} \int_M F([\check{A}]) \wedge \star F([\check{A}]) , \qquad (4.23)$$

and the hamiltonian derived from (4.23) is given by

$$H(t) := \frac{1}{2} \int_N \left(B \wedge \tilde{\star} B + E \wedge \tilde{\star} E \right) . \qquad (4.24)$$

The hamiltonian (4.24) is a non-negative function defined on the cotangent bundle $T^*\mathcal{G}$. At the identity, $T_0^*\mathcal{G}$ is the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. The canonical hamiltonian flow $\mathbb{R} \to T^*\mathcal{G}$ yields a family of maps $u_t : \mathfrak{g}^* \to \mathfrak{g}^*$ which is an action of $\mathbb{R}$ on $\mathfrak{g}^*$. If $\widehat{\mathcal{G}}$ is the Pontrjagin dual of $\mathcal{G}$ obtained through a non-degenerate pairing $\mathcal{G} \otimes \mathcal{G} \to \mathbb{T}$, then we obtain a family of operators acting on the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ of $\mathcal{G} \times \widehat{\mathcal{G}}$ which satisfies all the properties of a polarization. Quite generally, the choice of polarization also appears in Kähler quantization, wherein the Kähler form is given by the differential of the antisymmetric pairing; in this instance though one should clarify the origin of a suitable pre-quantum line bundle on the configuration groupoid.

In the self-dual case, this polarization does *not* induce a polarization on the diagonal subgroup. See [32] for a way to relate the self-dual gauge theory to a non-self-dual gauge theory in dimensions $\dim(M) = 4k + 2$, $k \in \mathbb{N}$; see also [4, 5, 44] where the complete pre-quantization data is specified. The problems with formulating self-dual higher abelian gauge theories which are both local and covariant go back to e.g. [36], see also [25].

In the following we will explicitly work out the cases where $\mathrm{E}^{\bullet} = \mathrm{H}^{\bullet}$ is ordinary cohomology and where $\mathrm{E}^{\bullet} = \mathrm{K}^{\bullet}$ is complex K-theory.

4.5. *Quantization of higher abelian gauge theory*

We will first apply this formalism to the Cheeger–Simons groups. Let us begin by giving the heuristic argument using canonical quantization of free fields. Let $(\check{A}, \Pi)$ denote local coordinates on

the phase space $T^*\check{\mathrm{H}}^p(N) = \check{\mathrm{H}}^p(N) \times \Omega_{\mathrm{cl}}^{n-p+1}(N)$ where $\Pi = \star F\big|_N$ with $[\star F]_{\mathrm{dR}} \in \mathrm{H}^{n-p+1}(N;\mathbb{R})$; we think of the "conjugate momentum" Π as the functional derivative operator $-\,\mathrm{i}\,\hbar\,\frac{\delta}{\delta \check{A}}$ which generates translations on the configuration space $\check{\mathrm{H}}^p(N)$. The differential cohomology $\check{\mathrm{H}}^p(N)$ is an abelian Lie group with a translation-invariant measure induced formally on $T\check{\mathrm{H}}^p(N)$ by the riemannian metric on N. By quantizing F, Π to operators $\hat{F}, \hat{\Pi}$ acting on the Hilbert space $\mathcal{H} := \mathrm{L}^2\big(\check{\mathrm{H}}^p(N)\big)$ we get the Heisenberg commutation relations

$$\left[\int_N \omega_1 \wedge \hat{F} , \int_N \omega_2 \wedge \hat{\Pi} \right] = \left(\mathrm{i}\,\hbar \int_N \omega_1 \wedge \mathrm{d}\omega_2 \right) \mathrm{id}_{\mathcal{H}}$$

for any pair of differential forms ω_1, ω_2. The right-hand side of these relations is just the pairing on globally defined forms in differential cohomology.

Now let us make this argument more precise. Using Pontrjagin duality

$$\check{\mathrm{H}}^p(N) \times \check{\mathrm{H}}^{n-p+1}(N) \longrightarrow \mathbb{T}$$

for N compact and oriented, we set

$$\widetilde{\mathcal{G}} = \check{\mathrm{H}}^p(N) \times \widehat{\check{\mathrm{H}}^p(N)} \cong \check{\mathrm{H}}^p(N) \times \check{\mathrm{H}}^{n-p+1}(N) \tag{4.25}$$

which is the phase space of an abelian gauge theory from §2.3. As the space of classical solutions on $M = \mathbb{R} \times N$ in this case is the tangent bundle on the Cheeger–Simons group $\check{\mathrm{H}}^p(N)$, the Hilbert space is heuristically $\mathcal{H} = \mathrm{L}^2\big(\check{\mathrm{H}}^p(N)\big)$. Since in this case the abelian group $\mathcal{G} = \check{\mathrm{H}}^p(N)$ is infinite-dimensional, the quantization must be specified by a polarization. In the hamiltonian formalism, a natural polarization is given by the energy operator $H = \mathrm{i}\,\frac{\partial}{\partial t}$; then the complexifcation of the space of classical solutions $\mathcal{M}$ is a sum of subspaces of positive and negative energy solutions. In this case the quantum (projective) Hilbert space $\mathcal{H}$ is the unique irreducible representation of the associated Heisenberg group which is compatible with the polarization. If N is non-compact, then this discussion needs to be modified using some (conjectural) analog of L^2-cohomology for the Cheeger–Simons groups.

