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Abstract

Let D be an irreducible hermitian symmetric domain of rank r in $\mathbb{C}^d$ and let $G := Aut(D)$ the group of all biholomorphic automorphisms of D. We construct explicit integral formulas for the G-invariant inner products on spaces of holomorphic functions on D associated with the discrete Wallach points by means of integration on G-orbits on the boundary ∂D of D.

0 Introduction

For an irreducible hermitian symmetric space D of non-compact type, the holomorphic automorphism group $G = Aut(D)$ has a (scalar) holomorphic discrete series whose analytic continuation is given by parameters forming the so-called "Wallach set". It is an important problem to give explicit realizations of the corresponding irreducible representations of G in terms of the (boundary) geometry of the underlying domain D. A standard reference using Lie theoretic methods is [RV76]. In our previous works ([AU97] and [AU98]) we considered mainly certain parameter values within the continuous part of the Wallach set and constructed realizations emphasizing the Jordan theoretic description of D [FK94]. In this paper we treat the more difficult discrete part and find explicit integral formulas using Lassalle's boundary measures [La87]. The paper contains also a new realization (and proof of existence) of Lassalle's measures, using only basic results from Jordan theory (Peirce decomposition).

1 Preliminaries

In this section we review some known results in analysis on Jordan algebras and triples and on the associated symmetric domains, and establish the notation. For more information consult [Hu63], [Gi64], [Lo77], [U87], [FK94] and [A95].

Let $D \subset \mathbb{C}^d$ be a *Cartan domain*, i.e. D is an irreducible bounded symmetric domain in the Harish-Chandra realization. This is equivalent to saying that D is the open unit ball of $\mathbb{C}^d$ with respect to a certain norm $\|\cdot\|$, such that the group $G := \mathrm{Aut}(D)$ of all biholomorphic automorphisms of D acts transitively on D. By [Lo77], [U87], there exists a *triple product* $\{\cdot,\cdot,\cdot\} : \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$ so that $Z := (\mathbb{C}^d, \|\cdot\|, \{\cdot,\cdot,\cdot\})$ is a *Jordan-Banach *-triple* (JB*-triple). The *maximal compact subgroup* of G is $K := \{\varphi \in G; \varphi(0) = 0\} = G \cap \mathrm{GL}(Z)$, and $D \equiv G/K$.

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Let (r, a, b) be the *type* of D (or, of Z), where r is the *rank* and a, b are the *characteristic multiplicities*. Thus the dimension d and the *genus* p are given by

$$d = r + \frac{r(r-1)}{2}\,a + r\,b, \qquad p = 2 + (r-1)\,a + b. \tag{1.2}$$

A *tripotent* $v \in Z$ is an element satisfying $\{v, v, v\} = v$. The *Peirce decomposition* associated with the tripotent v is

$$Z = Z_1(v) \oplus Z_{\frac{1}{2}}(v) \oplus Z_0(v), \tag{1.3}$$

where $Z_\nu(v) := \{z \in Z; \{v, v, z\} = \nu z\}, \quad \nu = 1, \frac{1}{2}, 0$. The associated *Peirce projection* $P_\nu(v)$, is the projection whose range is $Z_\nu(v)$ and whose kernel is the sum of the other two Peirce subspaces. We denote also

$$D_\nu(v) := D \cap Z_\nu(v). \tag{1.4}$$

The spaces $Z_\nu(v)$ are sub-triples of Z, and the rank of the tripotent v is by definition the rank of $Z_1(v)$. We define

$$S_j = \text{the set of tripotents of rank } j, \quad j = 0, 1, 2, \ldots, r. \tag{1.5}$$

$S := S_r$ is the *Shilov boundary* of D. Let us choose a *frame*

$$e_1, e_2, \ldots, e_r, \tag{1.6}$$

i.e. a maximal set of tripotents of rank one which are pairwise orthogonal, i.e. $\{e_i, e_i, e_j\} = 0$ whenever $i \neq j$. The tripotent

$$e = e_1 + e_2 + \ldots + e_r \tag{1.7}$$

is *maximal* (having rank r), and thus $Z_0(e) = 0$. The stabilizer of e in K, namely

$$L := \{k \in K; k(e) = e\}, \tag{1.8}$$

will play an important role in the sequel. Notice that since K acts transitively on S, we have $S \equiv K/L$. More generally, K acts transitively on the frames, and in particular it is transitive on each of the S_j. The sub-triple $Z_1(e)$ has the structure of a *JB*-algebra* with respect to the product $z \circ w := \{z, e, w\}$ and the involution $z^* := \{e, z, e\}$, and e is the unit of $Z_1(e)$. The real part of $Z_1(e)$, i.e. the subset $X = X_1(e) := \{x \in Z_1(e); x^* = x\}$ of *self-adjoint* elements of $Z_1(e)$ is a *Euclidean (or formally-real) Jordan algebra*, with *determinant* ("norm") and *trace* polynomials

$$N(z) = \det(z) \quad \text{and} \quad tr(z) :=< z, e > \tag{1.9}$$

respectively. Here $\langle z, w\rangle$ denotes the unique K-invariant scalar product on Z satisfying $\langle e_1, e_1\rangle = 1$. The set

$$\Omega := \{x^2; x \in X, N(x) \neq 0\} \tag{1.10}$$

is the *symmetric cone* associated with X. The group L, restricted to X, coincides with the Jordan-algebra automorphisms of X. In particular, it is transitive on the frames of orthogonal minimal idempotents in X whose sum is the unit element e.

For $1 \le j \le r$, let $u_j = e_1 + \ldots + e_j$ and let N_j denote the determinant polynomial of the Jordan sub-algebra $Z^{(j)} := Z_1(u_j)$, extended to all of Z via $N_j(z) := N_j(P_1(u_j)z)$. Note that $N_r \equiv N$. The *conical function* associated with $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ is defined by

$$N_{\mathbf{s}}(x) := N_1(x)^{s_1-s_2}\, N_2(x)^{s_2-s_3} \cdots N_{r-1}(x)^{s_{r-1}-s_r}\, N_r(x)^{s_r}, \quad \forall x \in \Omega. \tag{1.11}$$

A *partition* is a sequence $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ of integers so that $m_1 \ge m_2 \ge \ldots \ge m_r \ge 0$. Note that for any partition $\mathbf{m}$, $N_{\mathbf{m}}$ is a polynomial (called *conical*), and it extends to all of Z. Let us denote

$$P_{\mathbf{m}} := \operatorname{span}\{N_{\mathbf{m}} \circ k; k \in K\}. \tag{1.12}$$

A fundamental theorem [Sch69], (see also [U86]) says that the spaces $P_{\mathbf{m}}$ are irreducible and mutually inequivalent with respect to the action $\pi(k)(f) := f \circ k^{-1}$ of K, and that the space $\mathcal{P}$ of all holomorphic polynomials on Z is their direct sum: $\mathcal{P} = \sum_{\mathbf{m}}^{\oplus} P_{\mathbf{m}}$. Thus the $P_{\mathbf{m}}$ are mutually orthogonal with respect to any K-invariant inner-product on $\mathcal{P}$. The *Fischer-Fock* inner-product on $\mathcal{P}$ is given by

$$< f, g >_F = \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z)\, \overline{g(z)}\, e^{-|z|^2}\, dm(z), \tag{1.13}$$

where $|\cdot|$ is the Euclidean norm, and $dm(z)$ is the Lebesgue measure. The reproducing kernel of $P_{\mathbf{m}}$ with respect to $< \cdot, \cdot >_F$ is denoted by $K_{\mathbf{m}}(z, w)$. Thus, $\sum_{\mathbf{m}} K_{\mathbf{m}}(z, w) = e^{<z,w>}$.

The Gindikin-Koecher *Gamma function* associated with the cone Ω is defined for $\mathbf{s} = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ with $\Re s_j > (j-1)\frac{a}{2}$ by the convergent integral

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega e^{-tr(x)}\, N_{\mathbf{s}}(x)\; d\mu_\Omega(x), \tag{1.14}$$

where $d\mu_\Omega(x) := N(x)^{-\frac{d_1}{r}}\, dm(x)$ is the (unique up to a multiplicative constant) measure on Ω which is invariant under the group $\mathrm{GL}(\Omega) := \{g \in \mathrm{GL}(X); g(\Omega) = \Omega\}$, and $d_1 := \dim_{\mathbb{R}}(X) = \frac{r(r-1)}{2}\, a + r$. It is known that Γ_Ω can be expressed as a product of ordinary Gamma functions:

$$\Gamma_\Omega(\mathbf{s}) := (2\pi)^{\frac{d_1-r}{2}} \prod_{j=1}^{r} \Gamma(s_j - (j-1)\frac{a}{2}),$$

and this allows the extension of Γ_Ω to a meromorphic function on all of $\mathbb{C}^r$. The *Beta function* associated with the cone Ω is related to the Gamma function via

$$\mathrm{B}_\Omega(\mathbf{p}, \mathbf{q}) := \frac{\Gamma_\Omega(\mathbf{p})\, \Gamma_\Omega(\mathbf{q})}{\Gamma_\Omega(\mathbf{p}+\mathbf{q})}. \tag{1.15}$$

For $\lambda \in \mathbb{C}$ and any partition $\mathbf{m}$ we denote

$$(\lambda)_{\mathbf{m}} := \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)} = \prod_{j=1}^{r} (\lambda - (j-1)\frac{a}{2})_{m_j}, \tag{1.16}$$

where $(t)_m := t(t+1)(t+2)\cdots(t+m-1)$.

Let $h(z, w)$ be the unique K-invariant irreducible polynomial, which in holomorphic in z, anti-holomorphic in w, and satisfies $h(x, x) = N(e - x^2) \quad \forall x \in X$. It is known that

$$h(z, w)^{-\lambda} = \sum_{\mathbf{m}} (\lambda)_{\mathbf{m}}\, K_{\mathbf{m}}(z, w), \quad \forall z, w \in D, \;\; \forall \lambda \in \mathbb{C}, \tag{1.17}$$

and the series converges absolutely and uniformly on compact subsets of $D \times D \times \mathbb{C}$. The fundamental formula (1.17) (called the "binomial expansion") was proved in special cases in [Hu63] and [La86], and in full generality in [FK94]. The *Wallach set* $W(D)$ of D consists of all those $\lambda \in \mathbb{C}$ for which $(z, w) \mapsto h(z, w)^{-\lambda}$ is positive definite. Using the expansion (1.17) one sees that

$$W(D) = \{0, \frac{a}{2}, 2\frac{a}{2}, \ldots, (r-1)\frac{a}{2}\} \cup ((r-1)\frac{a}{2}, \infty). \tag{1.18}$$

This result was established by several authors using various techniques: [Be75], [RV76] (in the context of Siegel domains), [W79], [La87] and [FK90]. For each $\lambda \in W(D)$ we denote by $\mathcal{H}_\lambda$ the completion of $\text{span}\{h(\cdot, w)^{-\lambda}; w \in D\}$ with respect to the unique inner-product $< \cdot, \cdot >_\lambda$ determined by

$$< h(\cdot, w)^{-\lambda}, h(\cdot, z)^{-\lambda} >_\lambda = h(z, w)^{-\lambda}, \qquad \forall z, w \in D.$$

Point evaluations are continuous linear functionals on $\mathcal{H}_\lambda$ and the corresponding reproducing kernel is $h(z, w)^{-\lambda}$.

If $\lambda > (r-1)\frac{a}{2}$ then $\mathcal{H}_\lambda$ contains $\mathcal{P}$ as a dense subspace. On the other hand, for the discrete Wallach points (which are our main concern in this paper) $\ell\frac{a}{2}$, $0 \leq \ell \leq r-1$, $\mathcal{H}_{\ell\frac{a}{2}}$ is the completion of

$$\mathcal{P}_\ell := \sum_{m_1 \geq \cdots m_\ell \geq 0 = m_{\ell+1} = \cdots = m_r} P_{\mathbf{m}}. \tag{1.19}$$

Since K acts irreducibly on each $P_{\mathbf{m}}$, every K-invariant inner product on $P_{\mathbf{m}}$ is proportional to the Fischer inner product. The computation of the proportionality constants for the inner products $< \cdot, \cdot >_\lambda$ is one of the major steps in the proof of (1.17). Thus for every $\lambda \in W(D)$ and every partition $\mathbf{m}$ for which $P_{\mathbf{m}} \subset \mathcal{H}_\lambda$,

$$< f, g >_\lambda = \frac{< f, g >_F}{(\lambda)_{\mathbf{m}}}, \quad \forall f, g \in P_{\mathbf{m}}. \tag{1.20}$$

This implies for all functions $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$ and $g = \sum_{\mathbf{m}} g_{\mathbf{m}}$ in $\mathcal{H}_\lambda$ (with $f_{\mathbf{m}}, g_{\mathbf{m}} \in P_{\mathbf{m}} \ \forall \mathbf{m}$),

$$< f, g >_\lambda = \sum_{\mathbf{m}} \frac{< f_{\mathbf{m}}, g_{\mathbf{m}} >_F}{(\lambda)_{\mathbf{m}}}. \tag{1.21}$$

Let us define an action of G on functions on D via

$$(U^{(\lambda)}(\varphi^{-1})f)(z) := f(\varphi(z))\,(J\varphi(z))^{\frac{\lambda}{p}}, \quad \varphi \in G, \tag{1.22}$$

where $J\varphi(z) := \text{Det}(\varphi'(z))$. Then, for $\lambda \in W(D)$, $U^{(\lambda)}$ is a *projective representation* of G on $\mathcal{H}_\lambda$.

It is well known that for $\lambda > p-1$ $\mathcal{H}_\lambda$ is the weighted Bergman space $L^2_a(D, \mu_\lambda)$, i.e. the space of all analytic functions in $L^2(D, \mu_\lambda)$, where

$$d\mu_\lambda(z) := c_\lambda\ h(z, z)^{\lambda - p}\, dm(z), \quad c_\lambda := \frac{\Gamma_\Omega(\lambda)}{\pi^d\, \Gamma_\Omega(\lambda - \frac{d}{r})}.$$

The representations $\{U^{(\lambda)}; \lambda > p-1\}$ form the *holomorphic discrete series* of representations of G. The problem of concrete description of the analytic continuation of the holomorphic discrete series by means of Sobolev-type integral formulas attracted the attention of many mathematicians (see for instance [RV76], [O80], [A92-1], [A92-2], [FK90], [Y93], [AU97] and [AU98]). This problem is intimately connected with the problem of the concrete description of the analytic continuation of the *Riesz distribution* (see [Ri49], [Ga47], [O80], [Gi75] and [AU97]). One of the oldest results on the description of the analytic continuity of the holomorphic discrete series is the realization of $\mathcal{H}_{\frac{d}{r}}$ as the *Hardy space* $H^2(S) = L^2_a(S,\sigma)$ (where S is the Shilov boundary of D and σ is the unique K-invariant probability measure on S).

The *shifting method* of Yan (see [Y93] and [AU97]) enables one to give integral formulas of the form

$$< f, g >_\lambda = < S_{\lambda,\ell} f, g >_{\lambda+\ell} \tag{1.23}$$

for suitable $\ell \in \mathbb{N}$ and *shifting operator* $S_{\lambda,\ell}$ (which is a GL(Ω)-invariant differential operator). In particular, if $\lambda + \ell > p - 1$ or $\lambda + \ell = \frac{d}{r}$ one obtains integral formulas for $< f, g >_\lambda$ of the desired type. However, these integral formulas suffer from two main weaknesses:

1. They do not permit generalization to the infinite-rank case;

2. They use unnecessary large numbers of parameters. (i.e. the topological dimension of the set on which the integration is performed is too big compared to the Gelfand-Kirillov dimension of the representation).

Our main goal here is to obtain explicit, Sobolev-type integral formulas for the invariant inner products $< \cdot, \cdot >_{\ell \frac{a}{2}}$ associated with the discrete Wallach points $\ell\,\frac{a}{2}$, $\ell = 0, 1, 2, \ldots, r-1$, by means of integration on the G-orbits on the boundary ∂D. These formulas seem to use the optimal number of parameters (i.e. the topological dimension of the set on which the integration is performed is minimal), and allow the passage to the case of infinite rank domains. The paper is a continuation of [AU97] and [AU98], in which we develop the formulas of the desired type for $< f, g >_{\frac{a}{2}}$. The proofs in the general case given here use the Harish-Chandra isomorphism between the rings of invariant differential operators and the symmetric polynomials. They are simpler and more conceptual.

There is another type of integral formulas for $< \cdot, \cdot >_\lambda$, $\lambda \in W(D)$ which use the *Cayley transform* (which realizes D as a *symmetric Siegel domain*, denoted by $T(\Omega)$) and the Fourier transform (which realizes the weighted Bergman spaces on $T(\Omega)$ as weighted L^2-spaces on Ω). These formulas are extended to the discrete Wallach points $\ell\frac{a}{2}$ in Sections 6 and 7 below; they are relatively simple and quite natural, but they do not allow to work directly with the data coming from D.