In the present case the commutator map $s : \widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}} \to \mathrm{U}(1)$ can be constructed using the Pontrjagin–Poincaré duality property of ordinary differential cohomology to define the pairing

$$\langle -, - \rangle : \check{\mathrm{H}}^p(N) \otimes \check{\mathrm{H}}^{n-p+1}(N) \longrightarrow \mathbb{T} = \mathbb{R}/\mathbb{Z} , \qquad \langle [\check{A}], [\check{A}'] \rangle = \int\!\!\!\!\int_N^{\check{\mathrm{H}}} [\check{A}] \smile [\check{A}'] .$$

This pairing is *perfect*, i.e. it induces an isomorphism

$$\check{\mathrm{H}}^p(N) \cong \mathsf{Hom}_{\mathscr{A}b}\big(\check{\mathrm{H}}^{n-p+1}(N) , \mathbb{T}\big) ,$$

so that every homomorphism $\check{\mathrm{H}}^p(N) \to \mathbb{T}$ is given by pairing with an element of $\check{\mathrm{H}}^{n-p+1}(N)$. Let us sketch how to understand this isomorphism. Using the universal coefficient theorem and Poincaré duality, one shows that the pairing

$$\mathrm{H}^p(N;\mathbb{T}) \otimes \mathrm{H}^{n-p}(N) \longrightarrow \mathbb{T} , \qquad (\alpha, \alpha') \longmapsto \langle \alpha \smile \alpha', [N] \rangle$$

is perfect, where $[N]$ denotes the fundamental class of the manifold N. Moreover, one can define a pairing

$$\Omega_{\mathbb{Z}}^p(N) \otimes \Omega^{n-p}(N) \big/ \Omega_{\mathbb{Z}}^{n-p}(N) \longrightarrow \mathbb{T} , \qquad (F, [A]) \longmapsto \int_N F \wedge A \mod \mathbb{Z}$$

which is well-defined and perfect. Then by (2.16) there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^{p-1}(N;\mathbb{T}) & \longrightarrow & \check{\mathrm{H}}^p(N) & \xrightarrow{\;F\;} & \Omega^p_{\mathbb{Z}}(N) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\approx} & & \downarrow & & \downarrow{\scriptstyle\approx} & & \\
0 & \longrightarrow & \widehat{\mathrm{H}^{n-p+1}}(N;\mathbb{T}) & \longrightarrow & \widehat{\check{\mathrm{H}}^{n-p+1}}(N) & \longrightarrow & \widehat{\Omega^{n-p}(N)\big/\Omega^{n-p}_{\mathbb{Z}}}(N) & \longrightarrow & 0
\end{array}
$$

where $\widehat{G} := \mathsf{Hom}_{\mathscr{A}b}(G,\mathbb{T})$ and the vertical maps are the morphisms induced by the above pairings. It then follows that the middle arrow is an isomorphism as well.

By using this pairing we may define a non-degenerate commutator map $s : \widetilde{\mathcal{G}} \times \widetilde{\mathcal{G}} \to \mathrm{U}(1)$ by

$$
s\big(([\check{A}_1],[\check{A}'_1])\,,\,([\check{A}_2],[\check{A}'_2])\big) = \exp\Big(2\pi\,\mathrm{i}\,\big(\langle[\check{A}_2],[\check{A}'_1]\rangle - \langle[\check{A}_1],[\check{A}'_2]\rangle\big)\Big)
$$

which defines a central extension of the group (4.25). In this case, the map ε from (4.12) is given in terms of Wu classes which are polynomials in the Stiefel–Whitney classes of the tangent bundle of N [32]. After choosing a polarization, there thus exists a unique (up to isomorphism) irreducible Stone–von Neumann representation of the Heisenberg group $\mathtt{Heis}(\widetilde{\mathcal{G}})$ on which the central elements $(0,z)$ are realised as multiplication by $z \in \mathrm{U}(1)$; we identify this representation with the quantum Hilbert space $\mathcal{H} = \mathrm{L}^2\big(\check{\mathrm{H}}^p(N)\big)$ of the higher abelian gauge theory.

4.6. Quantization of Ramond–Ramond gauge theory

We will now consider the hamiltonian quantization of Ramond–Ramond gauge fields in the absence of D-brane sources, i.e. as induced solely by the closed string background. In the free Ramond–Ramond gauge theory on $M = \mathbb{R} \times N$ we have to contend with self-duality. In this case we set

$$
\widetilde{\mathcal{G}} = \mathcal{G} = \check{\mathrm{K}}^j(N)\;,
$$

and the commutator map s is obtained by restriction from $\check{\mathrm{K}}^j(N) \times \check{\mathrm{K}}^j(N)$ to its "diagonal" subgroup, in a sense that we now explain.

By composing the cup product with the integration map on differential K-theory, we define an intersection form

$$
(-,-) \,:\, \check{\mathrm{K}}^j(N) \otimes \check{\mathrm{K}}^j(N) \;\xrightarrow{\;\smile\;}\; \check{\mathrm{K}}^0(N) \;\xrightarrow{\;\int^{\check{\mathrm{K}}}\!\!\!\int_N\;}\; \check{\mathrm{K}}^{-1}(\mathrm{pt}) \cong \mathbb{T}\;,
$$

where we have used Bott periodicity and the assumption that $n = \dim(N)$ is odd. Explicitly, for Ramond–Ramond potentials $[\check{C}]$ and $[\check{C}']$ in complementary degrees, i.e. $\deg[\check{C}] + \deg[\check{C}'] = n+1$, one has

$$
([\check{C}],[\check{C}']) := \int^{\check{\mathrm{K}}}\!\!\!\int_N [\check{C}] \smile [\check{C}'] \;\in\; \mathbb{T}\;. \tag{4.26}
$$

By the general properties of Pontrjagin self-dual generalized cohomology theories, this pairing is *perfect*, i.e. it induces an isomorphism

$$
\check{\mathrm{K}}^j(N) \cong \mathsf{Hom}_{\mathscr{A}b}\big(\check{\mathrm{K}}^j(N)\,,\,\mathbb{T}\big)\;.
$$

But it is not necessarily *antisymmetric*, due to graded-commutativity of the cup product, e.g. in even degree $j = 0$ this pairing is symmetric. To this end we use the Adams operation (3.23) (lifting the complex conjugation map Ψ^{-1} of §3.6.) to define a new pairing by

$$
\langle[\check{C}],[\check{C}']\rangle := \Big[\int^{\check{\mathrm{K}}}\!\!\!\int_N [\check{C}] \smile \check{\Psi}^{-1}[\check{C}']\Big]_{u^0} \;\in\; \mathbb{T}\;, \tag{4.27}
$$

where again we regard differential forms on N as elements of the graded vector space $\Omega(N;K^\bullet)^\bullet$ with $K^\bullet = \mathbb{R}(u)$.

Theorem 4.28. *The pairing* $\langle -, - \rangle : \check{\mathrm{K}}^j(N) \otimes \check{\mathrm{K}}^j(N) \to \mathbb{T}$ *is non-degenerate, and it is antisymmetric in dimensions* $n \equiv 1 \bmod 4$.

Proof. The operator $u^{-\ell} \check{\Psi}^{-1}$ is an involution, and hence an isomorphism, and so since $(-, -)$ is non-degenerate, it follows that the pairing $\langle -, - \rangle$ is also non-degenerate. For illustration, we will prove antisymmetry for topologically trivial Ramond–Ramond fields in Type IIA string theory, i.e. potentials $[\check{C}] = [C] \in \check{\mathrm{K}}^0(N)$ which can be represented by odd degree differential forms $C \in \Omega(N; K^\bullet)^{-1}$ (modulo exact forms). Setting $n = 2k+1$, the pairing is then given modulo $\mathbb{Z}$ by

$$
\begin{aligned}
\langle [C], [C'] \rangle &= \Big[\int_N C \wedge \mathrm{d}\Psi^{-1}(C') \Big]_{u^0} \quad \bmod \mathbb{Z} \\[2mm]
&= \Big[\int_N \Big(\sum_{j=0}^{k} u^{-j-1} \otimes C_{2j+1} \Big) \wedge \Big(u^{k+1} \sum_{l=0}^{k} (-1)^l \, u^{-l-1} \otimes \mathrm{d}C'_{2l+1} \Big) \Big]_{u^0} \\[2mm]
&= (-1)^{k+1} \sum_{j=0}^{k} (-1)^j \int_N C_{2j+1} \wedge \mathrm{d}C'_{2(k-j)-1} \\[2mm]
&= (-1)^{k+1} \sum_{j=0}^{k} (-1)^j \int_N C'_{2(k-j)-1} \wedge \mathrm{d}C_{2j+1} \\[2mm]
&= (-1)^{k+1} \Big[\int_N C' \wedge \mathrm{d}\Psi^{-1}(C) \Big]_{u^0} = (-1)^{k+1} \langle [C'], [C] \rangle,
\end{aligned}
$$

where $C_p, C'_p \in \Omega^p(N)$; we have used the fact that N has no boundary and that the de Rham differential is a skew-derivation on forms. It follows that the pairing $\langle -, - \rangle$ is antisymmetric only when k is an even integer, i.e. when $n = \dim(N) = 1 + 4r$ with $r \in \mathbb{N}$. In the general case, the proof given in [31] involves the definition of an orthogonal version of differential K-theory, and its value on a point. $\square$