In order to formulate our main result in the context of D let us describe the structure of the boundary ∂D and introduce some more notation. The *boundary component* associated to a tripotent v is the set $B(v) := v + D_0(v)$ (see (1.4)). Its closure is a face of $\overline{D}$ and all the faces arise in this way. Notice that $D_1(v)$ and $D_0(v)$ are Cartan domains of type $(\ell, a, 0)$ and $(r-\ell, a, b)$ respectively, where $\ell := \text{rank}(v)$. Let us denote

$$\partial_\ell D := \cup_{v \in S_\ell} B(v), \quad 1 \leq \ell \leq r. \tag{1.24}$$

The sets $\partial_\ell D$ are the G-orbits on ∂D, and:

$$\partial_\ell\, D = G(u_\ell) = \{\varphi(u_\ell); \varphi \in G\}, \tag{1.25}$$

where $\{e_j\}_{j=1}^r)$ is the fixed frame and $u_\ell = e_1 + \cdots + e_\ell$. Thus

$$\partial D = \cup_{\ell=1}^r \partial_\ell D. \tag{1.26}$$

and the orbits of G in $\overline{D}$ are $\partial_0 D := D$, $\partial_1 D, \ldots$, and $\partial_r D = S$. Let us denote also

$$v_\ell = e - u_\ell = e_{\ell+1} + \cdots + e_r.$$

Then u_ℓ, v_ℓ are orthogonal tripotents of rank ℓ and $r - \ell$ respectively, and $u_\ell + v_\ell = e$. $Z^{(\ell)} := Z_1(u_\ell)$ is a JB*-sub-algebra of Z with unit u_ℓ, real part

$$X^{(\ell)} := \{z \in Z^{(\ell)}; z^* = z\},$$

and associated symmetric cone

$$\Omega^{(\ell)} := \{x^2; x \in X^{(\ell)}, N_\ell(x) \neq 0\}. \tag{1.27}$$

Consider the group of linear automorphisms of $\Omega^{(\ell)}$

$$\mathrm{GL}(\Omega^{(\ell)}) := \{g \in \mathrm{GL}(X^{(\ell)}); g(\Omega^{(\ell)}) = \Omega^{(\ell)}\}$$

and the associated ring of $\mathrm{GL}(\Omega^{(\ell)})$-*invariant differential operators*

$$\mathcal{D}_\ell := \mathrm{Diff}(\Omega^{(\ell)})^{\mathrm{GL}(\Omega^{(\ell)})}. \tag{1.28}$$

Thus, $\mathcal{D}_\ell$ consists of all differential operators T on $\Omega^{(\ell)}$ so that $TC_g = C_g T$ for all $g \in \mathrm{GL}(\Omega^{(\ell)})$, where $C_g(f) := f \circ g$. Let us denote

$$L^{(\ell)} := \{k \in K; k(u_\ell) = u_\ell\}. \tag{1.29}$$

Then $k_{|\Omega^{(\ell)}} \in \mathrm{GL}(\Omega^{(\ell)})$ $\forall k \in L^{(\ell)}$, and in particular $T(f \circ k) = (Tf) \circ k$ for all $T \in \mathcal{D}_\ell$ and $f \in C^\infty(\Omega^{(\ell)})$. Let

$$K^{(\ell)} := \{k \in K; k(Z_\nu(v_\ell)) = Z_\nu(v_\ell), \quad \nu = 1, \frac{1}{2}, 0\}. \tag{1.30}$$

Clearly, $\{k \in K; k(v_\ell) = v_\ell\} \subset K^{(\ell)}$. Also, every triple-automorphism of $Z_\nu(v_\ell)$ for *some* $\nu = 1, \frac{1}{2}, 0$ extends to a triple-automorphism of Z which preserves *all* the $Z_\nu(v_\ell)$, i.e. to an element of $K^{(\ell)}$. Let $K^{(\ell)^{\mathbb{C}}}$ denote the complexification of $K^{(\ell)}$. One of the technical results that will be established below is the following.

Lemma 1.1 *Every $T \in \mathcal{D}_\ell$ extends uniquely to a differential operator on $Z_0(v_\ell)$ which is invariant under the group $K^{(\ell)^{\mathbb{C}}}$.*

Let $T \in \mathcal{D}_\ell$ be extended to a $K^{(\ell)^{\mathbb{C}}}$-invariant differential operator on $Z_0(v_\ell)$. Given a tripotent $v \in S_{r-l}$, we define a differential operator T_v on $Z_0(v)$ in the following way. Since K acts transitively on $S_{r-\ell}$, there exists $k \in K$ for which $k(v_\ell) = v$. We define

$$T_v := C_k^{-1} T C_k, \tag{1.31}$$

where $C_k(f) := f \circ k$. T_v is well-defined, i.e. independent of the particular $k \in K$ for which $k(v_\ell) = v$. Indeed, if $k_1, k_2 \in K$ satisfy $k_1(v_\ell) = k_2(v_\ell) = v$, then $k_1^{-1}(k_2(v_\ell)) = v_\ell$, and so $k_2 = k_1 k$ for some $k \in K$ for which $k(v_\ell) = v_\ell$. As we remarked above, $k \in K^{(\ell)}$, and therefore

$$C_{k_2} T C_{k_2} = C_{k_1}^{-1} C_k^{-1} T C_k C_{k_1} = C_{k_1} T C_{k_1}.$$

For any function f on $\overline{D}$ and any tripotent v, the restriction of f to $B(v)$ yields a function f_v on $D_0(v)$ via

$$f_v(z) := f(v + z), \quad z \in D_0(v). \tag{1.32}$$

For any $1 \leq \ell \leq r$ let ν_ℓ be the unique K-invariant probability measure on S_ℓ, defined via

$$\int_{S_\ell} f \, d\nu_\ell := \int_K f(k(u_\ell)) \, dk. \tag{1.33}$$

Our main result in this framework is the following theorem (compare Theorem 3.2)

Theorem *Let $1 \leq \ell \leq r - 1$ and let $\lambda > (\ell - 1)\frac{a}{2}$. Then there exists $T = T^{(\ell,\lambda)} \in \mathcal{D}_\ell$ so that for every $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$ which are analytic in a neighborhood of $\overline{D}$,*

$$< f, g >_{\ell\frac{a}{2}} = \int_{S_{r-\ell}} < T_v f_v, g_v >_{\mathcal{H}_\lambda(D_0(v))} \; d\nu_{r-\ell}(v). \tag{1.34}$$

For general $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$ the integral (1.34) is an improper Riemann integral, namely

$$< f, g >_{\ell\frac{a}{2}} = \lim_{t \nearrow 1} \int_{S_{r-\ell}} < T_v(f^t)_v, (g^t)_v >_{\mathcal{H}_\lambda(D_0(v))} \; d\nu_{r-\ell}(v),$$

where $f^t(z) := f(tz)$, $g^t(z) := g(tz)$.

We remark that the case $\ell = 0$ in the above theorem (and in subsequent results) is trivial since $\mathcal{H}_0$ consists of constant functions.

The paper is organized in the following way. Section 2 is devoted to the construction of the tools needed to prove the above mentioned and related results. In subsection 2.1 we survey the Harish-Chandra isomorphism between the rings of invariant differential operators on symmetric cones and and of the symmetric polynomials. Using the spectral theory we extend this result to more general invariant operators. In subsection 2.2 we use the conical polar decomposition $Z = K \cdot \Omega$ to study K-averaging of certain functions on D, (a process we call "conialization"). In subsection 2.3 we construct for each $\ell \in \{1, 2, \ldots, r - 1\}$ two K-orbits on $\bar{D}$ and natural measures on them. After these preparatory sections we prove the above mentioned theorem, in section 2.3 (see Theorem 3.2 for the exact formulation). Some related results are established as well.

Section 4, is devoted to the development of canonical integral formulas for the inner products $< \cdot, \cdot >_{\ell\frac{a}{2}}$, $1 \leq \ell \leq r-1$, in the framework of the symmetric Siegel domain $T(\Omega)$ associated with the Cartan domain D via the Cayley transform. The case of symmetric Siegel domains of type I (i.e. tubes over the symmetric cones Ω) is treated first, where we use in an essential way the semi-invariant Lassalle measures on the boundary orbits $\partial_\ell\Omega$ of the cone Ω. The development of the analogous integral formulas in the context of symmetric Siegel domains of type II is technically harder and requires additional efforts. In section 5 we use the Lassalle measures to construct integral formulas for the invariant inner products associated with the continuous Wallach points $\alpha_\ell := \frac{d}{r} + \ell\frac{a}{2}$, $0 \le \ell \le r-1$, in the context of symmetric Siegel domains of type II, which generalize the analogous formulas in the context of symmetric Siegel domains of type I constructed in [AU97]. Finally, in Section 6 we present a new construction of the Lassalle measures. Unlike the original construction of Lassalle (see [La87]) which uses local coordinates (coming from the subgroup AN of $GL(\Omega)$), our formulas use global coordinates and make the semi-invariance apparent.

2 Preparation

2.1 Invariant differential operators on symmetric cones and symmetric polynomials

In this section we review briefly the connection between the ring $\mathcal{D} = \mathrm{Diff}(\Omega)^{\mathrm{GL}(\Omega)}$ of $\mathrm{GL}(\Omega)$-invariant differential operators on Ω and the ring $\mathcal{S}$ of symmetric polynomials in r variables. See [FK94] for more details and [He78] for the general theory.

We denote the half-sum of the strongly orthogonal positive roots by

$$\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_r) \text{ where } \rho_j := (2j - r - 1)\frac{a}{4}, \ \ 1 \le j \le r. \tag{2.1}$$

The *L-spherical functions* are the L-averages of the conical functions:

$$\Phi_{\boldsymbol{\lambda}}(x) := \int_L N_{\boldsymbol{\lambda}}(\ell(x))\, d\ell. \tag{2.2}$$

They are L-invariant and normalized by the condition $\Phi_{\boldsymbol{\lambda}}(e) = 1$. It is known that the $\Phi_{\boldsymbol{\lambda}}$ are the spherical functions associated with the Riemannian symmetric space Ω in the usual sense. The Weyl group W_r in this case is simply the permutation group, acting naturally on $\mathbb{C}^r$ and thus on the $\Phi_{\boldsymbol{\lambda}}$'s. It is known that $\Phi_{\boldsymbol{\lambda}} = \Phi_{\boldsymbol{\mu}}$ if and only if $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are in the same orbit of W_r.

For each partition $\mathbf{m}$ the function $\Phi_{\mathbf{m}}$ is an L-invariant polynomial which belongs to $P_{\mathbf{m}}$ and in particular extends to a polynomial on Z. Every L-invariant polynomial in $P_{\mathbf{m}}$ is proportional to $\Phi_{\mathbf{m}}$. The ring

$$\mathcal{S} = \mathbb{C}[\lambda_1, \lambda_2, \ldots, \lambda_r]^{W_r} \tag{2.3}$$

of symmetric (i.e. permutation invariant) polynomials in $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is isomorphic to the full polynomial ring $\mathbb{C}[\sigma_1, \sigma_2, \ldots, \sigma_r]$ via the *elementary symmetric polynomials* $\{\sigma_j\}_{j=1}^r$ defined by

$$\sigma_j(\boldsymbol{\lambda}) := \sum_{1 \le i_1 < i_2 < \cdots < i_j \le r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_j}. \tag{2.4}$$

Thus, for each $p \in \mathcal{S}$ there is a unique polynomial $q \in \mathbb{C}[\sigma_1, \sigma_2, \ldots, \sigma_r]$ so that

$$p(\boldsymbol{\lambda}) = q(\sigma_1(\boldsymbol{\lambda}), \sigma_2(\boldsymbol{\lambda}), \ldots, \sigma_r(\boldsymbol{\lambda})).$$

Thus, $\{\sigma_j\}_{j=1}^r$ are algebraically independent generators of $\mathcal{S}$.

A fundamental property of the spherical functions is that they are the joint eigenfunctions of the operators in $\mathcal{D}$.

Theorem 2.1 **(i)** *The conical and the spherical functions are eigenfunctions of every $T \in \mathcal{D}$: For all $\boldsymbol{\lambda} \in \mathbb{C}^r$ we have*

$$T(N_{\boldsymbol{\lambda}+\boldsymbol{\rho}}) = \gamma_T(\boldsymbol{\lambda}) N_{\boldsymbol{\lambda}+\boldsymbol{\rho}}\,, \quad T(\Phi_{\boldsymbol{\lambda}+\boldsymbol{\rho}}) = \gamma_T(\boldsymbol{\lambda}) \Phi_{\boldsymbol{\lambda}+\boldsymbol{\rho}}\,. \tag{2.5}$$

(ii) *$\gamma_T(\boldsymbol{\lambda})$ is a symmetric polynomial in $\lambda_1, , \lambda_2, , \ldots, \lambda_r$, thus $\gamma_T \in \mathcal{S}$.*

(iii) *The map $\gamma : \mathcal{D} \to \mathcal{S}$ defined via $\mathcal{D} \ni T \mapsto \gamma_T \in \mathcal{S}$ is a surjective ring isomorphism, called the Harish-Chandra isomorphism.*

(iv) *$\mathcal{D}$ is commutative.*

Definition 2.1 *For $1 \leq j \leq r$ we define $\Delta_j := \gamma^{-1}(\sigma_j)$. Namely, for every $\boldsymbol{\lambda} \in \mathbb{C}^r$:*

$$\Delta_j(N_{\boldsymbol{\lambda}+\boldsymbol{\rho}}) = \sigma_j(\boldsymbol{\lambda})\, N_{\boldsymbol{\lambda}+\boldsymbol{\rho}}\,, \;\; \Delta_j(\Phi_{\boldsymbol{\lambda}+\boldsymbol{\rho}}) = \sigma_j(\boldsymbol{\lambda})\, \Phi_{\boldsymbol{\lambda}+\boldsymbol{\rho}}\,. \tag{2.6}$$

Corollary 2.1 *The operators $\{\Delta_j\}_{j=1}^r$ are algebraically independent generators of $\mathcal{D}$.*

Since $\Omega = \mathrm{GL}(\Omega)/L$ is a Riemannian symmetric space (more precisely, a direct product of $\mathbb{R}_+$ with an irreducible symmetric space $\Omega' := \{x \in \Omega : N(x) = 1\}$ of non-compact type), one has a direct integral decomposition

$$L^2(\Omega) = \int\limits_{\mathbb{R}^r/W_r} H_{\boldsymbol{\lambda}}\, |c(\boldsymbol{\lambda})|^2\, d\boldsymbol{\lambda} \tag{2.7}$$

where $c(\boldsymbol{\lambda})$ is Harish-Chandra's c-function and $H_{\boldsymbol{\lambda}}$ is the Hilbert space completion of the space spanned by all $\mathrm{GL}(\Omega)$-translates of $\Phi_{\boldsymbol{\lambda}}$, endowed with its natural inner product [He84]. Via (2.7), the translation representation T of $\mathrm{GL}(\Omega)$ on $L^2(\Omega)$ has a decomposition

$$T = \int\limits_{\mathbb{R}^r/W_r} T_{\boldsymbol{\lambda}}\, |c(\boldsymbol{\lambda})|^{-2} d\boldsymbol{\lambda}$$

where $T_{\boldsymbol{\lambda}}$ is the (irreducible) spherical representation of $\mathrm{GL}(\Omega)$ on $H_{\boldsymbol{\lambda}}$. For any continuous W_r-invariant function $F : \mathbb{R}^r \to \mathbb{R}$ one can define a $\mathrm{GL}(\Omega)$-invariant self-adjoint operator $\hat{F}$ on $L^2(\Omega)$ by the formula

$$\hat{F}\, f = \int F(\boldsymbol{\lambda})\;\, f_{\boldsymbol{\lambda}}\, |c(\boldsymbol{\lambda})|^{-2}\, d\boldsymbol{\lambda} \tag{2.8}$$

for

$$f = \int f_{\boldsymbol{\lambda}} |c(\boldsymbol{\lambda})|^{-2}\, d\boldsymbol{\lambda}, \qquad f_{\boldsymbol{\lambda}} \in H_{\boldsymbol{\lambda}}. \tag{2.9}$$

The domain of $\hat{F}$ is defined as the space of functions f such that

$$\int |F(\boldsymbol{\lambda})|^2 \ \| f_{\boldsymbol{\lambda}} \|^2_{\boldsymbol{\lambda}} \ |c(\boldsymbol{\lambda})|^{-2} \, d\boldsymbol{\lambda} \ < \ +\infty.$$

Thus $\hat{F}$ is bounded if F is a bounded function. Let

$$\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_r) : \mathbb{C}^r \to \mathbb{C}^r,$$

where the σ_j are defined by (2.4). The direct integral decomposition above diagonalizes simultaneously the (commuting) operators Δ_k. Writing

$$F = f \circ \sigma$$

for some continuous bounded function $f : \mathbb{R}^r \to \mathbb{R}$, the bounded operator $\hat{F}$ can be expressed as a function

$$\hat{F} = f(\Delta_1, \ldots, \Delta_r),$$

in the spectral-theoretic sense, of $\Delta_1, \ldots, \Delta_r$.

Remark: There are many other natural choices of r algebraically independent generators of $\mathcal{S}$, and each such choice yields r algebraically independent generators of $\mathcal{D}$ via the Harish-Chandra isomorphism. See [FK94], [N89], [M87], and [M95].

Lemma 2.1 *Let $U \subset \mathbb{C}^r$ be a W_r-invariant domain, and let F be a W_r-invariant holomorphic function on U.*

(i) *The associated $GL(\Omega)$-invariant operator $T = \hat{F}$ satisfies*

$$T(\Phi_{\boldsymbol{\lambda}+\boldsymbol{\rho}}) = F(\boldsymbol{\lambda}) \ \Phi_{\boldsymbol{\lambda}+\boldsymbol{\rho}} \quad \forall \boldsymbol{\lambda} \in U. \tag{2.10}$$

(ii) *There exists a unique holomorphic function f on $\sigma(U)$ so that $F = f \circ \sigma$, i.e.*

$$F(\boldsymbol{\lambda}) = f(\sigma_1(\boldsymbol{\lambda}), \sigma_2(\boldsymbol{\lambda}), \ldots, \sigma_r(\boldsymbol{\lambda})) \quad \forall \boldsymbol{\lambda} \in U.$$

(iii) *In terms of the L^2-functional calculus associated with $\{\Delta_j\}_{j=1}^r$,*

$$T = f(\Delta_1, \Delta_2, \ldots, \Delta_r). \tag{2.11}$$

The results described above are valid in the context of the cones $\Omega^{(\ell)}$, $1 \le \ell \le r$. Thus the ring $\mathcal{S}_\ell := \mathbb{C}[\lambda_1, \ldots, \lambda_\ell]^{W_\ell}$ of the symmetric polynomials in $\boldsymbol{\lambda}^{(\ell)} := (\lambda_1, \ldots, \lambda_\ell)$ is isomorphic to the full polynomial ring $\mathbb{C}[\sigma_1, \ldots, \sigma_\ell]$, and the elementary symmetric polynomials

$$\sigma_j^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}) := \sum_{1 \le i_1 < i_2 < \cdots < i_j \le \ell} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j} \quad 1 \le j \le \ell \tag{2.12}$$

are algebraically independent generators of $\mathcal{S}_\ell$. The spherical functions in the context of $\Omega^{(\ell)}$ are parametrized by $\mathbb{C}^\ell$ and are defined as before via

$$\Phi^{(\ell)}_{\boldsymbol{\lambda}^{(\ell)}}(x) := \int_{L^{(\ell)}} N_{\boldsymbol{\lambda}^{(\ell)}}(k(x)) \, dk, \quad x \in \Omega^{(\ell)}.$$

The Harish-Chandra isomorphism between $\mathcal{D}_\ell = \mathrm{Diff}(\Omega^{(\ell)})^{\mathrm{GL}(\Omega^{(\ell)})}$ and $\mathcal{S}_\ell$ is given via

$$T(\Phi^{(\ell)}_{\boldsymbol{\lambda}^{(\ell)}+\boldsymbol{\rho}^{(\ell)}}) = \gamma_T^{(\ell)}(\boldsymbol{\lambda}^{(\ell)})\ \Phi^{(\ell)}_{\boldsymbol{\lambda}^{(\ell)}+\boldsymbol{\rho}^{(\ell)}}, \qquad \boldsymbol{\lambda}^{(\ell)} \in \mathbb{C}^{(\ell)},$$

where

$$\boldsymbol{\rho}^{(\ell)} := (\rho_1^{(\ell)}, \rho_2^{(\ell)}, \ldots, \rho_\ell^{(\ell)}), \quad \text{and} \quad \rho_j^{(\ell)} := \frac{a}{4}(2j-\ell-1). \tag{2.13}$$

The algebraically independent generators of $\mathcal{D}_\ell$ are

$$\Delta_j^{(\ell)} := (\gamma^{(\ell)})^{-1}(\sigma_j^{(\ell)}), \quad 1 \le j \le \ell. \tag{2.14}$$

Lemma 2.1 is valid in the context of $\Omega^{(\ell)}$ with obvious notational changes.

2.2 Conialization of functions

In this section we study *conialization* (i.e. "conical polarization") of functions on Z. The basic fact used here is that every $z \in Z$ admits a *conical polar decomposition* $z = k(x)$ with $k \in K$ and a unique $x \in \Omega$. Thus $Z = K \cdot \Omega$, and we have a formula for *integration in conical polar coordinates* for functions $f \in L^1(Z, m)$:

$$\int_Z f(z)\ dm(z) = c_0 \int_\Omega \left(\int_K f(k(x^{\frac{1}{2}}))\ dk \right)\ N(x)^b\ dx \tag{2.15}$$

where m is Lebesgue measure, and $c_0 = \pi^d/\Gamma_\Omega(\frac{d}{r})$. The function

$$\tilde{f}(x) := \int_K f(k(x^{\frac{1}{2}}))\ dk, \qquad x \in \Omega, \tag{2.16}$$

is called the *conialization* of f. The map $E(f)(x) := \tilde{f}(x^2)$ can be considered as the averaging projection (i.e. conditional expectation) from $L^1(Z, m)$ onto its subspace of K-invariant functions.

Lemma 2.2 **(i)** *For every partition* $\mathbf{m}$ *and every* $x \in \Omega$

$$\int_K |\Phi_{\mathbf{m}}(k(x^{\frac{1}{2}}))|^2\ dk = \frac{\Phi_{\mathbf{m}}(x)}{d_{\mathbf{m}}}, \tag{2.17}$$

where $d_{\mathbf{m}} := \dim(P_{\mathbf{m}})$.

(ii) *For every* $x \in \Omega$ *and all polynomials* $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$ *and* $g = \sum_{\mathbf{m}} g_{\mathbf{m}}$ *with* $f_{\mathbf{m}}, g_{\mathbf{m}} \in P_{\mathbf{m}}$ *for all* $\mathbf{m}$,

$$\widetilde{(f\overline{g})}(x) = \sum_{\mathbf{m}} < f_{\mathbf{m}}, g_{\mathbf{m}} >_{\frac{d}{r}} \Phi_{\mathbf{m}}(x) \tag{2.18}$$

Proof: Formula (2.17) is proved in [FK94], Proposition XI.4.1 in the case where Z is a JB*-algebra, and in [FK90] in the case where Z is a JB*-triple. Notice that (2.17) with $x = e$ yields for every $\mathbf{m}$

$$\|\Phi_{\mathbf{m}}\|^2_{\frac{d}{r}} = \int_K |\Phi_{\mathbf{m}}(k(e))|^2\ dk = \frac{1}{d_{\mathbf{m}}}.$$

To prove (2.18), consider the K-invariant inner product

$$< f, g >_x := \widetilde{(f\overline{g})}(x) = \int_K f(k(x^{\frac{1}{2}}))\, \overline{g(k(x^{\frac{1}{2}}))}\, dk \tag{2.19}$$

on $\mathcal{P}$. Using the fact that the actions of K on the $P_{\mathbf{m}}$ are irreducible and pair-wise inequivalent, we see that the $P_{\mathbf{m}}$ are pair-wise orthogonal with respect to $< \cdot, \cdot >_x$, and that there exist positive constants $c_{\mathbf{m}}(x)$ so that

$$< f_{\mathbf{m}}, g_{\mathbf{m}} >_x = c_{\mathbf{m}}(x) < f_{\mathbf{m}}, g_{\mathbf{m}} >_{\frac{d}{r}}, \quad \forall f_{\mathbf{m}}, g_{\mathbf{m}} \in P_{\mathbf{m}}.$$

The proportionality constants are computed by taking $f_{\mathbf{m}} = g_{\mathbf{m}} = \Phi_{\mathbf{m}}$ and using (2.17) for x and e. ∎

Let $1 \leq \ell \leq r$ and denote the vectors in $\mathbb{C}^\ell$ by $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \ldots, \lambda_\ell)$. For notational simplicity we shall adopt the convention that $\Phi_{\boldsymbol{\lambda}^{(\ell)}} = \Phi_{(\lambda_1,\ldots,\lambda_\ell,0,\ldots,0)}$, and similarly for the conical functions. Recall that the spherical functions associated with the symmetric cone $\Omega^{(\ell)}$ of $X^{(\ell)}$ are denoted by $\Phi^{(\ell)}_{\boldsymbol{\lambda}^{(\ell)}}$.

Proposition 2.1 *Let $1 \leq \ell \leq r$ and let $\mathbf{m}^{(\ell)} = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell$ be a partition. Then for every $x \in X^{(\ell)}$*

$$\Phi_{\mathbf{m}^{(\ell)}}(x) = \gamma_{\mathbf{m}^{(\ell)}}\, \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x), \tag{2.20}$$

where

$$\gamma_{\mathbf{m}^{(\ell)}} = \frac{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}}} = \frac{\Gamma_{\Omega^{(\ell)}}(r\frac{a}{2})}{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})} \prod_{j=1}^{\ell} \frac{\Gamma(m_j + (\ell+1-j)\frac{a}{2})}{\Gamma(m_j + (r+1-j)\frac{a}{2})}. \tag{2.21}$$

Proof: Recall that for every $y \in X$ and $\lambda \in \mathbb{C}$,

$$N(e-y)^{-\lambda} = \sum_{\mathbf{m}} (\lambda)_{\mathbf{m}} \frac{\Phi_{\mathbf{m}}(y)}{\|\Phi_{\mathbf{m}}\|_F^2}. \tag{2.22}$$

Similarly, for $x \in X^{(\ell)}$ and $\lambda \in \mathbb{C}$,

$$N(e-x)^{-\lambda} = N_\ell(u_\ell - x)^{-\lambda} \sum_{\mathbf{m}^{(\ell)}} (\lambda)_{\mathbf{m}^{(\ell)}} \frac{\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x)}{\|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|_F^2}. \tag{2.23}$$

In order to continue the proof of the proposition, we need the following result.

Lemma 2.3 *Let $1 \leq \ell \leq r$ and let $y \in X$ be an element of rank at most ℓ. If $\mathbf{n} = (n_1, \ldots, n_r)$ is a partition with $n_{\ell+1} \geq 1$, then $N_{\mathbf{n}}(y) = \Phi_{\mathbf{n}}(y) = 0$.*

Proof of the Lemma: The condition $n_{\ell+1} \geq 1$ guarantees that for some $j > \ell$, $N_{\mathbf{n}}$ is divisible by N_j to a positive power. Notice that $\operatorname{rank}(P_1(u_j)y) \leq \operatorname{rank}(y) \leq \ell$. Hence, $N_j(y) = N_j(P_1(u_j)y) = 0$ (because in the Jordan algebra $X^{(j)}$ elements of rank smaller than j have zero determinant). In particular, $N_{\mathbf{n}}(y) = 0$. If $k \in L$ then $\operatorname{rank}(k(y)) = \operatorname{rank}(y) \leq \ell$, and therefore $N_{\mathbf{n}}(k(y)) = 0$. Finally, $\Phi_{\mathbf{n}}(y) = \int_L N_{\mathbf{n}}(k(y))\, dk = 0$. ∎

Using Lemma 2.3 we see that (2.22) for $x \in X^{(\ell)}$ yields

$$N(e-x)^{-\lambda} = \sum_{\mathbf{m}^{(\ell)}} (\lambda)_{\mathbf{m}^{(\ell)}} \frac{\Phi_{\mathbf{m}^{(\ell)}}(x)}{\|\Phi_{\mathbf{m}^{(\ell)}}\|_F^2}. \tag{2.24}$$

Since $\Phi_{\mathbf{m}^{(\ell)}|Z_0(v_\ell)} \in P_{\mathbf{m}}^{(\ell)}$, we obtain by comparing the expansions (2.23) and (2.24) that

$$\Phi_{\mathbf{m}^{(\ell)}}(x) = \frac{\|\Phi_{\mathbf{m}^{(\ell)}}\|_F^2}{\|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|_F^2}\, \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x) = \gamma_{\mathbf{m}^{(\ell)}}\, \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x) \quad \forall x \in X^{(\ell)}.$$

In order to compute $\gamma_{\mathbf{m}^{(\ell)}}$ we use the known fact (see [FK90]) that

$$\|\Phi_{\mathbf{m}^{(\ell)}}\|_F^2 = \frac{(\frac{d}{r})_{\mathbf{m}^{(\ell)}}}{d_{\mathbf{m}^{(\ell)}}} \quad \text{and} \quad \|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|_F^2 = \frac{(\frac{d_\ell}{\ell})_{\mathbf{m}^{(\ell)}}}{d^{(\ell)}_{\mathbf{m}^{(\ell)}}},$$

where $d_\ell := \dim\ Z_1(u_\ell) = \ell + \ell(\ell-1)\frac{a}{2}$, $d_{\mathbf{m}^{(\ell)}} = \dim(P_{\mathbf{m}^{(\ell)}})$, and $d^{(\ell)}_{\mathbf{m}^{(\ell)}}$ has the same meaning with respect to the algebra $Z_1(u_\ell)$. Quite generally, the dimensions $d_{\mathbf{m}}$ are expressed by

$$d_{\mathbf{m}} = \prod_{1 \le i < j \le r} \frac{B((j-i)\frac{a}{2}, \frac{a}{2})}{B(m_i - m_j + (j-i)\frac{a}{2}, \frac{a}{2})}\, \frac{B((i-j)\frac{a}{2}, \frac{a}{2})}{B(m_j - m_i + (i-j)\frac{a}{2}, \frac{a}{2})} \tag{2.25}$$

where $B(x,y) := \Gamma(x)\,\Gamma(y)/\Gamma(x+y)$ is the ordinary Beta function. (see [U83] for the general case, and [FK94], p. 315 for the case of JB*-algebras).

A straightforward computation yields the expression (2.21) for $\gamma_{\mathbf{m}^{(\ell)}}$. ∎

Remark: One can prove Proposition 2.1 using the connection between the spherical polynomials and the *Jack symmetric functions* $J_{\mathbf{m}}^{(\alpha)}$, where $\alpha := \frac{2}{a}$, and $\mathbf{m}$ ranges over all finite partitions. (See [M87], [M95] and [St89] for the study of Jack symmetric functions). $J_{\boldsymbol{\lambda}}^{(\alpha)}$ is defined on all finite sequences (identified with infinite sequences which contain only finitely many non-zero terms), and it is permutation invariant. The connecting formula is

$$\Phi_{\mathbf{m}}(\sum_{j=1}^{r} t_j e_j) = \frac{J_{\mathbf{m}}^{(\alpha)}(t_1, \ldots, t_r, 0, \ldots, 0, \ldots)}{J_{\mathbf{m}}^{(\alpha)}(1^r)} \qquad \forall t_1, \ldots, t_r > 0, \tag{2.26}$$

where $1^r := (1, \ldots, 1, 0, \ldots, 0, \ldots)$ has r "1". A similar formula is valid also for the spherical functions $\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}$ associated with $\Omega^{(\ell)}$:

$$\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(\sum_{j=1}^{\ell} t_j e_j) = \frac{J^{(\alpha)}_{\mathbf{m}^{(\ell)}}(t_1, \ldots, t_\ell, 0, \ldots, 0, \ldots)}{J^{(\alpha)}_{\mathbf{m}^{(\ell)}}(1^\ell)} \qquad \forall t_1, \ldots, t_\ell > 0, \tag{2.27}$$

It follows that for every $t_1, \ldots, t_\ell > 0$,

$$\frac{\Phi_{\mathbf{m}^{(\ell)}}(\sum_{j=1}^{\ell} t_j e_j)}{\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(\sum_{j=1}^{\ell} t_j e_j)} = \frac{J^{(\alpha)}_{\mathbf{m}^{(\ell)}}(1^\ell)}{J^{(\alpha)}_{\mathbf{m}^{(\ell)}}(1^r)} = \gamma_{\mathbf{m}^{(\ell)}}. \tag{2.28}$$

The numbers $J_{\mathbf{m}}^{(\alpha)}(1^\nu)$, $\nu \in \mathbb{N}$, are known in full generality (see [St89] Th. 5.4 and [M95]):

$$J_{\mathbf{m}}^{(\alpha)}(1^\nu) = \prod_{i=1}^{\ell(\mathbf{m})} \prod_{j=1}^{m_i} (\nu + 1 - i + \alpha(j-1)), \tag{2.29}$$

where $\ell(\mathbf{m}) := \max\{k; m_k \neq 0\}$ is the *length* of $\mathbf{m}$. It follows that if $\ell(\mathbf{m}) \leq r$ and $\alpha = \frac{2}{a}$, then

$$J^{(\alpha)}_{\mathbf{m}}(1^\nu) = (\frac{2}{a})^{|\mathbf{m}|} \prod_{j=1}^{r} \frac{\Gamma(m_j + (\nu + 1 - j)\frac{a}{2})}{\Gamma((\nu + 1 - j)\frac{a}{2})} = (\frac{2}{a})^{|\mathbf{m}|}(\nu \frac{a}{2})_{\mathbf{m}}.$$

In particular,

$$\gamma_{\mathbf{m}^{(\ell)}} = \frac{J^{(\alpha)}_{\mathbf{m}^{(\ell)}}(1^\ell)}{J^{(\alpha)}_{\mathbf{m}^{(\ell)}}(1^r)} = \frac{(\ell \frac{a}{2})_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}}} = \frac{\Gamma_{\Omega^{(\ell)}}(r\,\frac{a}{2})}{\Gamma_{\Omega^{(\ell)}}(\ell\,\frac{a}{2})} \prod_{j=1}^{\ell} \frac{\Gamma(m_j + (\ell + 1 - j)\frac{a}{2})}{\Gamma(m_j + (r + 1 - j)\frac{a}{2})}, \tag{2.30}$$

where $\Gamma_{\Omega^{(\ell)}}$ is the Gamma function associated with the cone $\Omega^{(\ell)}$. The spectral theorem in X [Lo77] and the fact that L acts transitively on the frames of primitive idempotents in X imply that every spherical polynomial $\Phi_{\mathbf{m}}$ is determined by its restriction to $\text{span}\{e_j\}_{j=1}^r$. Thus (2.20) in general follows from (2.20) for $x = \sum_{j=1}^{\ell} t_j e_j$, i.e. from (2.28). ∎

Remark: Recall that the half sum of the strongly orthogonal positive roots associated with $\Omega^{(\ell)}$ is $\boldsymbol{\rho}^{(\ell)} := (\rho_1^{(\ell)}, \ldots \rho_\ell^{(\ell)}), \quad \rho_j^{(\ell)} := (2j - \ell - 1)\,\frac{a}{4}$. For any partition $\mathbf{m}^{(\ell)} = (m_1, \ldots, m_\ell)$ define $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \ldots, \lambda_\ell)$ via the "$\boldsymbol{\rho}^{(\ell)}$-shift"

$$\boldsymbol{\lambda}^{(\ell)} := \mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)}, \quad \text{namely} \quad \lambda_j := m_j - \rho_j^{(\ell)} = m_j - (2j - \ell - 1)\,\frac{a}{4}, \quad 1 \leq j \leq \ell.$$

Then $\gamma_{\mathbf{m}^{(\ell)}}$ can be written as a *symmetric function* of $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \ldots, \lambda_\ell)$:

$$\gamma_{\mathbf{m}^{(\ell)}} = \frac{\Gamma_{\Omega^{(\ell)}}(r\,\frac{a}{2})}{\Gamma_{\Omega^{(\ell)}}(\ell\,\frac{a}{2})} \prod_{j=1}^{\ell} \frac{\Gamma(\lambda_j + (\ell + 1)\,\frac{a}{4})}{\Gamma(\lambda_j + (2r - \ell + 1)\,\frac{a}{4})}. \tag{2.31}$$

This will be crucial in the sequel.