Thm. 4.28 applies in particular to the case relevant to the physical Type II superstring theory, where N is a nine-dimensional spin manifold. Using it we can now define the Type IIA/IIB commutator map

$$
s : \check{\mathrm{K}}^j(N) \times \check{\mathrm{K}}^j(N) \longrightarrow \mathrm{U}(1), \qquad s([\check{C}], [\check{C}']) = \exp\Big(2\pi \, \mathrm{i} \, \langle [\check{C}], [\check{C}'] \rangle \Big)
$$

in even/odd degree. It is bilinear, skew-symmetric, and non-degenerate, but it is *not* necessarily alternating. One has $s(g, g)^2 = 1$ for all $g \in \mathcal{G}$, so that $s(g, g) = \exp(\pi \, \mathrm{i} \, \varepsilon(g))$, where $\varepsilon : \mathcal{G} \to \mathbb{Z}_2$ is a group homomorphism which defines a $\mathbb{Z}_2$-*grading*. An alternating commutator map $\widetilde{s}$ is then defined by

$$
\widetilde{s}([\check{C}], [\check{C}']) = \exp\Big(2\pi \, \mathrm{i} \, \big(\langle [\check{C}], [\check{C}'] \rangle - \tfrac{1}{2} \varepsilon[\check{C}] \varepsilon[\check{C}'] \big) \Big). \tag{4.29}
$$

This definition does not depend on the chosen lifts. The degree

$$
\varepsilon[\check{C}] = 2 \langle [\check{C}], [\check{C}] \rangle = 2 \Big[\iint_N^{\check{\mathrm{K}}} [\check{C}] \smile \check{\Psi}^{-1}[\check{C}] \Big]_{u^0} \in \mathbb{Z}_2
$$

depends only on the characteristic class of $[\check{C}]$, i.e. $\varepsilon \in \mathrm{Hom}_{\mathscr{A}b}(\mathrm{K}^j(N), \mathbb{Z}_2)$; for $j = 0$ it can be identified with the mod 2 index of the Dirac operator on N coupled to the virtual bundle $\xi \otimes \overline{\xi}$, where $\xi = c([\check{C}])$.

Using Prop. 4.13 we may now define the Heisenberg group extension $\mathtt{Heis}\big(\check{\mathrm{K}}^j(N) \big)$ of the differential K-theory group $\mathcal{G} = \check{\mathrm{K}}^j(N)$ associated to the commutator map s, which is unique up

to (non-canonical) isomorphism. It is also $\mathbb{Z}_2$-graded with degree map $\varepsilon : \mathtt{Heis}\big(\check{\mathrm{K}}^j(N)\big) \to \mathbb{Z}_2$, with the maps in the central extension $\mathbb{Z}_2$-graded and the U(1) subgroup of even degree. The quantum Hilbert space $\mathcal{H}$ is also $\mathbb{Z}_2$-graded, and the unique $\mathbb{Z}_2$-graded irreducible representation of $\mathtt{Heis}\big(\check{\mathrm{K}}^j(N)\big)$ (after a choice of polarization) is compatible with the $\mathbb{Z}_2$-grading on $\mathsf{End}_{\mathbb{C}}(\mathcal{H})$.

Definition 4.30. *The quantum Hilbert space of the Ramond–Ramond gauge theory $\mathcal{H}_{\mathrm{RR}}$ is the unique irreducible $\mathbb{Z}_2$-graded unitary representation of the Heisenberg group $\mathtt{Heis}\big(\check{\mathrm{K}}^j(N)\big)$ of positive energy defined by the commutator map (4.29) which is compatible with the polarization discussed in §4.4., with the property that the central subgroup $\mathrm{U}(1)$ acts by scalar multiplication, where $j = 0/-1$ for the Type IIA/IIB string theory respectively.*

The $\mathbb{Z}_2$-grading implies, in particular, that the quantum Hilbert space $\mathcal{H}_{\mathrm{RR}}$ contains both bosonic and fermionic states. Let

$$\mathcal{O} : \mathtt{Heis}\big(\check{\mathrm{K}}^j(N)\big) \longrightarrow \mathsf{End}_{\mathbb{C}}(\mathcal{H}_{\mathrm{RR}})$$

denote the irreducible representation of Def. 4.30. Given classes $[\check{C}], [\check{C}'] \in \check{\mathrm{K}}^j(N)$, let $\check{\mathcal{C}} = ([\check{C}], z)$ and $\check{\mathcal{C}}' = ([\check{C}'], z')$ be lifts to the Heisenberg group $\mathtt{Heis}\big(\check{\mathrm{K}}^j(N)\big)$. From (4.10), with the commutator understood as the graded group commutator in $\mathtt{Heis}\big(\check{\mathrm{K}}^j(N)\big)$, together with $\mathcal{O}\big((0, z)\big) = z \, \mathrm{id}_{\mathcal{H}_{\mathrm{RR}}}$ for all $z \in \mathrm{U}(1)$, it follows that the commutation relations among the corresponding unitary operators on the (infinite-dimensional) quantum Hilbert space $\mathcal{H}_{\mathrm{RR}}$ of the Ramond–Ramond gauge theory are given by

$$\big[\mathcal{O}(\check{\mathcal{C}}), \, \mathcal{O}(\check{\mathcal{C}}')\big] = \widetilde{s}\big([\check{C}], [\check{C}']\big) \, \mathrm{id}_{\mathcal{H}_{\mathrm{RR}}} \tag{4.31}$$

in $\mathsf{End}_{\mathbb{C}}(\mathcal{H}_{\mathrm{RR}})$.