Recall that $\mathcal{H}_{\ell\frac{a}{2}}$ is the completion of $\mathcal{P}_\ell$ (see (1.19)) with respect to the inner product (1.21) with $\lambda = \ell\,\frac{a}{2}$.

Corollary 2.2 *For all functions $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$ with expansions $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ and $g = \sum_{\mathbf{m}^{(\ell)}} g_{\mathbf{m}^{(\ell)}}$, and for every $x \in \Omega^{(\ell)}$,*

$$\begin{aligned}
\widetilde{(f\overline{g})}(x) &= \sum_{\mathbf{m}^{(\ell)}} < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_{\frac{d}{r}} \frac{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}}}\, \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x) \\
&= \frac{\Gamma_{\Omega^{(\ell)}}(r\,\frac{a}{2})}{\Gamma_{\Omega^{(\ell)}}(\ell\,\frac{a}{2})} \sum_{\mathbf{m}^{(\ell)}} \left(\prod_{j=1}^{\ell} \frac{\Gamma(\lambda_j + (\ell + 1)\,\frac{a}{4})}{\Gamma(\lambda_j + (2r - \ell + 1)\,\frac{a}{4})}\, \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x) \right) < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_{\frac{d}{r}},
\end{aligned}$$

where, as before,

$$\lambda_j = m_j - \rho_j^{(\ell)} = m_j - (2j - \ell - 1)\,\frac{a}{4}, \quad 1 \leq j \leq \ell.$$

The point is that the coefficients of $< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_{\frac{d}{r}}$ in the expansion of $\widetilde{(f\overline{g})}(x)$ are symmetric functions of $\boldsymbol{\lambda}^\ell = (\lambda_1, \ldots, \lambda_\ell)$.

2.3 Integration on K-orbits

In this section we will be interested in two sequences of K-orbits. The first sequence is the G-orbits $\{\partial_j D\}_{j=1}^r$ on ∂D. Notice that $\partial_{r-\ell} D = K(B(v_\ell))$, where $v_\ell = e_{\ell+1} + \cdots + e_r$. Recall that $u_\ell = e_1 + \cdots + e_\ell$, and denote the open unit interval in the cone $\Omega^{(\ell)}$ by

$$I^{(\ell)} := \Omega^{(\ell)} \cap (u_\ell - \Omega^{(\ell)}) = \{x \in X^{(\ell)}; 0 < x < u_\ell\}. \tag{2.32}$$

The second sequence of K-orbits that we shall need is

$$\mathcal{O}_\ell := K(I^{(\ell)}), \quad 1 \le \ell \le r. \tag{2.33}$$

Note that $\partial_{r-\ell} D = K(v_\ell + I^{(\ell)})$ and $\mathcal{O}_\ell$ are the K-orbits of the opposite faces $I^{(\ell)}$ and $v_\ell + I^{(\ell)}$ of the unit interval $I := \Omega \cap (e - \Omega)$ of the cone Ω.

We shall use the subgroup

$$G_{v_\ell} := \{\varphi \in G; \varphi(v_\ell) = v_\ell\} \tag{2.34}$$

of G, identified naturally with $\mathrm{Aut}(D_0(v_\ell))$, and the subgroups $K_{v_\ell} := K \cap G_{v_\ell}$ and $K^{(\ell)}$ (defined via (1.30)) of K.

We describe now a construction which assigns to a measure ν on $\overline{I^{(\ell)}}$ measures $\hat{\mu}$ and $\tilde{\mu}$ on the orbits $\partial_{r-\ell} D$ and $\mathcal{O}_\ell$ respectively. The construction uses as an intermediate step a construction of a measure μ on $\overline{D_0(v_\ell)}$.

Let ν be a measure on $\overline{I^{(\ell)}}$, and define a measure μ (depending on ν) on $\overline{D_0(v_\ell)}$ via

$$\int_{\overline{D_0(v_\ell)}} f \, d\mu = \int_{\overline{I^{(\ell)}}} \left(\int_{K_{\nu_\ell}} f(k(x^{\frac{1}{2}})) \, dk \right) \, d\nu(x). \tag{2.35}$$

We call ν the *conical part* of μ. Using μ we construct measures $\tilde{\mu}$ and $\hat{\mu}$ on the K-orbits $\mathcal{O}_\ell$ and $\partial_{r-\ell} D$ in a canonical way.

Construction of $\tilde{\mu}$: We define

$$\begin{aligned} \int_{\mathcal{O}_\ell} f \, d\tilde{\mu} &:= \int_{\overline{D_0(v_\ell)}} \left(\int_K f(k(z)) \, dk \right) \, d\mu(z) \\ &= \int_{\overline{I^{(\ell)}}} \left(\int_K f(k(x^{\frac{1}{2}})) \, dk \right) \, d\nu(x) = \int_{\overline{I^{(\ell)}}} \tilde{f}(x) \, d\nu(x). \end{aligned}$$

Example 2.1 *Let $\lambda > p_\ell - 1$ (where $p_\ell := (\ell - 1)\, a + 2 + b$ is the genus of $D_0(v_\ell)$), and consider the probability measure*

$$d\mu_\lambda^{(\ell)}(z) := c^{(\ell)} \, h_\ell(z, z)^{\lambda - p_\ell} \, dm(z), \quad c^{(\ell)} = \frac{\Gamma_{\Omega^{(\ell)}}(\lambda)}{\pi^\ell \, \Gamma_{\Omega^{(\ell)}}(\lambda - \frac{d^{(\ell)}}{\ell})} \tag{2.36}$$

on $D_0(v_\ell)$, where $d^{(\ell)} := \dim \ Z_0(v_\ell) = \ell(\ell - 1)\,\frac{a}{2} + \ell + \ell\, b$, and

$$h_\ell(k(x^{\frac{1}{2}}), k(x^{\frac{1}{2}})) = N_\ell(u_\ell - x), \quad \forall x \in I^{(\ell)}, \ \forall k \in K^{(\ell)}.$$

The conical part of $d\mu_\lambda^{(\ell)}$ is the probability measure

$$d\nu_\lambda^{(\ell)}(x) := \frac{1}{B_{\Omega^{(\ell)}}(\frac{d^{(\ell)}}{\ell}, \lambda - \frac{d^{(\ell)}}{\ell})} \, N_\ell(u_\ell - x)^{\lambda - p_\ell} \, N_\ell(x)^b \, dm(x) \tag{2.37}$$

on $I^{(\ell)}$, where $B_{\Omega^{(\ell)}}$ is the Beta function associated with the cone $\Omega^{(\ell)}$ (see (1.15)).

Example 2.2 *For $\lambda = \frac{d^{(\ell)}}{\ell}$ we consider the probability measure σ_ℓ on the Shilov boundary $\partial_\ell D_0(v_\ell)$ of $D_0(v_\ell)$:*

$$\int_{\partial_\ell D_0(v_\ell)} f \, d\sigma_\ell := \int_{K^{(\ell)}} f(k(u_\ell)) \, dk. \tag{2.38}$$

Its conical part is the Dirac measure δ_{u_ℓ}.

Note that with respect to the measures $\mu_\lambda^{(\ell)}$ and σ_ℓ considered in Examples 2.1 and 2.2, we have

$$\|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|^2_{L^2(\mu_\lambda^{(\ell)})} = \frac{1}{d^{(\ell)}_{\mathbf{m}^{(\ell)}}} \int_{I^{(\ell)}} \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x) \, d\nu_\lambda^{(\ell)}(x) = \frac{\|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|^2_F}{(\lambda)_{\mathbf{m}^{(\ell)}}} \tag{2.39}$$

and

$$\|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|^2_{L^2(\sigma_\ell)} = \frac{\|\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}\|^2_F}{(\frac{d^{(\ell)}}{\ell})_{\mathbf{m}^{(\ell)}}} = \frac{1}{(\frac{d^{(\ell)}}{\ell})_{\mathbf{m}^{(\ell)}}}. \tag{2.40}$$

Applying Corollary 2.2, and using (2.39) and (2.40), we obtain

Corollary 2.3 *Let $f, g \in \mathcal{P}_\ell$ have expansions $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ and $g = \sum_{\mathbf{m}^{(\ell)}} g_{\mathbf{m}^{(\ell)}}$. Then*

(i)

$$< f, g >_{L^2(\mathcal{O},\tilde{\mu})} = \sum_{\mathbf{m}^{(\ell)}} \frac{(\ell\, \frac{a}{2})_{\mathbf{m}^{(\ell)}}}{(r\, \frac{a}{2})_{\mathbf{m}^{(\ell)}}} \int_{\overline{I^{(\ell)}}} \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x) \, d\nu(x) \, \frac{< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F}{(\frac{d}{r})_{\mathbf{m}^{(\ell)}}}. \tag{2.41}$$

(ii) *For any $\lambda > p_\ell - 1$,*

$$< f, g >_{L^2(\mathcal{O}_\ell, \widetilde{\mu_\lambda^{(\ell)}})} = \sum_{\mathbf{m}^{(\ell)}} \frac{(\ell\, \frac{a}{2})_{\mathbf{m}^{(\ell)}} \, (\frac{d^{(\ell)}}{\ell})_{\mathbf{m}^{(\ell)}}}{(r\, \frac{a}{2})_{\mathbf{m}^{(\ell)}} \, (\frac{d}{r})_{\mathbf{m}^{(\ell)}}} \, \frac{< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F}{(\lambda)_{\mathbf{m}^{(\ell)}}}. \tag{2.42}$$

(iii)

$$< f, g >_{L^2(\mathcal{O}_\ell,\sigma_\ell)} = \sum_{\mathbf{m}^{(\ell)}} \frac{(\ell\, \frac{a}{2})_{\mathbf{m}^{(\ell)}}}{(r\, \frac{a}{2})_{\mathbf{m}^{(\ell)}} \, (\frac{d}{r})_{\mathbf{m}^{(\ell)}}} \, < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F . \tag{2.43}$$

Construction of $\hat{\mu}$: The $K^{(\ell)}$-invariant measure μ on $\overline{D_0(v_\ell)}$ is used to define a measure $\hat{\mu}$ on $\partial_{r-\ell} D$:

$$\int_{\partial_{r-\ell} D} f \, d\hat{\mu} = \int_{\overline{D_0(v_\ell)}} \left(\int_K f(k(v_\ell + z)) \, dk \right) \, d\mu(z). \tag{2.44}$$

Obviously,

$$\int_{\partial_{r-\ell} D} f \, d\hat{\mu} = \int_{\overline{I^{(\ell)}}} \left(\int_K f(k(v_\ell + x^{\frac{1}{2}})) \, dk \right) \, d\nu(x) = \int_{\overline{I^{(\ell)}}} \tilde{f}(v_\ell + x) \, d\nu(x). \tag{2.45}$$

3 Integral formulas for the invariant inner products $< \cdot, \cdot >_{\ell\frac{a}{2}}$

In this section we obtain the formulas for the inner products $< f, g >_{\ell\frac{a}{2}}$, $1 \le \ell \le r-1$, via integration on the K-orbits $\partial_{r-\ell}$ and $\mathcal{O}_\ell$.

Let $\{\sigma_j^{(\ell)}\}_{j=1}^{\ell}$ be the elementary symmetric polynomials (2.4) in the variables $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, and let $\boldsymbol{\sigma}^{(\ell)}$ be the vector map $\boldsymbol{\lambda}^{(\ell)} \mapsto (\sigma_1^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}), \dots, \sigma_\ell^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}))$. Following the remark after Lemma 2.1, let $\gamma^{(\ell)} : \mathcal{D}_\ell \to \mathcal{S}_\ell$ be the Harish-Chandra isomorphism, and let

$$\Delta_j^{(\ell)} = \left(\gamma^{(\ell)}\right)^{-1}\left(\sigma_j^{(\ell)}\right), \quad 1 \le j \le \ell. \tag{3.1}$$

We define also $\sigma_0^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}) \equiv 1$, $\Delta_0^{(\ell)} = I$, and let W_ℓ be the permutation group of the coordinates in $\mathbb{C}^\ell$. Thus if U is a W_ℓ-invariant domain and f is an analytic function in $\sigma^{(\ell)}(U)$, then the operator

$$f(\boldsymbol{\Delta}^{(\ell)}) = f(\Delta_1^{(\ell)}, \dots, \Delta_\ell^{(\ell)}) \tag{3.2}$$

(defined via the functional calculus analogous to Lemma 2.1) is $\mathrm{GL}(\Omega^{(\ell)})$-invariant and satisfies

$$f(\boldsymbol{\Delta}^{(\ell)})(\Phi^{(\ell)}_{\boldsymbol{\lambda}^{(\ell)}+\boldsymbol{\rho}^{(\ell)}}) = f(\sigma^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}))\Phi^{(\ell)}_{\boldsymbol{\lambda}^{(\ell)}+\boldsymbol{\rho}^{(\ell)}} \tag{3.3}$$

for every $\boldsymbol{\lambda}^{(\ell)} \in U$, where $\boldsymbol{\rho}^{(\ell)}$ is given by (2.13). In particular, for every partition $\mathbf{m}^{(\ell)} = (m_1, \dots, m_\ell, 0, 0, \dots, 0) \ge 0$ we obtain

$$f(\boldsymbol{\Delta}^{(\ell)})(\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}) = f(\sigma^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}))\Phi^{(\ell)}_{\mathbf{m}^{(\ell)}} \tag{3.4}$$

where $\boldsymbol{\lambda}^{(\ell)} := \mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)}$.

Lemma 3.1 *Let $\alpha > (\ell-1)\frac{a}{2}$. Then for every partition $\mathbf{m}^{(\ell)} = (m_1, \dots, m_\ell, 0, 0, \dots, 0)$ we have*

$$\Gamma_{\Omega^{(\ell)}}(\alpha + \mathbf{m}^{(\ell)}) = (2\pi)^{\ell(\ell-1)\frac{a}{4}} \prod_{j=1}^{\ell} \Gamma(\lambda_j + \alpha - \frac{a}{4}(\ell-1)) \tag{3.5}$$

where $\lambda_j := m_j - \rho_j^{(\ell)} = m_j - (2j-\ell-1)\frac{a}{4}$. Thus $\Gamma_{\Omega^{(\ell)}}(\alpha + \mathbf{m}^{(\ell)})$ and $(\alpha)_{\mathbf{m}^{(\ell)}} = \Gamma_{\Omega^{(\ell)}}(\alpha + \mathbf{m}^{(\ell)})/\Gamma_{\Omega^{(\ell)}}(\alpha)$ are symmetric functions of $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. Moreover, for any $s \in \mathbb{N}$

$$\begin{aligned}\frac{(\alpha+s)_{\mathbf{m}^{(\ell)}}}{(\alpha)_{\mathbf{m}^{(\ell)}}} &= \prod_{\nu=0}^{s-1}\prod_{j=1}^{\ell}(\lambda_j + \alpha + \nu - \frac{a}{4}(\ell-1)) \\ &= \prod_{\nu=0}^{s-1}\sum_{k=0}^{\ell}(\alpha + \nu - \frac{a}{4}(\ell-1))^{\ell-k}\sigma_k^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}). \end{aligned}\tag{3.6}$$

Thus $(\alpha+s)_{\mathbf{m}^{(\ell)}}/(\alpha)_{\mathbf{m}^{(\ell)}}$ is a symmetric polynomial in $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. Hence the operator

$$T := \prod_{\nu=0}^{s-1}\sum_{k=0}^{\ell}(\alpha + \nu - \frac{a}{4}(\ell-1))^{\ell-k}\Delta_k^{(\ell)} \tag{3.7}$$

belongs to $\mathcal{D}_\ell$ and satisfies

$$T f_{\mathbf{m}^{(\ell)}} = \frac{(\alpha+s)_{\mathbf{m}^{(\ell)}}}{(\alpha)_{\mathbf{m}^{(\ell)}}} f_{\mathbf{m}^{(\ell)}} \tag{3.8}$$

for every $\mathbf{m}^{(\ell)} = (m_1, \dots, m_\ell, 0, \dots, 0) \ge 0$ and $f_{\mathbf{m}^{(\ell)}} \in P_{\mathbf{m}^{(\ell)}}$.