4.7. Noncommutative quantum flux sectors

The quantum flux sectors of the free abelian gauge theory can be described as follows. A state $\psi \in \mathrm{L}^2\big(\check{\mathrm{H}}^p(N)\big)$ of definite electric flux $E \in \mathrm{H}^{n-p+1}(N; \mathbb{Z})$ is an eigenstate of translation by flat fields, i.e.

$$\psi(\check{A} + \phi) = \exp\left(2\pi\,\mathrm{i} \int\!\!\!\!\!\int_N^{\mathrm{H}} E \smile \phi\right) \psi(\check{A}) \qquad \text{for} \quad \phi \in \mathrm{H}^{p-1}(N; \mathbb{T}) \ .$$

This defines a decomposition of the Hilbert space $\mathcal{H}$ into electric flux sectors labelled by $E \in \mathrm{H}^{n-p+1}(N; \mathbb{Z})$. Note that an analogous definition using electric fields $\check{E} \in \check{\mathrm{H}}^{n-p+1}(N)$ and arbitrary translations $\check{\phi} \in \check{\mathrm{H}}^{p-1}(N)$ would lead to wavefunctionals ψ which are neither compactly supported nor decaying; in particular, $\check{E}_1$ is homotopic to $\check{E}_2$ if and only if $\int_N^{\check{\mathrm{H}}} \check{\phi} \smile \check{E}_1 = \int_N^{\check{\mathrm{H}}} \check{\phi} \smile \check{E}_2$ for $\phi \in \mathrm{H}^{p-1}(N; \mathbb{T})$, or equivalently if and only if $\int_N^{\check{\mathrm{H}}} \phi \smile c(\check{E}_1) = \int_N^{\check{\mathrm{H}}} \phi \smile c(\check{E}_2)$. Using dual flat fields, we also get a decomposition of $\mathcal{H}$ into magnetic flux sectors labelled by $B \in \mathrm{H}^p(N; \mathbb{Z})$. However, one cannot simultaneously decompose $\mathcal{H}$ into both electric and magnetic flux sectors because of the Heisenberg commutation relations

$$\big[\mathcal{U}_E(\eta_e), \, \mathcal{U}_B(\eta_m)\big] = \exp\left(2\pi\,\mathrm{i} \int\!\!\!\!\!\int_N^{\mathrm{H}} \eta_e \smile \beta(\eta_m)\right) \mathrm{id}_{\mathcal{H}} \ ,$$

where $\mathcal{U}_E : \mathrm{H}^{p-1}(N; \mathbb{T}) \to \mathcal{H}$ and $\mathcal{U}_B : \mathrm{H}^{n-p}(N; \mathbb{T}) \to \mathcal{H}$ are quantization maps corresponding to the electric and magnetic gradings of the Hilbert space $\mathcal{H}$, and β is the Bockstein homomorphism (see (2.17)). Hence non-trivial commutators are related to torsion in the cohomology $\mathrm{H}^{\bullet}(N; \mathbb{Z})$; this means that the Hilbert space $\mathcal{H}$ can be simultaneously graded by electric and magnetic fluxes only modulo torsion. See [31, 32] for further details.

For $p = 1$ and $M = S^1$, the generalized abelian gauge theory is the theory of a periodic scalar field $g : S^1 \to S^1$; in this case the magnetic flux is the winding number and the electric flux is the momentum of the field g.

For $p = 2$ and $N = L_k = S^3/\mathbb{Z}_k$ a lens space, there is only torsion in the relevant cohomology groups $\mathrm{H}^2(L_k; \mathbb{Z}) = \mathbb{Z}_k$ and $\mathrm{H}^1(L_k; \mathbb{T}) = \mathbb{Z}_k$. The quantum Hilbert space is the unique finite-dimensional irreducible representation of the Heisenberg group extension

$$ 1 \; \longrightarrow \; \mathbb{Z}_k \; \longrightarrow \; \mathtt{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k) \; \longrightarrow \; \mathbb{Z}_k \times \mathbb{Z}_k \; \longrightarrow \; 0 \ , $$

and hence one cannot simultaneously measure electric and magnetic flux in this case. A potential experimental test of this phenomenon is described in [41]: Although any embedded codimension zero three-manifold in $\mathbb{R}^3$ has torsion-free cohomology, it might be possible to find a configuration where the effective space is only immersed, with a line of double points, by using Josephson junctions.