Proof: (3.5) is a consequence of (1.14) for the cone $\Omega^{(\ell)}$. The first equality in (3.6) is a consequence of (3.5) and the fact that $\Gamma(z+1) = z\Gamma(z)$, and the second is a well-known property of the $\{\sigma_k^{(\ell)}\}_{k=0}^{\ell}$. The rest follows from (3.3). ∎

Remark: For every $\beta \in \mathbb{C}$ define

$$D^{(\ell)}(\beta) := N_\ell^{\beta+1}\,\partial_{N_\ell}\,N_\ell^{-\beta} \in \mathcal{D}_\ell \tag{3.9}$$

It is well-known (see [FK94] Chapter XIV and [AU97]) that

$$\gamma^{(\ell)}_{D^{(\ell)}(\beta)}(\boldsymbol{\lambda}^{(\ell)}) = \prod_{j=1}^{\ell}(\lambda_j + \frac{a}{4}(\ell-1) - \beta). \tag{3.10}$$

It follows from (3.6) that if $\alpha > (\ell-1)\frac{a}{2}$ and $s \in \mathbb{N}$ then

$$\frac{(\alpha+s)_{\mathbf{m}^{(\ell)}}}{(\alpha)_{\mathbf{m}^{(\ell)}}} = \prod_{\nu=0}^{s-1}\gamma^{(\ell)}_{D^{(\ell)}(\frac{a}{2}(\ell-1)-\alpha-\nu)}(\boldsymbol{\lambda}^{(\ell)}).$$

Since $\gamma^{(\ell)} : \mathcal{D}_\ell \to \mathcal{S}_\ell$ is a (surjective) ring isomorphism, it follows that the operator (3.7) admits the following expression

$$T = N_\ell^{\frac{a}{2}(\ell-1)-\alpha} N_\ell \left(\frac{d}{dx}\right)^s N_\ell^{\alpha+s-\frac{a}{2}(\ell-1)}. \tag{3.11}$$

Theorem 3.1 *Let $0 \le \ell \le r-1$ and let $\beta > \frac{a}{2}(\ell-1)$. Then there exists an operator $T = T^{(\ell,\beta)}$ on $C^\infty(\Omega^{(\ell)})$ which is invariant under $\mathrm{GL}(\Omega^{(\ell)})$, so that for every $f \in \mathcal{H}_{\ell\frac{a}{2}}$ with Peter-Weyl expansion $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$,*

$$Tf = \sum_{\mathbf{m}^{(\ell)}} \frac{(\beta)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} f_{\mathbf{m}^{(\ell)}}. \tag{3.12}$$

Hence, for all $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$,

$$< f, g >_{\ell\frac{a}{2}} = < Tf, g >_\beta\ . \tag{3.13}$$

Moreover, if $\beta - \ell\frac{a}{2} \in \mathbb{N}$ then $T \in \mathcal{D}_\ell$ (i.e. T is a $\mathrm{GL}(\Omega^{(\ell)})$-invariant differential operator).

Remark: Strictly speaking, the meaning of (3.13) is that $T^{\frac{1}{2}}$ (defined in general via the functional calculus (2.11), and for holomorphic functions via $T^{\frac{1}{2}}(\sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}) = \sum_{\mathbf{m}^{(\ell)}} (\frac{(\beta)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}})^{\frac{1}{2}} f_{\mathbf{m}^{(\ell)}})$ maps $\mathcal{H}_{\ell\frac{a}{2}}$ isometrically into $\mathcal{H}_\beta$. Formula (3.13) is valid for all polynomials f, g.

Proof: We define an operator T_0 on holomorphic functions of the form $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ via $T_0 f = \sum_{\mathbf{m}^{(\ell)}} \frac{(\beta)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} f_{\mathbf{m}^{(\ell)}}$. Then T_0 is well defined and continuous with respect to the topology of uniform convergence on compact subsets of D (see [A96]). Notice that the eigenvalues $(\beta)_{\mathbf{m}^{(\ell)}}/(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}$ are positive (since $m_{\ell+1} = \cdots = m_r = 0$ and $\beta > \frac{a}{2}(\ell-1)$). If $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ and $g = \sum_{\mathbf{m}^{(\ell)}} g_{\mathbf{m}^{(\ell)}}$ are polynomials then

$$< Tf, g >_\beta = \sum_{\mathbf{m}^{(\ell)}} \frac{1}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F = < f, g >_{\ell\frac{a}{2}}\ .$$

Thus $T_0^{\frac{1}{2}}$ maps $\mathcal{H}_{\ell\frac{a}{2}}$ into $\mathcal{H}_\beta$ isometrically. Using the notation $\boldsymbol{\lambda}^{(\ell)} = \mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)}$, Lemma 3.1 guarantees that there exists a symmetric function of $\boldsymbol{\lambda}^{(\ell)}$ of the form $p(\boldsymbol{\sigma}^{(\ell)}(\boldsymbol{\lambda}^{(\ell)})) = p(\sigma_1^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}), \ldots, \sigma_\ell^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}))$, so that for all $\mathbf{m}^{(\ell)} \geq 0$

$$\frac{(\beta)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} = p(\boldsymbol{\sigma}^{(\ell)}(\boldsymbol{\lambda}^{(\ell)})).$$

Hence $T := p(\boldsymbol{\Delta}^{(\ell)}) = p(\Delta_1^{(\ell)}, \ldots, \Delta_\ell^{(\ell)})$ is a $\mathrm{GL}(\Omega^{(\ell)})$-invariant operator whose restriction to the holomorphic functions of the form $\sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ is T_0. If $n := \beta - \ell\frac{a}{2} \in \mathbb{N}$, then (3.6) shows that p is the polynomial

$$p(x_1, \ldots, x_\ell) = \prod_{\nu=0}^{n-1} \left(\sum_{k=0}^{\ell} (\ell\frac{a}{2} + \nu - \frac{a}{4}(\ell-1))^{\ell-k} x_k \right)$$

where $x_0 := 1$. Hence

$$T = p(\Delta_1^{(\ell)}, \ldots, \Delta_\ell^{(\ell)}) = \prod_{\nu=0}^{n-1} \left(\sum_{k=0}^{\ell} (\ell\frac{a}{2} + \nu - \frac{a}{4}(\ell-1))^{\ell-k} \Delta_k^{(\ell)} \right)$$

is a member of $\mathcal{D}_\ell$ (i.e. a *polynomial* in the generators $\Delta_1^{(\ell)}, \ldots, \Delta_\ell^{(\ell)}$). ∎

Using (3.4) and Corollary 2.2 we obtain the following result.

Corollary 3.1 *Let $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$ have Peter-Weyl expansions $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ and $g = \sum_{\mathbf{m}^{(\ell)}} g_{\mathbf{m}^{(\ell)}}$. Then for every symmetric function of $\boldsymbol{\lambda}^{(\ell)}$ of the form*

$$p(\boldsymbol{\sigma}^{(\ell)}(\boldsymbol{\lambda}^{(\ell)})) = p(\sigma_1^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}), \ldots, \sigma_\ell^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}))$$

the corresponding differential operator

$$p(\boldsymbol{\Delta}^{(\ell)}) = p(\Delta_1^{(\ell)}, \ldots, \Delta_\ell^{(\ell)}) \in \mathcal{D}_\ell$$

satisfies for every $x \in \Omega^{(\ell)}$:

$$p(\boldsymbol{\Delta}^{(\ell)}) \left(\widetilde{f\bar{g}}(x) \right) = \tag{3.14}$$
$$= c_\ell \sum_{\mathbf{m}^{(\ell)}} \frac{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d}{r})_{\mathbf{m}^{(\ell)}}} \; p(\boldsymbol{\sigma}^{(\ell)}(\mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)})) < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F \Phi_{\mathbf{m}^{(\ell)}}^{(\ell)}(x).$$

If $\alpha > p_\ell - 1 = (\ell-1)a + 1 + b$, then

$$\int_{I_\ell} p(\boldsymbol{\Delta}^{(\ell)}) \left(\widetilde{f\bar{g}} \right) d\nu_\alpha^{(\ell)} = \tag{3.15}$$
$$= c_\ell \sum_{\mathbf{m}^{(\ell)}} \frac{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d_\ell}{\ell})_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d}{r})_{\mathbf{m}^{(\ell)}} (\alpha)_{\mathbf{m}^{(\ell)}}} \; p(\boldsymbol{\sigma}^{(\ell)}(\mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)})) < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F .$$

Here $c_\ell = \Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})/\Gamma_{\Omega^{(\ell)}}(r\frac{a}{2})$, $d_\ell = \dim_{\mathbb{R}}(X_1(u_\ell)) = \ell(\ell-1)\frac{a}{2} + 1$, and $\nu_\alpha^{(\ell)}$ is the measure defined in (2.37).

Notice that by Lemma 3.1 the coefficients of $< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x)$ in (3.14) and (3.15) are symmetric functions of $\boldsymbol{\lambda}^{(\ell)} := \mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)}$.

Remarks: (i) If one uses (3.14) with $x = u_\ell$, one obtains (with $T = p(\boldsymbol{\Delta}^{(\ell)})$)

$$\begin{aligned} (T\tilde{f} \cdot \bar{g})(u_\ell) = T(f \tilde{\cdot} \bar{g})(u_\ell) = \\ = c_\ell \sum_{\mathbf{m}^{(\ell)}} \frac{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}} t_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d}{r})_{\mathbf{m}^{(\ell)}}} < f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F . \end{aligned} \tag{3.16}$$

(ii) If we choose T so that its eigenvalues satisfy

$$c_\ell \frac{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}} t_{\mathbf{m}^{(\ell)}}}{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d}{r})_{\mathbf{m}^{(\ell)}}} = \frac{1}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}}$$

then for every $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$

$$\widetilde{Tf \cdot \bar{g}}(u_\ell) = \sum_{\mathbf{m}^{(\ell)}} \frac{< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} =< f, g >_{\ell\frac{a}{2}} .$$

Namely

$$< f, g >_{\ell\frac{a}{2}} = \int_{S_\ell} (Tf \cdot \bar{g})(v) \, d\sigma_\ell(v). \tag{3.17}$$

This realizes $\mathcal{H}_{\ell\frac{a}{2}}$ as a Hardy-type space on S_ℓ

(iii) It would be interesting to exhibit T in concrete terms (not only via its eigenvalues). If a is even then $T \in \mathcal{D}_\ell$, i.e. T is a *polynomial* in the generators $\Delta_1^{(\ell)}, \Delta_2^{(\ell)}, \ldots, \Delta_\ell^{(\ell)}$ of $\mathcal{D}_\ell$. It would be interesting also to exhibit T as a linear combination of Yan's operators (see [AU97]). If a is odd then either D is of type IV_n with n odd (a case which was considered in [AU97] and [AU98] since $\ell = 1$), or D is of type III_r (with $a = 1$).

Theorem 3.2 *Let $0 \leq \ell \leq r-1$ and let $\alpha > p_\ell - 1 = (\ell-1)a + 1 + b$. Let $p(\sigma^{(\ell)}(\boldsymbol{\lambda}^{(\ell)}))$ be the symmetric function of $\boldsymbol{\lambda}^{(\ell)} = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ so that*

$$p(\sigma^{(\ell)}(\boldsymbol{\lambda}^{(\ell)})) = \frac{1}{c_\ell} \frac{(r\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d}{r})_{\mathbf{m}^{(\ell)}} (\alpha)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})^2_{\mathbf{m}^{(\ell)}} (\frac{d_\ell}{\ell})_{\mathbf{m}^{(\ell)}}} \tag{3.18}$$

for every $\boldsymbol{\lambda}^{(\ell)} := \mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)}$. Let $T = p(\Delta_1^{(\ell)}, \ldots, \Delta_\ell^{(\ell)})$ be the $\mathrm{GL}(\Omega^{(\ell)})$-invariant operator defined via the functional calculus (Lemma 2.1). Then for every $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$

$$< f, g >_{\ell\frac{a}{2}} = \int_{I_\ell} T\left(\widetilde{f\bar{g}}\right) d\nu_\alpha^{(\ell)}, \tag{3.19}$$

where $\nu_\alpha^{(\ell)}$ is the measure defined in (2.37). Moreover, if $s := \alpha - \ell\frac{a}{2} \in \mathbb{N}$, then p is a polynomial in $\boldsymbol{\lambda}^{(\ell)}$ and $T \in \mathcal{D}_\ell$, i.e. T is a $\mathrm{GL}(\Omega^{(\ell)})$-invariant differential operator on Ω.

Proof: The right hand side of (3.18) is symmetric in $\boldsymbol{\lambda}^{(\ell)} := \mathbf{m}^{(\ell)} - \boldsymbol{\rho}^{(\ell)}$ by Lemma 3.1. Thus (3.14) yields for any $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$ with Peter-Weyl expansions $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$ and $g =$

$\sum_{\mathbf{m}^{(\ell)}} g_{\mathbf{m}^{(\ell)}}$,

$$\begin{aligned}\int_{I_\ell} T\left(\widetilde{f\bar{g}}\right) d\nu_\alpha^{(\ell)} &= \sum_{\mathbf{m}^{(\ell)}} \frac{< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F (\alpha)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}} (\frac{d_\ell}{\ell})_{\mathbf{m}^{(\ell)}}} \int_{I_\ell} \Phi^{(\ell)}_{\mathbf{m}^{(\ell)}}(x)\, d\nu_\alpha^{(\ell)}(x) \\ &= \sum_{\mathbf{m}^{(\ell)}} \frac{< f_{\mathbf{m}^{(\ell)}}, g_{\mathbf{m}^{(\ell)}} >_F}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} =< f, g >_{\ell\frac{a}{2}} .\end{aligned}$$

Assume that $s := \alpha - \ell\frac{a}{2} \in \mathbb{N}$. If also $n := (r-\ell)\frac{a}{2} \in \mathbb{N}$ then

$$r\frac{a}{2} - \ell\frac{a}{2} = \frac{d}{r} - \frac{d_\ell}{\ell} = n,$$

and Lemma 3.1 guarantees that p is a symmetric *polynomial* of degree $\ell(s+2n)$ in $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. If $(r-\ell)\frac{a}{2} \notin \mathbb{N}$, then necessarily $b = 0$, and both

$$n_1 := (r-\ell+1)\frac{a}{2} - 1 \quad \text{and} \quad n_2 := \frac{d}{r} - \ell\frac{a}{2} = (r-1-\ell)\frac{a}{2} + 1$$

are in $\mathbb{N}$. Again, Lemma 3.1 guarantees that p is a polynomial of degree $\ell(s+n_1+n_2)$ in $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. This completes the proof. ∎

Remarks:

(i) Using Lemma 2.2 and Proposition 2.1 it follows that if T is a $\mathrm{GL}(\Omega^{(\ell)})$-invariant operator on $C^\infty(\Omega^{(\ell)})$, then for every $f, g \in \mathcal{H}_{\ell\frac{a}{2}}$

$$T\left(\widetilde{f\bar{g}}\right) = \widetilde{Tf \cdot \bar{g}} = \widetilde{f \cdot \overline{Tg}}. \tag{3.20}$$

Theorems 3.1 and 3.2 can be reformulated accordingly. For instance, (3.19) can be rewritten as

$$< f, g >_{\ell\frac{a}{2}} = \int_{I_\ell} \widetilde{Tf \cdot \bar{g}}\, d\nu_\alpha^{(\ell)} = \int_{I_\ell} \widetilde{f \cdot \overline{Tg}}\, d\nu_\alpha^{(\ell)}. \tag{3.21}$$

(ii) Formula (3.18) can be rewritten as

$$< f, g >_{\ell\frac{a}{2}} = \int_K \left(\int_{D_0(v_\ell)} (Tf \cdot \bar{g})(k(z)) d\mu_\alpha^{(\ell)}(z) \right) dk$$

4 Integral formulas in the context of symmetric Siegel domains

In this section we develop integral formulas for the inner products in the spaces $\mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$, (where $T(\Omega)$ is the symmetric Siegel associated to D via the Cayley transform) in terms of the Fourier transform of the functions. We begin with the relatively simple case of a Siegel domain of type I. The results presented below for the discrete Wallach points $\{\ell\frac{a}{2}\}_{\ell=0}^{r-1}$ will be somewhat analogous to our earlier results [AU97] for the continuous Wallach points $\lambda > (r-1)\frac{a}{2}$. The development of the integral formulas in the context of a general symmetric Siegel domain of type II requires additional machinery, and will be treated separately.

The case of a symmetric Siegel domain of type I

Assume that Z is a JB^*-algebra with a unit e. The open unit ball of Z is holomorphically equivalent to the tube domain

$$T(\Omega) = X + i\Omega$$

via the *Cayley transform* $c(z) = i(e+z)(e-z)^{-1}, \;\; z \in D$. $T(\Omega)$ is a symmetric Siegel domain of type I. For any $\lambda \in W(D)$ the operator $V^{(\lambda)} f = (f \circ c^{-1})(J\, c^{-1})^{\lambda/p}$ maps the space $\mathcal{H}_\lambda(\mathcal{D})$ isometrically onto a Hilbert space of analytic functions on $T(\Omega)$, denoted by $\mathcal{H}_\lambda(T(\Omega))$. The inner product in $\mathcal{H}_\lambda(T(\Omega))$ is defined by

$$\langle f, g\rangle_\lambda = \langle f, g\rangle_{\mathcal{H}_\lambda(T(\Omega))} = \langle {V^{(\lambda)}}^{-1}(f), \;\; {V^{(\lambda)}}^{-1} g\rangle_{\mathcal{H}_\lambda(\mathcal{D})} \; . \tag{4.1}$$

The description of $\mathcal{H}_\lambda(T(\Omega))$ is therefore equivalent to the description of $\mathcal{H}_\lambda(\mathcal{D})$.

The reproducing kernel of $\mathcal{H}_\lambda(T(\Omega))$ is

$$K_\lambda(z, w) = N\Big(\frac{z - w^*}{i}\Big)^{-\lambda}, \quad z, w \in T(\Omega) \; . \tag{4.2}$$

Namely, for all $z, w \in T(\Omega)$

$$(J(c^{-1})(z)^{\lambda/p}\, h(c^{-1}(z), c^{-1}(w))^{-\lambda}\, \overline{(J\, c^{-1}(w))}^{\lambda/p} = N\Big(\frac{z - w^*}{i}\Big)^{-\lambda} \; .$$

It is known that for $\lambda > p - 1$ $\mathcal{H}_\lambda(T(\Omega))$ is the weighted Bergman space

$$\mathcal{H}_\lambda(T(\Omega)) = L^2_a(T(\Omega), m_\lambda) = L^2(T(\Omega), m_\lambda) \cap \{\text{analytic function}\}$$

where

$$d\, m_\lambda(z) = c_\lambda\, dx\;\, N(2y)^{\lambda - p} dy, \quad z = x + iy, \; x \in X, \; y \in \Omega$$

and

$$c_\lambda = \frac{\Gamma_\Omega(\lambda)}{\pi^d\, \Gamma_\Omega(y - \frac{d}{r})}$$

Also, the Shilov boundary of $T(\Omega)$ is $X := \{z \in Z; \; z^* = z\}$, and $\mathcal{H}_{\frac{d}{r}}(T(\Omega))$ coincides with the Hardy space $H^2(X)$ (consisting of all analytic functions f in $T(\Omega)$ for which $\| f \|^2_{H^2(X)} := \sup\limits_{y \in \Omega} \int\limits_X |f(x + iy)|^2 dx < \infty$).

Using the Fourier transform (with respect to x) one obtains the following result. Here for $\lambda > (r-1)\frac{g}{2}$ we consider on Ω the measure

$$d\, \sigma_\lambda(v) = \beta_\lambda\, N(v)^{\frac{d}{r} - \lambda}\, dv, \qquad \beta_\lambda = (2\pi)^{-2d}\Gamma_\Omega(\lambda). \tag{4.3}$$

Proposition 4.1 [AU97; Proposition 6.1] *Let $\lambda > (r-1)\frac{a}{2}$, and let f be a holomorphic function in $T(\Omega)$. Then the following are equivalent:*

(i) *$f \in \mathcal{H}_\lambda(T(\Omega))$;*

(ii) *The boundary values $f(x) := \lim\limits_{\Omega \ni y \to 0} f(x + iy)$ exist almost everywhere on X, and the Fourier transform $\hat{f}$ of $f(x)$ is supported in $\overline{\Omega}$ and belongs to $L^2(\Omega, \sigma_\lambda)$;*
Moreover, the map $f \mapsto \hat{f}$ is an isometry of $\mathcal{H}_\lambda(T(\Omega))$ onto $L^2(\Omega, \sigma_\lambda)$. Consequently, for all $f, g \in \mathcal{H}_\lambda(T(\Omega))$

$$\langle f, g\rangle_\lambda = \int\limits_\Omega \hat{f}\; \overline{\hat{g}}\; d\, \sigma_\lambda \tag{4.4}$$

Our goal here is to extend Proposition 4.1 to the discrete Wallach points $\ell\frac{a}{2}$, $\ell = 0, 1, 2, \ldots, r-1$. With respect to the fixed frame $\{e_j\}_{j=1}^r$ of minimal, pairwise orthogonal idempotents, we denote $u_\ell = \sum_{j=1}^\ell e_j$, $v_\ell = \sum_{j=\ell+1}^r e_j$, $0 \le \ell \le r-1$. Recall that the orbits of $\mathrm{GL}(\Omega)$ on $\partial\Omega$ are exactly

$$\begin{aligned} \partial_\ell\Omega &= \mathrm{GL}(\Omega)(u_\ell) = \{\varphi(u_\ell);\ \varphi \in \mathrm{GL}(\Omega)\} \\ &= \{x \in \overline{\Omega};\ \mathrm{rank}(x) = \ell\},\ \ell = 0, 1, 2, \ldots, r-1 \end{aligned} \tag{4.5}$$

The following fundamental fact is established in [RV76] and [La87]. An explicit direct proof will be given in Section 6 bellow.

Theorem 4.1 *Let $0 \le \ell \le r-1$. There exists a unique measure μ_ℓ on $\partial_\ell\Omega$, having the following properties:*

$$d\mu_\ell(\varphi(x)) = Det(\varphi)^{\ell\frac{a}{2}/\frac{d_1}{r}}\, d\mu_\ell(x), \quad \forall \varphi \in GL(\Omega) \tag{4.6}$$

where $d_1 = \dim(Z_1(e)) = r(r-1)\frac{a}{2} + r$, and

$$\int_{\partial_\ell\Omega} e^{-<x,y>}\, d\mu_\ell(y) = \gamma_\ell\ N_\ell(x)^{-\ell\,\frac{a}{2}} \quad \forall x \in \Omega, \tag{4.7}$$

where $\gamma_\ell = (2\pi)^{\ell(r-\ell)\frac{a}{2}}\Gamma_{\Omega(\ell)}(\ell\frac{a}{2})$.

Let $\mathrm{GL}(\Omega) = L\, N_\Omega\, A$ be the Iwasawa decomposition. Then it is known that the set

$$N_\Omega\, A(u_\ell) = \{x \in \partial_\ell(\Omega);\ N_\ell(x) > 0\} \tag{4.8}$$

is open and dense in $\partial_\ell\,\Omega$ and $\mu_\ell(\partial_\ell\,\Omega \setminus N_\Omega\, A(u_\ell)) = 0$. The following result is established in [La87].

Lemma 4.1 *An element $x \in \partial_\ell\,\Omega$ belongs to $N_\Omega\, A(u_\ell)$ if and only if in its Peirce decomposition relative to $u_\ell : x = x_1 + x_{1/2} + x_0$, x_1 is invertible in $X_1(u_\ell)$ and*

$$x_0 = 2\, v_\ell(x_{1/2}(x_{1/2}\, x_1^{-1})) \tag{4.9}$$

The expression of μ_ℓ in the coordinates $(x_1, x_{1/2})$ of $x \in N_\Omega\, A(u_\ell)$ is

$$d\mu_\ell(x) = N_\ell(x_1)^{\ell\frac{a}{2}-\frac{d}{r}}\ dx_1\ dx_{1/2} \tag{4.10}$$

The properties of μ_ℓ enable us to describe the space $\mathcal{H}_{\ell\frac{a}{2}}$.

Lemma 4.2 *Fix $w \in T(\Omega)$ and $0 \le \ell \le r-1$. Then the Fourier transform with respect to x of the function $K_w^{(\ell\frac{a}{2})}(x) = K^{(\ell\frac{a}{2})}(x, w) = N(\frac{x-w^*}{i})^{-\ell\frac{a}{2}}$ is the following measure with support $\partial_\ell\,\Omega$:*

$$\widehat{K_w^{(\ell\frac{a}{2})}}(t) = \frac{(2\pi)^d}{\gamma_\ell}\ e^{-i\langle w^*|t\rangle}\ d\mu_\ell(t) \tag{4.11}$$

where $\gamma_\ell = (2\pi)^{\ell(r-\ell)\frac{a}{2}}\Gamma_{\Omega(\ell)}(\frac{a}{2})$, as in Theorem 4.1.

Proof: Theorem 4.1 and the fact that Ω is a set of uniqueness for holomorphic functions on $T(\Omega)$ imply that for all $z \in T(\Omega)$

$$\int_{\partial_\ell \Omega} e^{i\langle z|t\rangle} d\,\mu_\ell(t) = \gamma_\ell \; N(\frac{z}{i})^{\ell\frac{a}{2}}$$

It follows that for all $z, w \in T(\Omega)$

$$K^{(\ell\frac{a}{2})}(z,w) = \left(N\Big(\frac{z-w^*}{i}\Big)\right)^{-\ell\frac{a}{2}} = \frac{1}{\gamma_\ell} \int_{\Omega_\ell} e^{-\langle \frac{z-w^*}{i}|t\rangle} \; d\,\mu_\ell(t) \; .$$

Hence

$$K_w^{(\ell\frac{a}{2})}(x) = \frac{1}{\gamma_\ell} \int_{\partial_\ell\Omega} e^{i\langle x|t\rangle} e^{-i\langle w^*|t\rangle} \; d\,\mu_\ell(t), \quad w \in T(\Omega) \tag{4.12}$$

Thus $K_w^{(\ell\frac{a}{2})}(x)$ is the inverse Fourier transform of the measure $\frac{(2\pi)^d}{\gamma_\ell} \; e^{-i\langle w^*|t\rangle} \; d\,\mu_\ell(t)$, which is supported on $\partial_\ell \Omega$, and (4.11) follows. ∎

Lemma 4.2 can be reformulated by saying that $\widehat{K_w^{(\ell\frac{a}{2})}}$ is a measure supported in $\partial_\ell \Omega$ which is absolutely continuous with respect to μ_ℓ, with Radon-Nikodym derivative

$$\frac{d\,\widehat{K_w^{(\ell\frac{a}{2})}}}{d\,\mu_\ell}(t) = \frac{(2\pi)^d}{\gamma_\ell} \; e^{-i\langle w^*|t\rangle} \tag{4.13}$$

Lemma 4.3 *For every $z, w \in T(\Omega)$ and $0 \le \ell \le r-1$*

$$\Big\langle \frac{d\,\widehat{K_w^{(\ell\frac{a}{2})}}}{d\,\mu_\ell}, \; \frac{d\,\widehat{K_z^{(\ell\frac{a}{2})}}}{d\,\mu_\ell} \Big\rangle_{L^2(\partial_\ell\Omega,\mu_\ell)} = \frac{(2\pi)^{2d}}{\gamma_\ell} K^{(\ell\frac{a}{2})}(z,w) \tag{4.14}$$

Proof: Both sides of (4.14) are holomorphic in z and anti-holomorphic in w. Therefore it suffices to prove (4.14) for $z = w = u + iv, \;\; u \in X, \;\; v \in \Omega$. In this case we obtain by Lemma 4.1

$$\begin{aligned} \left\| \frac{d\,\widehat{K^{(\ell\frac{a}{2})}}}{d\,\mu_\ell} \right\|^2_{L^2(\partial_\ell\Omega,\mu_\ell)} &= \frac{(2\pi)^{2d}}{\gamma_\ell^2} \int_{\partial_\ell\Omega} \left| e^{-i\langle w^*|t\rangle} \right|^2 d\,\mu_\ell(t) = \frac{(2\pi)^{2d}}{\gamma_\ell^2} \int_{\partial_\ell\Omega} e^{-i\langle 2v|t\rangle} \, d\,\mu_\ell(t) \\ &= \frac{(2\pi)^{2d}}{\gamma_\ell} \; N(2v)^{-\ell\frac{a}{2}} = \frac{(2\pi)^{2d}}{\gamma_\ell} \; K^{(\ell\frac{a}{2})}(w,w) \; . \end{aligned}$$ ∎

Fix $0 \le \ell \le r-1$ and consider the space

$$\mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega)) := \text{span} \left\{ K_w^{(\ell\frac{a}{2})}; \; w \in T(\Omega) \right\} \; . \tag{4.15}$$

We define a map $V_\ell^{(0)}$ on $\mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$ via

$$V_\ell^{(0)} f = \frac{\gamma_\ell^{\frac{1}{2}}}{(2\pi)^d} \; \frac{d\,\hat{f}}{d\,\mu_\ell} \; , \tag{4.16}$$

where $\hat{f}$ is the Fourier transform of the restriction of f to the Shilov boundary X, and $\frac{d\hat{f}}{d\mu_\ell}$ is the Radon-Nikodym derivative of the measure $\hat{f}$ with respect to μ_ℓ, which exists in view of Lemma 4.2 and the fact that $f \in \mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$.

Lemma 4.4 *$V^{(0)}_\ell$ is an isometry of $\mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$ into $L^2(\partial_\ell\Omega, \mu_\ell)$, and it has a dense range.*

Proof: : Let $f = \sum_{j=1}^n c_j K^{(\ell\frac{a}{2})}_{w_j} \in \mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T\Omega)$. Then

$$\| f \|^2_{\ell\frac{a}{2}} = \sum_{i,j=1}^n c_i \overline{c}_j \langle K^{(\ell\frac{a}{2})}_{w_i}, K^{\ell\frac{a}{2}}_{w_j} \rangle_{\ell\frac{a}{2}} = \sum_{i,j=1}^n c_i \overline{c}_j \; K^{(\ell\frac{a}{2})}(w_j, w_i) \; .$$

Also, Lemma 4.3 implies

$$\begin{aligned} \| V^{(0)}_\ell \|^2_{L^2(\partial_\ell\Omega\,\mu_\ell)} &= \frac{\gamma_\ell}{(2\pi)^{2d}} \sum_{i,j=1}^n c_i \overline{c}_j \left\langle \frac{d\,\widehat{K^{(\ell\frac{a}{2})}_{w_i}}}{d\,\mu_\ell}, \frac{d\,\widehat{K^{(\ell\frac{a}{2})}_{w_j}}}{d\,\mu_\ell} \right\rangle_{L^2(\partial_\ell\Omega,\mu_\ell)} \\ &= \sum_{i,j=1}^n c_i \overline{c}_j \; K^{(\ell\frac{a}{2})}(w_j, w_i) = \| f \|^2_{\ell\frac{a}{2}} \end{aligned}$$

Thus $V^{(0)}_\ell$ is an isometry. The range of $V^{(0)}_\ell$ contains all the functions

$$V^{(0)}_\ell \left(\gamma^{1/2}_\ell \; K^{(\ell\frac{a}{2})}_w\right)(t) = e^{-i\langle w^* | t\rangle}, \;\; w \in T(\Omega) \; .$$

The linear span of these functions is a self-adjoint sub-algebra of $C(\partial_\ell\Omega)$, which separates the points of $\partial_\ell\Omega$. Therefore $V^{(0)}_\ell(\mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$ is dense in $C_0(\partial_\ell\Omega)$ by the Stone-Weierstrass theorem. Since μ_ℓ is mutually absolutely continuous with respect to Lebesgue measure on $\partial_\ell\Omega$, the density of $V^{(0)}_\ell(\mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$ in $L^2(\partial_\ell\Omega, \mu_\ell)$ follows now by standard arguments. ∎

It follows from Lemma 4.4 that $V^{(0)}_\ell$ extends an isometry V_ℓ of $\mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$ onto $L^2(\partial_\ell\Omega, \mu_\ell)$. The exact statement is the following result.