Now let us look more closely at the quantum flux sectors of the Ramond–Ramond gauge theory. As first pointed out in [45], the quantization of *flat* Ramond–Ramond fields is of particular interest; these fluxes are described by classes in the K-module theory which is isomorphic to the kernel of the curvature morphism

$$ \mathrm{K}^{j-1}(N; \mathbb{T}) = \ker \left(\check{\mathrm{K}}^j(N) \xrightarrow{F} \Omega(N; K^\bullet)^j \right) \xrightarrow{\check{\imath}} \check{\mathrm{K}}^j(N) \ , \qquad (4.32) $$

where $\check{\imath}$ denotes the embedding and we assume $n = \dim(N) = 4k + 1$ for some $k \in \mathbb{N}$. The Chern character $\mathsf{ch} : \mathrm{K}^j(N) \to \mathrm{H}(N; K^\bullet)^j$ becomes an isomorphism after tensoring over $\mathbb{R}$; its kernel coincides with the image of the connecting homomorphism $\beta : \mathrm{K}^{j-1}(N; \mathbb{T}) \to \mathrm{K}^j(N)$, so that $\mathrm{im}(\beta) = \mathsf{Tor}\, \mathrm{K}^j(N) \subseteq \mathrm{K}^j(N)$ is the torsion subgroup. We will show that the quantum commutators (4.31) restrict non-degenerately to the flat Ramond–Ramond fields if and only if the K-theory $\mathrm{K}^j(N)$ has non-trivial torsion subgroup (and the real cohomology vanishes in the opposite parity). On general grounds, the non-degenerate pairing on the torsion group arises from the fact that there exists a non-degenerate pairing between the group of components of the flat part and the torsion subgroup of topological K-theory. Note that this sector of the Ramond–Ramond gauge theory is "topological", in the sense that the corresponding hamiltonian vanishes and there is no time evolution. In this case the Heisenberg group $\mathtt{Heis}\big(\mathrm{K}^{j-1}(N; \mathbb{T}) \big)$ is a finite-dimensional torus extended by a finite abelian group which plays the role of the (finite-dimensional) configuration space of fields; by Prop. 4.15 it is represented uniquely on a $\mathbb{Z}_2$-graded finite-dimensional Hilbert space $\mathcal{H}$. These torsion fluxes arise entirely from Dirac quantization, and the corresponding quantum operators do not all commute in the quantization by the full K-theory group $\mathrm{K}^j(N)$.

Proposition 4.33. *Suppose that* $\mathsf{Tor}\, \mathrm{K}^j(N) = 0$. *Let* $\omega, \omega' \in \mathrm{K}^{j-1}(N; \mathbb{T})$ *be classes of flat fields with lifts* $\tilde{\omega}, \tilde{\omega}'$ *to the Heisenberg group* $\mathtt{Heis}\big(\mathrm{K}^{j-1}(N; \mathbb{T}) \big)$. *Then*

$$ \big[\mathcal{O}(\tilde{\omega}) \, , \, \mathcal{O}(\tilde{\omega}') \big] = \mathrm{id}_{\mathcal{H}_{\mathrm{RR}}} \ . $$

Proof. Since $\mathsf{ch} \circ c = [-]_{\mathrm{dR}} \circ F$, we have

$$ c \circ \check{\imath}(\omega) \ \in \ \mathsf{Tor}\, \mathrm{K}^j(N) $$

for $\omega \in \mathrm{K}^{j-1}(N; \mathbb{T})$. One also has

$$ \check{\imath}(\omega) \smile [\check{C}] = \check{\imath}\big(\omega \smile c([\check{C}]) \big) $$

for $\omega \in \mathrm{K}^{j-1}(N;\mathbb{T})$ and $[\check{C}] \in \check{\mathrm{K}}^{j'}(N)$, where

$$\smile \; : \; \mathrm{K}^{j-1}(N;\mathbb{T}) \otimes \mathrm{K}^{j'}(N) \; \longrightarrow \; \mathrm{K}^{j+j'-1}(N;\mathbb{T})$$

denotes the restriction of the cup product. Then

$$\begin{aligned}
\big\langle \, \check{\imath}(\omega) \, , \, \check{\imath}(\omega') \, \big\rangle &= \left[\int\!\!\!\!\int_N^{\check{\mathrm{K}}} \check{\imath}(\omega) \smile \check{\Psi}^{-1}\big(\check{\imath}(\omega')\big) \right]_{u^0} \\
&= \left[\int\!\!\!\!\int_N^{\check{\mathrm{K}}} \check{\imath}(\omega) \smile \check{\imath}\big(\Psi^{-1}(\omega')\big) \right]_{u^0} \\
&= \left[\int\!\!\!\!\int_N^{\check{\mathrm{K}}} \check{\imath}\Big(\omega \smile c \circ \check{\imath}\big(\Psi^{-1}(\omega')\big)\Big) \right]_{u^0} \; = \; 0 \; ,
\end{aligned}$$

where we have used the fact that the Adams operation $\check{\Psi}^{-1}$ commutes with the embedding $\check{\imath}$ and the last equality follows from $\mathsf{Tor}\,\mathrm{K}^j(N) = 0$. $\qquad\square$

The converse of Prop. 4.33 gives an explicit criterion and formula for the non-trivial quantum commutators of flat fields in the Ramond–Ramond gauge theory; its crux is the fact that the pairing (4.27) factors to a non-degenerate pairing on the torsion part of topological K-theory.

Proposition 4.34. *If* $\mathsf{Tor}\,\mathrm{K}^j(N) \neq 0$, *then there exist classes* $\omega, \omega' \in \mathrm{K}^{j-1}(N;\mathbb{T})$ *such that*

$$\big[\mathcal{O}(\tilde{\omega})\,,\,\mathcal{O}(\tilde{\omega}')\big] \neq \mathrm{id}_{\mathcal{H}_{\mathrm{RR}}} \; .$$

Proof. We show that the pairing $\langle -, - \rangle$ on differential K-theory restricts non-degenerately to the subring $\mathrm{K}^{j-1}(N;\mathbb{T}) \subset \check{\mathrm{K}}^j(N)$. Since $\mathrm{K}^{j-1}(N;\mathbb{T}) \cong \mathsf{Hom}_{\mathscr{A}b}(\mathrm{K}^j(N),\mathbb{T})$, there is a non-degenerate pairing

$$(-,-)_{\mathrm{K}} \; : \; \mathrm{K}^{j-1}(N;\mathbb{T}) \otimes \mathrm{K}^j(N) \; \longrightarrow \; \mathbb{T}$$

given by a formula like that in (4.26). Combined with the pairing

$$\Omega_{\mathbb{Z}}(N;K^\bullet)^j \otimes \big(\Omega(N;K^\bullet)^{j-1} \,/\, \Omega_{\mathbb{Z}}(N;K^\bullet)^{j-1}\big) \; \longrightarrow \; \mathbb{T}\,, \qquad \big(F\,,\,[C]\big) \;\longmapsto\; \left[\int_N F \wedge C \right]_{u^0}$$

and the two exact sequences (3.10) of differential K-theory, one proves explicitly that the pairing (4.26) is a perfect pairing. Recall from §3.4. that the group of flat fields (4.32) sits in the long exact sequence

$$\cdots \;\longrightarrow\; \mathrm{K}^{j-1}(N) \;\xrightarrow{\mathrm{ch}}\; \mathrm{H}(N;K^\bullet)^{j-1} \;\longrightarrow\; \mathrm{K}^{j-1}(N;\mathbb{T}) \;\xrightarrow{\delta}\; \mathrm{K}^j(N) \;\xrightarrow{\mathrm{ch}}\; \mathrm{H}(N;K^\bullet)^j \;\longrightarrow\; \cdots$$

which induces the short exact sequence

$$0 \;\longrightarrow\; \mathrm{H}(N;K^\bullet)^{j-1}\,/\,\mathrm{ch}\big(\mathrm{K}^{j-1}(N)\big) \;\longrightarrow\; \mathrm{K}^{j-1}(N;\mathbb{T}) \;\xrightarrow{\beta}\; \mathsf{Tor}\,\mathrm{K}^j(N) \;\longrightarrow\; 0 \; . \qquad (4.35)$$

There is also a short exact sequence

$$\begin{aligned}
0 \;\longrightarrow\; \mathsf{Hom}_{\mathscr{A}b}\big(\mathrm{H}(N;K^\bullet)^j\,,\,\mathbb{T}\big) \;&\longrightarrow\; \mathsf{Hom}_{\mathscr{A}b}\big(\mathrm{K}^j(N)\,,\,\mathbb{T}\big) \;\longrightarrow\; \\
&\longrightarrow\; \mathsf{Hom}_{\mathscr{A}b}\big(\mathsf{Tor}\,\mathrm{K}^j(N)\,,\,\mathbb{T}\big) \;\longrightarrow\; 0
\end{aligned}$$

obtained by applying the exact contravariant functor $\mathsf{Hom}_{\mathscr{A}b}(-,\mathbb{T})$ to the short exact sequence induced by taking the kernel of the Chern character homomorphism $\mathsf{ch} : \mathrm{K}^j(N) \to \mathrm{H}(N; K^\bullet)^j$. We then obtain a commutative diagram with exact horizontal sequences given by

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathrm{H}(N;K^\bullet)^{j-1}}{\mathsf{ch}\big(\mathrm{K}^{j-1}(N)\big)} & \longrightarrow & \mathrm{K}^{j-1}(N;\mathbb{T}) & \xrightarrow{\ \beta\ } & \mathsf{Tor}\ \mathrm{K}^j(N) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle\approx} & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Hom}_{\mathscr{A}b}\big(\mathrm{H}(N;K^\bullet)^j,\mathbb{T}\big) & \longrightarrow & \mathsf{Hom}_{\mathscr{A}b}\big(\mathrm{K}^j(N),\mathbb{T}\big) & \longrightarrow & \mathsf{Hom}_{\mathscr{A}b}\big(\mathsf{Tor}\ \mathrm{K}^j(N),\mathbb{T}\big) & \longrightarrow & 0
\end{array}
$$

where the vertical morphism on the left is given by composing wedge product, integration, and reduction modulo $\mathbb{Z}$, and in the middle isomorphism we have used Pontrjagin duality of the $\mathbb{R}/\mathbb{Z}$ K-theory $\mathrm{K}^{j-1}(N;\mathbb{T})$. By Pontrjagin duality on $K^\bullet$-valued cohomology $\mathrm{H}(N;K^\bullet)^j$ it follows that the left vertical morphism is an isomorphism, and hence so is the right vertical morphism. Denote by