Theorem 4.2 *Let $0 \le \ell \le r-1$, and let f be a holomorphic function in $T(\Omega)$. The following conditions are equivalent:*

(i) *$f \in \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$;*

(ii) *The boundary values $f(x) = \lim_{\Omega\ni y\to 0} f(x+iy)$ exist almost everywhere on X, the Fourier transform $\hat{f}$ of $f(x)$ is a measure with support in $\partial_\ell\Omega$ which is absolutely continuous with respect to μ_ℓ, and the Radon-Nikodym derivative $\frac{d\hat{f}}{d\mu_\ell}$ belongs to $L^2(\partial_\ell\Omega, \mu_\ell)$. Moreover, the map $V_\ell f = \frac{d\hat{f}}{d\mu_\ell}$ is an isometry of $\mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$ onto $L^2(\partial_\ell\Omega, \mu_\ell)$. Thus for all $f, g \in \mathcal{H}^{(0)}_{\ell\frac{a}{2}}(T(\Omega))$*

$$\langle f, g\rangle_{\ell\frac{a}{2}} = \frac{\Gamma_{\Omega(\ell)}(\ell\frac{a}{2})}{(2\pi)^{2\delta_\ell}} \int_{\partial_\ell\Omega} \frac{d\hat{f}}{d\mu_\ell}(t) \; \overline{\frac{d\hat{g}}{d\mu_\ell}}(t) \; d\mu_\ell(t) \tag{4.17}$$

where $\delta_\ell = d - \ell(r-\ell)\frac{a}{2}$.

Expressing μ_ℓ via (4.10) on $N_\Omega\, A(u_\ell)$, we obtain

$$\langle f, g\rangle_{\ell\frac{a}{2}} = \frac{\Gamma_{\Omega(\ell)}(\ell\frac{a}{2})}{(2\pi)^{2\delta_\ell}} \int\limits_{N_\Omega\, A(u_\ell)} \frac{d\,\hat{f}}{d\,\mu_\ell}(t)\, \overline{\frac{d\,\hat{g}}{d\,\mu_\ell}}(t)\; N_\ell(t_1)^{\ell\frac{a}{2}-\frac{d}{r}}\, dt_1\, dt_2 \tag{4.18}$$

Remark 4.1 *In the case where $\lambda > (r-1)\,\frac{a}{2}$, (4.4) can be written in the form*

$$\langle f, g\rangle_\lambda = \frac{\Gamma_\Omega(\lambda)}{(2\pi)^{2d}} \int\limits_\Omega \frac{d\,\hat{f}}{d\,\mu_\lambda}(t)\, \overline{\frac{d\,\hat{g}}{d\,\mu_\lambda}}(t)\; N(t)^{\lambda-\frac{d}{r}}\, dt \tag{4.19}$$

(where $\hat{f}, \hat{g}$ are considered as the measures $\hat{f}(t)\,dt$ and $\hat{g}(t)\,dt$). Thus (4.18) is the right analogue of (4.19), and therefore of (4.4). It is an interesting problem to obtain (4.18) from (4.19) by analytic continuation in the parameter λ.

The case of a symmetric Siegel domain of type II

Assume now that e is a maximal tripotent in Z which is not unitary. Thus $Z_1(e) + Z_{1/2}(e)$ and $Z_{1/2}(e) \neq 0$. Thus

$$d_1 := \dim\, Z_1(e) = r + r(r-1)\,\frac{a}{2},\;\; d_{1/2} := \dim\, Z_{1/2}(e) = rb$$

(where $1 \leq b \in \mathbb{N}$). $Z_1(e)$ is a JB^*-algebra which operates on $Z_{1/2}(e)$ via

$$R(z)\,w = 2\{z, e, \eta\}, \quad z \in Z_1(e),\;\; \eta \in Z_{1/2}(e)\;. \tag{4.20}$$

$R : Z_1(e) \to \mathrm{End}\,(Z_{1/2}(e))$ is a monomorphism of Jordan $*$-algebras, where the involution in $\mathrm{End}\,(Z_{1/2}(e))$ is induced by the given K-invariant inner product $\langle\xi|\eta\rangle$ (see [Lo75], Lemma 8.1, p.75). Let us denote

$$F(\xi,\eta) = \{\xi, \eta, e\}, \quad \xi, \eta \in Z_{1/2}(e)\;. \tag{4.21}$$

Then $F : Z_{1/2}(e) \times Z_{1/2}(e) \to Z_1(e)$ is sesquilinear, and $F(\xi,\xi) \in \overline{\Omega}$ for all $\xi \in Z_{1/2}(e)$. We denote also $F(\xi) := F(\xi,\xi)$. Let us define $\tau : Z \times Z \to Z_1(e)$ by

$$\tau(z,w) = \frac{z_1 - w_1^*}{i} - 2\,F(z_{1/2}, w_{1/2}) \tag{4.22}$$

where $z = z_1 + z_{1/2},\;\; w = w_1 + w_{1/2}\;\; (z_1, w_1 \in Z_1^{(e)}$ and $z_{1/2}, w_{1/2} \in Z_{1/2}(e))$. For convenience we denote $\tau(z) = \tau(z,z)$. The associated Siegel domain of type II is

$$T(\Omega) := \{z \in Z;\quad \tau(z) \in \Omega\}\;. \tag{4.23}$$

It is known that the *Cayley transform*

$$c(z) = i\,\frac{e + z_1}{e - z_1} + \sqrt{2}\;R((e - z_1)^{-1})(z_{1/2}), \quad z = z_1 + z_{1/2} \tag{4.24}$$

maps the Cartan domain D (i.e. the open unit ball of Z) biholomorphically onto $T(\Omega)$. Again, for $\lambda \in W(D)$ the operator $V^{(\lambda)}f = (f \circ c^{-1})(J\,c^{-1})^{\lambda/p}$ maps $\mathcal{H}_\lambda(D)$ isometrically onto $\mathcal{H}_\lambda(T(D))$, which is endowed with the inner product (4.1). Also, the reproducing kernel of $\mathcal{H}_\lambda(T(D))$ is

$$K^{(\lambda)}(z,w) = N(\tau(z,w))^{-\lambda}, \quad z, w \in T(\Omega)\;. \tag{4.25}$$

Our main goal here is to describe the inner product of $\mathcal{H}_\lambda(T(D))$ concretely.

The *Shilov boundary* of $T(\Omega)$ is the set

$$H = \{z \in T(\Omega); \tau(z) = 0\} = \{x + i\,F(\xi) + \xi;\ \ x \in X_1(e) + \xi \in Z_{1/2}(e)\}. \tag{4.26}$$

Proposition 4.2 *Let $\xi, \eta \in Z_{1/2}(e)$. Then for every $v \in \Omega$*

$$|\langle F(\eta,\xi)|v\rangle| \le \langle F(\xi|v)\rangle^{1/2}\langle F(\eta|v)\rangle^{1/2} \le \frac{1}{2}\langle F(\xi) + F(\eta)|v\rangle. \tag{4.27}$$

Thus

$$Re\, F(\eta,\xi) \le \frac{1}{2}\langle (F(\xi) + F(\eta)). \tag{4.28}$$

The straightforward proof is based on the positivity of F (i.e. the fact that $F(\xi) \in \overline{\Omega}$ for all $\xi \in Z_{1/2}(e)$), and it is omitted.

Corollary 4.1 *For all $z, w \in T(\Omega)$*

$$Re\,(\tau(z,w)) \ge \frac{1}{2}(\tau(z) + \tau(w))\ . \tag{4.29}$$

In particular $Re\,(\tau(z,w)) \in \Omega$, and this is true even if $z \in H$ and $w \in T(\Omega)$.

Proof: Using (4.28) we have

$$\begin{aligned} 2R\,\tau(z,w) &= \frac{z_1 - z_1^*}{i} + \frac{w_1 - w_1^*}{i} - 4Re\,F(z_{1/2}, w_{1/2}) \\ &\ge \frac{z_1 - z_1^*}{i} + \frac{w_1 - w_1^*}{i} - 2F(z_{1/2}) - 2F(w_{1/2}) = \tau(z) + \tau(w)\ . \quad \blacksquare \end{aligned}$$

For $\lambda > (r-1)\,\frac{a}{2}$ consider the measure $d\mu_\lambda(x) = N(x)^{\lambda - \frac{d_1}{r}}\,dx$ on Ω. For $\lambda = \ell\frac{a}{2},\ \ 0 \le \ell \le r-1$, let $\mu_\lambda := \mu_\ell$ be the Lassalle measure (see Theorem 4.1 and Section 6 bellow). Then for all $\lambda \in W(D)$

$$\int_\Omega e^{-\langle y|x\rangle}\,d\mu_\lambda(x) = \gamma_\lambda\,N(y)^{-\lambda}, \tag{4.30}$$

with $\gamma_\lambda = \Gamma_\Omega(\lambda)$ for $\lambda > (r-1)\frac{a}{2}$, and $\gamma_\lambda = \gamma_\ell = (2\pi)^{\ell(\ell-1)\frac{a}{2}}\,\Gamma_{\Omega(\ell)}\,(\ell\frac{a}{2})$ for $\lambda = \ell_{\frac{a}{2}},\ \ 0 \le \ell \le r-1$.

Corollary 4.2 *Let $z, w \in T(\Omega)$ and let $\lambda \in W(D)$. Then*

$$K^{(\lambda)}(z,w) = \frac{1}{\gamma_\lambda}\int_\Omega e^{-\langle \tau(z,w)|t\rangle}\,d\,\mu_\lambda(t)\ . \tag{4.31}$$

The formula holds also for $z \in H$ and $w \in T(\Omega)$.

Proof: Since $Re\,\tau(z,w) \in \Omega$, the integral converges absolutely, and uniformly on compact subsets of $T(\Omega) \times T(\Omega)$. Therefore, the integral is holomorphic in z and anti-holomorphic in w. Since $K^{(\lambda)}(z,w)$ is also sesqui-holomorphic, it is enough to show that (4.31) holds for $z = w \in T(\Omega)$. Writing $z = x + iy + \xi$ $(x \in X,(e),\ y \in \Omega,\ \xi \in Z_{1/2}(e))$, and using (4.30), we obtain

$$\int_\Omega e^{-\langle \tau(z)|t\rangle}\,d\,\mu_\lambda(t) \quad = \quad \gamma_\lambda\,N(\tau(z))^{-\lambda} = \gamma_\lambda\,K^{(\lambda)}(z,z)\ .$$

Thus (4.31) is established for all $z, w \in T(\Omega)$. Letting $\tau(z) \to 0$ in (4.31) for fixed $w \in T(\Omega)$ (i.e. $z \to H$), we obtain (4.31) also for $z \in H$ and $w \in T(\Omega)$. $\blacksquare$

Lemma 4.5 *Let $\lambda \in W(D)$, fix $w = u + iv + \eta \in T(\Omega)$ (with $u \in X_1(e)$, $v \in \Omega$, $\eta \in Z_{1/2}(e)$) and $\xi \in Z_{1/2}(e)$, and consider the function*

$$K^{(\lambda)}_{w,\xi}(x) = K^{(\lambda)}(x + i\,F(\xi) + \xi,\ w), \quad x \in X_1(e)\ . \tag{4.32}$$

Then the Fourier transform of $K^{(\lambda)}_{w,\xi}$, considered as measure, has support in $\overline{\Omega}$, and is given by

$$\overline{K^{(\lambda)}_{w,\xi}}(t) = \frac{(2\pi)^{d_1}}{\gamma_\lambda}\ \exp\{-\langle F(\xi) + v - 2\,F(\xi,\eta) + iu|t\rangle\}\,d\mu_\lambda(t). \tag{4.33}$$

Proof: Using (4.31) for $w \in T(\Omega)$ and $z = x + i\,F(\xi) + \xi \in H$, we obtain

$$K^{(\lambda)}_{w,\xi}(x) = \frac{1}{\gamma_\lambda}\int_\Omega e^{i\langle x|t\rangle}\,e^{-\langle F(\xi)+v-2\,F(\xi,\eta)+iu|t\rangle}\,d\,\mu_\lambda(t)\ .$$

Thus $K^{(\lambda)}_{w,\xi}$ is the inverse Fourier transform of the measure $\gamma_\lambda^{-1}\ \exp\{-\langle F(\xi) + v - 2\,F(\xi,\eta) + iu|t\rangle\}\ d\,\mu_\lambda(t)$, whose support is contained in $\overline{\Omega}$. From this (4.33) follows by inverting the Fourier transform. ∎

For $\lambda > (r-1)\,\frac{a}{2}$ we consider on $\Omega \times Z_{1/2}(e)$ the measure

$$d\,\sigma_\lambda(t,\xi) = N(t)^b\ d\,\mu_\lambda(t)\ d\xi = N(t)^{\lambda - \frac{d_1}{r} + b}\ dt\,d\xi\ . \tag{4.34}$$

Lemma 4.6 *For every $w \in T(\Omega)$*

$$\iint\limits_{\Omega\times Z_{1/2}(e)} \Big|\frac{\widehat{K^{(\lambda)}_{w,\xi}}}{d\,\mu_\lambda}\ (t)\Big|^2\ d\,\sigma_\lambda(t,\xi) = \frac{(2\pi)^{rp}}{\Gamma_\Omega(\lambda)}\ K^{(\lambda)}(w,w)\ . \tag{4.35}$$

Proof: Writing $w = u + iv + \eta$ as in Lemma 4.5, we obtain μ_λ from (4.33)

$$\begin{aligned}
\int\limits_{Z_{1/2}(e)} \Big|\frac{d\,\widehat{K^{(\lambda)}_{w,\xi}}}{d\,\mu_\lambda}\ (t)\Big|^2\ d\xi &= \frac{(2\pi)^{2d_1}}{\gamma_\lambda^2} e^{-2\langle v|t\rangle} \int\limits_{Z_{1/2}(e)} e^{-2\langle F(\xi) - 2\,Re\,F(\xi,\eta)|t\rangle}\,d\xi \\
&= \frac{(2\pi)^{2d_1}}{\gamma_\lambda^2} e^{-2\langle v|t\rangle}\ e^{2\langle F(\eta)|t\rangle} \int\limits_{Z_{1/2}(e)} e^{-2\langle F(\xi-\eta)|t\rangle}\,d\xi \\
&= \frac{(2\pi)^{2d_1}}{\gamma_\lambda^2} e^{-\langle \tau(w)|t\rangle} \int\limits_{Z_{1/2}(e)} e^{-\|R(t^{1/2})\xi\|^2}\,d\xi \\
&= \frac{(2\pi)^{2d_1+rb}}{\gamma_\lambda^2}\,N(t)^{-b}\ e^{-\langle\tau(w)|t\rangle}\ .
\end{aligned}$$

Here we used the well-known formula

$$\langle x|\{y,\,z,\,w\}\rangle = \langle\{x,\,y,\,z\}|w\rangle, \quad \forall\ x,y,z,w \in Z \tag{4.36}$$

to obtain

$$\|\ R(t^{1/2})\,\xi\ \|^2 = \langle\xi|R(t)\,\xi\rangle = \langle\xi|2\{t,\,e,\,\xi\}\rangle = 2\langle\{\xi,\xi,e\}|t\rangle = 2\langle F(\xi)|t\rangle\ .$$

It follows that

$$\begin{aligned}
&\int\!\!\int_{\Omega\times Z_{1/2}(e)} \Big|\frac{d\,\widehat{K^{(\lambda)}_{w,\xi}}}{d\,\mu_\lambda}\,(t)\Big|^2\,d\xi\;N(t)^b\;d\,\mu_\lambda(t) = \\
&= \frac{(2\pi)^{rp}}{\gamma_\lambda^2}\int_\Omega e^{-\langle\tau(w)|t\rangle}\,d\,\mu_\lambda(t) \\
&= \frac{(2\pi)^{rp}}{\Gamma_\Omega(\lambda)}\cdot N(\tau(w))^{-\lambda} = \frac{(2\pi)^{rp}}{\Gamma_\Omega(\lambda)}\;K^{(\lambda)}(w,w)\;. \qquad \blacksquare
\end{aligned}$$

Corollary 4.3 *Let $\lambda > (r-1)\frac{a}{2}$. For all $z, w \in T(\Omega)$*

$$\int_\Omega\int_{Z_{\frac{1}{2}}(e)} \frac{d\widehat{K^{(\lambda)}_{w,\xi}}}{d\nu_\lambda}(t)\overline{\frac{d\widehat{K^{(\lambda)}_{z,\xi}}}{d\nu_\lambda}(t)}\;d\sigma_\lambda(t,\xi) = \frac{(2\pi)^{rp}}{\Gamma_\Omega(\lambda)}\;K^{(\lambda)}(z,w). \tag{4.37}$$

Also, considering $\widehat{K^{(\lambda)}_{w,\xi}}(t)$ as a function, we have

$$\int_\Omega\int_{Z_{\frac{1}{2}}(e)} \widehat{K^{(\lambda)}_{w,\xi}}(t)\;\overline{\widehat{K^{(\lambda)}_{z,\xi}}}(t)\;d\xi\;N(t)^{\frac{d}{r}-\lambda}\;dt = \frac{(2\pi)^{rp}}{\Gamma_\Omega(\lambda)}\;K^{(\lambda)}(z,w). \tag{4.38}$$

Proof: Both sides of (4.37) are sesqui-holomorphic in (z,w) and coincide on the "diagonal" $z = w$ by Lemma 4.6. Hence they coincide for all $z, w \in T(\Omega)$. (4.38) is an obvious consequence of (4.37), since

$$\frac{d\widehat{K^{(\lambda)}_{w,\xi}}}{d\nu_\lambda}(t)\cdot N(t)^{\lambda-\frac{d}{r}} = \widehat{K^{(\lambda)}_{w,\xi}}(t). \qquad \blacksquare \tag{4.39}$$

The generalization of Proposition 4.1 to Siegel domains of type II is the following result.