$$(-,-)_{\mathsf{Tor}} \ : \ \mathsf{Tor}\ \mathrm{K}^j(N) \otimes \mathsf{Tor}\ \mathrm{K}^j(N) \ \longrightarrow \ \mathbb{T} \tag{4.36}$$

the *torsion pairing* associated to these morphisms. Then the calculation in the proof of Prop. 4.33 shows that

$$\big\langle\, \check{\imath}(\omega)\,,\, \check{\imath}(\omega')\,\big\rangle = \big(\omega\,,\, c(\,\check{\imath}(\omega')\,)\big)_{\mathrm{K}}\,,$$

and since $\mathsf{ch} \circ c = [-]_{\mathrm{dR}} \circ F$, we have $c(\,\check{\imath}(\omega')\,) \in \mathsf{Tor}\ \mathrm{K}^j(N)$ and it follows that

$$\big\langle\, \check{\imath}(\omega)\,,\, \check{\imath}(\omega')\,\big\rangle = \big(\beta(\omega)\,,\, c(\,\check{\imath}(\omega')\,)\big)_{\mathsf{Tor}}\,,$$

where we have used the fact that the kernel torus $\ker(\beta)$ of the Bockstein homomorphism has trivial cup product with the torsion elements in $\mathsf{Tor}\ \mathrm{K}^j(N)$. Since the pairing $(-,-)_{\mathsf{Tor}}$ is non-degenerate, and $\mathsf{Tor}\ \mathrm{K}^j(N) \neq 0$, we can always find classes ω and ω' in $\mathrm{K}^{j-1}(N;\mathbb{T})$ such that $\langle \check{\imath}(\omega), \check{\imath}(\omega') \rangle \neq 0$. $\qquad\square$

An explicit formula for the torsion pairing (4.36) can be written as follows. Let $E \to N$ be a complex vector bundle and $\mathtt{rank}(E)$ the trivial vector bundle over N of the same rank as E. Then the K-theory class $\xi = [E] - [\mathtt{rank}(E)]$ is torsion, so there exits an integer k and an isomorphism on K-theory $\psi : E^{\oplus k} \to k\,[\mathtt{rank}(E)]$. Let ∇ be a connection on E, and ∇_0 the trivial connection with vanishing holonomy. Then the K-theory integral of the class ξ can be expressed as

$$\int\!\!\!\!\!\int_N^{\mathrm{K}} \xi = \eta(\slashed{D}_\nabla) - \eta(\slashed{D}_{\nabla_0}) - \frac{1}{k}\int_N \mathsf{CS}\big(\psi^*(\nabla^{\oplus k}),\nabla_0^{\oplus k}\big) \wedge \widehat{A}(N) \quad \mathrm{mod}\ \mathbb{Z}\,,$$

where $\eta(\slashed{D}_\nabla)$ is the spectral asymmetry of the Dirac operator on N coupled to the bundle E.

The flat fluxes play a crucial role in the grading on the quantum Hilbert space $\mathcal{H}_{\mathrm{RR}}$ of the Ramond–Ramond gauge theory into topological sectors [31, 32]. As explained by [45, 31, 32], the subgroup of differential K-theory comprising flat cocycles is the group of unbroken gauge symmetries of the Ramond–Ramond gauge theory. The equivalence classes comprising shifts of cycles by flat fields define the topological classes of Ramond–Ramond fluxes. Recall that the characteristic classes $\xi \in \mathrm{K}^j(N)$ label the connected components of $\check{\mathrm{K}}^j(N)$. Hence there is a natural grading of the quantum Hilbert space $\mathcal{H}_{\mathrm{RR}}$ of the Ramond–Ramond gauge theory into topological sectors by the K-theory group $\mathrm{K}^j(N)$ modulo torsion; it is induced by diagonalising the translation action by the flat Ramond–Ramond fields. The group of components of this subgroup, which is isomorphic to the torsion part of the topological K-theory, can shift the Hilbert space gradings. In that case, the grading can only be defined modulo these torsion subgroups. By Prop. 4.34, if the topological K-theory $\mathrm{K}^j(N)$ of the space N has non-trivial torsion subgroup,

then elements of the subgroup of flat Ramond–Ramond fields do not commute in the Heisenberg extension of the differential K-theory group and quantize to operators which do not all commute among themselves; this expresses an uncertainty principle which asserts that the K-theory class of a Ramond–Ramond field cannot be measured. Explicit examples exhibiting this phenomenon can be found in [31, Ex. 2.21] and [32, §5]. Flat Ramond–Ramond fluxes also give rise to novel effects in certain flux compactifications of Type IIA string theory [20, §4.5]. If N is any smooth manifold of dimension $4k + 1$ with finite abelian fundamental group, then the K-theory $\mathrm{K}^0(N)$ contains a non-trivial torsion subgroup. A simple example is the manifold $N = L_k \times S^6$ where $L_k \cong S^3/\mathbb{Z}_k$ is a three-dimensional lens space; then $\mathsf{Tor}\,\mathrm{K}^0(N) = \mathbb{Z}_k \oplus \mathbb{Z}_k$.

Acknowledgments

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