Theorem 4.3 *Let $T(\Omega)$ be a symmetric Siegel domain of type II, let $\lambda > (r-1)\frac{a}{2}$, and let f be a holomorphic function on $T(\Omega)$. Then the following conditions are equivalent*

(i) *$f \in \mathcal{H}_\lambda(T(\Omega))$;*

(ii) *The boundary values of f at points $z = x + iF(\xi) + \xi$ ($x \in X_1(e), \xi \in Z_{\frac{1}{2}}(e)$) of the Shilov boundary H, i.e.*

$$f_\xi(x) = f(x + iF(\xi) + \xi) = \lim_{\Omega\ni y\to 0} f(x + iy + iF(\xi) + \xi)$$

exist almost everywhere, the Fourier transform $\hat{f}_\xi(t)$ is supported in $\bar{\Omega}$, and

$$\int_\Omega\int_{Z_{\frac{1}{2}}(e)} |\hat{f}_\xi(t)|^2\,d\xi\,N(t)^{\frac{d}{r}-\lambda}\,dt < \infty.$$

Moreover, the operator $V_\lambda : \mathcal{H}_\lambda(T(\Omega)) \to L^2(\Omega \times Z_{\frac{1}{2}}(e), N(t)^{\frac{d}{r}-\lambda}\, dt\, d\xi)$ *defined by*

$$(V_\lambda f)(t,\xi) = \frac{\Gamma_\Omega(\lambda)^{\frac{1}{2}}}{(2\pi)^{\frac{rp}{2}}}\, \hat{f}_\xi(t) \tag{4.40}$$

is an isometry of $\mathcal{H}_\lambda(T(\Omega))$ *onto* $L^2(\Omega \times Z_{\frac{1}{2}}(e), N(t)^{\frac{d}{r}-\lambda}\, dt\, d\xi)$*. In particular, for every* $f, g \in \mathcal{H}_\lambda(T(\Omega))$

$$< f, g >_\lambda = \frac{\Gamma_\Omega(\lambda)}{(2\pi)^{rp}} \int_\Omega \int_{Z_{\frac{1}{2}}(e)} \hat{f}_\xi(t)\, \overline{\hat{g}_\xi(t)}\, d\xi\, N(t)^{\frac{d}{r}-\lambda}\, dt. \tag{4.41}$$

The proof uses (4.38) and is analogous to the proofs of Proposition 4.1 (i.e. Proposition 6.1 and Theorem 6.1 of [AU97]) and to the proof of Theorem 4.2. Therefore we omit it. We remark that in view of (4.37),(4.41) can be written also in the form

$$< f, g >_\lambda = \frac{\Gamma_\Omega(\lambda)}{(2\pi)^{rp}} \int_\Omega \int_{Z_{\frac{1}{2}}(e)} \frac{d\hat{f}_\xi(t)}{d\nu_\lambda}\, \overline{\frac{d\hat{g}_\xi(t)}{d\nu_\lambda}}\, d\sigma_\lambda(t,\xi), \tag{4.42}$$

where σ_λ is the measure defined by (4.34).

We turn now to the case where $\lambda = \ell\frac{a}{2}$, $0 \le \ell \le r-1$ (and for simplicity denote $\nu_{\ell\frac{a}{2}} = \nu_\ell$ and $\gamma_{\ell\frac{a}{2}} = \gamma_\ell$).

Let $t \in \partial_\ell\Omega$, then its support idempotent $s(t)$ has rank ℓ. Thus $Z_{\frac{1}{2}}(e)$ is the direct sum

$$Z_{\frac{1}{2}}(e) = \left(Z_{\frac{1}{2}}(e) \cap Z_{\frac{1}{2}}(s(t))\right) + \left(Z_{\frac{1}{2}}(e) \cap Z_0(s(t))\right). \tag{4.43}$$

Let us denote

$$\partial_\ell\left(\widehat{T(\Omega)}\right) = \{t + \xi;\, t \in \partial_\ell\Omega,\, \xi \in Z_{\frac{1}{2}}(e) \cap Z_{\frac{1}{2}}(s(t))\}. \tag{4.44}$$

(The notation is chosen as to indicate that the Fourier transforms of functions in $\mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$ are supported in $\partial_\ell\left(\widehat{T(\Omega)}\right)$).

$\partial_\ell\left(\widehat{T(\Omega)}\right)$ can be viewed as a bundle whose base is $\partial_\ell\Omega$, and the fiber over $t \in \partial_\ell\Omega$ is $Z_{\frac{1}{2}}(e) \cap Z_{\frac{1}{2}}(s(t))$. Let us consider on $\partial_\ell\left(\widehat{T(\Omega)}\right)$ the measure $\tilde{\mu_\ell}$, defined by

$$\int_{\partial_\ell\left(\widehat{T(\Omega)}\right)} f\, d\tilde{\mu_\ell} = \int_{\partial_\ell\Omega} \left(\int_{Z_{\frac{1}{2}}(e)\cap Z_{\frac{1}{2}}(s(t))} f(t+\xi)\, d\xi \right) d\mu_\ell(t). \tag{4.45}$$

For every $t \in \partial_\ell\Omega$ let $\det(t) = N_{X_1(s(t))}(t)$ be the determinant of t in the Jordan algebra $X_1(s(t))$. We define a measure σ_ℓ on $\partial_\ell\left(\widehat{T(\Omega)}\right)$ via

$$\int_{\partial_\ell\left(\widehat{T(\Omega)}\right)} f\, d\sigma_\ell = \int_{\partial_\ell\Omega} \left(\int_{Z_{\frac{1}{2}}(e)\cap Z_{\frac{1}{2}}(s(t))} f(t+\xi)\, d\xi \right) \det(t)^b d\mu_\ell(t), \tag{4.46}$$

i.e., $d\sigma_\ell(t,\xi) = \det(t)^b d\tilde{\mu_\ell}(t,\xi)$. Namely, on the base $\partial_\ell\Omega$ we use the measure μ_ℓ and at the fiber above $t \in \partial_\ell\Omega$ we use the measure $\det(t)^b d\xi$. Notice the analogy between σ_ℓ and σ_λ for $\lambda > (r-1)\frac{a}{2}$.

Lemma 4.7 *Let $0 \le \ell \le r-1$ and fix $w = u + iv + \eta \in T(\Omega)$ (where $u \in X_1(e)$, $v \in \Omega$ and $\eta \in Z_{\frac{1}{2}}(e)$). Then the Fourier transform (with respect to x) of $K_{w,\xi}^{(\ell\frac{a}{2})}(x) = K_w^{(\ell\frac{a}{2})}(x + iF(\xi) + \xi)$ is a measure on $\partial_\ell\left(\widehat{T(\Omega)}\right)$ which is absolutely continuous with respect to $\tilde{\mu}_\ell$, and*

$$\frac{d\widehat{K_w^{(\ell\frac{a}{2})}}}{d\tilde{\mu}_\ell}(t,\xi) = \frac{(2\pi)^{d_1}}{\gamma_\ell} \exp\left(-\left\langle iu - 2i\Im F(\xi,\eta) + \frac{1}{2}\tau(w) + F(\xi - \eta)\middle| t\right\rangle\right). \qquad (4.47)$$

Moreover, with $\chi_\ell = (2\pi)^{2d_1 + \ell b - \ell(r-\ell)\frac{a}{2}} \cdot 2^{-\ell b}$ we have

$$\int_{\partial_\ell\left(\widehat{T(\Omega)}\right)} \left|\frac{d\widehat{K_w^{(\ell\frac{a}{2})}}}{d\tilde{\mu}_\ell}\right|^2 d\sigma_\ell = \frac{\chi_\ell}{\gamma_\ell} K^{(\ell\frac{a}{2})}(w,w). \qquad (4.48)$$

Proof: Using Lemma 4.5 for $\lambda = \ell\frac{a}{2}$, we see that for $t \in \partial_\ell\Omega$ and $\xi \in Z_{\frac{1}{2}}(e)$,

$$\widehat{K_{w,\xi}^{(\ell\frac{a}{2})}}(t) = \frac{(2\pi)^{d_1}}{\gamma_\ell} \exp\left(-\langle iu + v + F(\xi) - 2F(\xi,\eta)|t\rangle\right) d\mu_\ell(t).$$

It is easy to see that for all $\xi, \eta \in Z_{\frac{1}{2}}(e)$

$$\langle F(\xi,\eta)|t\rangle = \langle F(P_{\frac{1}{2}}(s(t))\xi, P_{\frac{1}{2}}(s(t))\eta)|t\rangle. \qquad (4.49)$$

Hence, the measure $\widehat{K_w^{(\ell\frac{a}{2})}}$ is supported in $\partial_\ell\left(\widehat{T(\Omega)}\right)$, it is absolutely continuous with respect to $\tilde{\mu}_\ell$, and its Radon-Nikodym derivative with respect to $\tilde{\mu}_\ell$ is given by (4.47). Next, using (4.49) we see that for fixed $t \in \partial_\ell\Omega$

$$\begin{aligned}
\int_{Z_{\frac{1}{2}}(e)\cap Z_{\frac{1}{2}}(s(t))} \left|\frac{d\widehat{K_w^{(\ell\frac{a}{2})}}}{d\tilde{\mu}_\ell}(t,\xi)\right|^2 d\xi &= \frac{(2\pi)^{2d_1}}{\gamma_\ell^2} e^{-\langle\tau(w)|t\rangle} \int_{Z_{\frac{1}{2}}(e)\cap Z_{\frac{1}{2}}(s(t))} e^{-2\langle F(\xi)|t\rangle} d\xi \\
&= \frac{(2\pi)^{2d_1}\pi^{\ell b}}{\gamma_\ell^2} e^{-\langle\tau(w)|t\rangle} \det(t)^{-b}.
\end{aligned}$$

Hence, using Corollary 4.2, we obtain

$$\begin{aligned}
\int_{\partial_\ell\left(\widehat{T(\Omega)}\right)} \left|\frac{d\widehat{K_w^{(\ell\frac{a}{2})}}}{d\tilde{\mu}_\ell}(t,\xi)\right|^2 d\sigma_\ell(t,\xi) &= \frac{(2\pi)^{2d_1}\pi^{\ell b}}{\gamma_\ell^2} \int_{\partial_\ell\Omega} e^{-\langle\tau(w)|t\rangle} d\mu_\ell(t) \\
&= \frac{\chi_\ell}{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})} N(\tau(w))^{-\ell\frac{a}{2}} = \frac{\chi_\ell}{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})} K^{(\ell\frac{a}{2})}(w,w). \qquad \blacksquare
\end{aligned}$$

Corollary 4.4 *Let $0 \le \ell \le r-1$. For every $z, w \in T(\Omega)$,*

$$\int_{\partial_\ell\left(\widehat{T(\Omega)}\right)} \frac{d\widehat{K_w^{(\ell\frac{a}{2})}}}{d\tilde{\mu}_\ell} \cdot \overline{\frac{d\widehat{K_z^{(\ell\frac{a}{2})}}}{d\tilde{\mu}_\ell}} d\sigma_\ell = \frac{\chi_\ell}{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})} K^{(\ell\frac{a}{2})}(z,w). \qquad (4.50)$$

Proof: Both sides of (4.50) are holomorphic in z and anti-holomorphic in w, and they coincide on the "diagonal" $z = w$. Hence they coincide for all $z, w \in T(\Omega)$. $\blacksquare$

Theorem 4.4 *Let $T(\Omega)$ be a symmetric Siegel domain of type II, let $0 \le \ell \le r-1$, and let f be a holomorphic function on $T(\Omega)$. The following conditions are equivalent*

(i) $f \in \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$;

(ii) *The boundary values of f*

$$f_\xi(x) = f(x+iF(\xi)+\xi) = \lim_{\Omega \ni y \to 0} f(x+iy+iF(\xi)+\xi)$$

exist for almost all points $x+iF(\xi)+\xi$ of the Shilov boundary H, the Fourier transform $\hat{f}_\xi(t) := \int_{X_1(e)} e^{-i\langle x|t\rangle} f_\xi(x)dx$ is a measure with support in $\partial_\ell\left(\widehat{T(\Omega)}\right)$ which is absolutely continuous with respect to $\tilde{\mu}_\ell$, and the Radon-Nikodym derivative $\frac{\partial \hat{f}}{\partial \tilde{\mu}_\ell}$ belongs to $L^2\left(\partial_\ell\left(\widehat{T(\Omega)}\right), \sigma_\ell\right)$.

Moreover, the operator $V_\ell : \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega)) \to L^2\left(\partial_\ell\left(\widehat{T(\Omega)}\right), \sigma_\ell\right)$ defined via

$$(V_\ell f)(t,\xi) = \left(\frac{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})}{\chi_\ell}\right)^{\frac{1}{2}} \frac{\partial \hat{f}}{\partial \tilde{\mu}_\ell}(t,\xi) \tag{4.51}$$

is a surjective isometry. Thus, for all $f, g \in \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$,

$$\begin{aligned} < f,g >_{\ell\frac{a}{2}} &= \frac{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})}{\chi_\ell} \int_{\partial_\ell\left(\widehat{T(\Omega)}\right)} \frac{\partial \hat{f}}{\partial \tilde{\mu}_\ell} \cdot \overline{\frac{\partial \hat{g}}{\partial \tilde{\mu}_\ell}} d\sigma_\ell \qquad (4.52) \\ &= \frac{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})}{\chi_\ell} \int_{\partial_\ell \Omega} \left(\int_{Z_{\frac{1}{2}}(e)\cap Z_{\frac{1}{2}}(s(t))} \frac{\partial \hat{f}}{\partial \tilde{\mu}_\ell}(t,\xi) \cdot \overline{\frac{\partial \hat{g}}{\partial \tilde{\mu}_\ell}(t,\xi)} \det(t)^b \, d\xi \right) d\nu_\ell(t). \end{aligned}$$

The proof of Theorem 4.4 uses Lemma 4.7 and Corollary 4.4, as well as the standard arguments used in the proofs of Proposition 4.1 and Theorem 4.2; it is therefore omitted.

Although the bundle $\partial_\ell\left(\widehat{T(\Omega)}\right)$ and the measure σ_ℓ give natural and canonical description of the space $\mathcal{H}_{\ell\frac{a}{2}}$ and its inner product (Theorem 4.4), they are not easy to use in some concrete computation. We therefore develop now a formula for $< f,g >_{\ell\frac{a}{2}}$ analogous to (4.52) with more concrete space and measure, which are however not invariant.

Recall that $u_\ell = \sum_{j=1}^{\ell} e_j$, $v_\ell = \sum_{j=\ell+1}^{r} e_j$. We write

$$Z_{\frac{1}{2}}^{(\frac{1}{2})} = Z_{\frac{1}{2}}(e) \cap Z_{\frac{1}{2}}(u_\ell), \quad Z_{\frac{1}{2}}^{(0)} = Z_{\frac{1}{2}}(e) \cap Z_0(u_\ell). \tag{4.53}$$

Thus $Z_{\frac{1}{2}}(e) = Z_{\frac{1}{2}}^{(\frac{1}{2})} + Z_{\frac{1}{2}}^{(0)}$. Recall also (see Lemma 4.1) that every $t \in N_\Omega A(u_\ell) \subset \partial_\ell \Omega$ has Peirce decomposition $t = t_1 + t_{\frac{1}{2}} + t_0$, where $t_1 \in X_1(u_\ell)$ positive and invertible, $t_{\frac{1}{2}} \in X_{\frac{1}{2}}(u_\ell) = X_{\frac{1}{2}}(v_\ell)$, and $t_0 \in X_1(v_\ell)$ depends on t_1 and $t_{\frac{1}{2}}$ via

$$t_0 = 2v_\ell(t_{\frac{1}{2}}(t_{\frac{1}{2}} t_1^{-1})), \tag{4.54}$$

where t_1^{-1} is the inverse of t_1 in $X_1(u_\ell)$.

Lemma 4.8 *For every* $\xi = \xi_{\frac{1}{2}} + \xi_0 \in Z_{\frac{1}{2}}$ *(with* $\xi_{\frac{1}{2}} \in Z_{\frac{1}{2}}^{(\frac{1}{2})}$ *and* $\xi_0 \in Z_{\frac{1}{2}}^{(0)}$*) and every* $t = t_1 + t_{\frac{1}{2}} + t_0 \in N_\Omega A(u_\ell)$*,*

$$2\langle F(\xi)|t\rangle = \|R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}} + R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\|^2 \tag{4.55}$$

where $t_1^{-\frac{1}{2}}$ *is the inverse of* $t_1^{\frac{1}{2}}$ *in* $X_1(u_\ell)$*.*

Proof: (4.55) will follow as soon as we prove that

$$\begin{aligned} 2\langle\{\xi_{\frac{1}{2}},\xi_{\frac{1}{2}},u_\ell\}|t_1\rangle &= \|R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}}\|^2 & (4.56)\\ 2\langle\{\xi_0,\xi_{\frac{1}{2}},u_\ell\}|t_{\frac{1}{2}}\rangle &= \langle R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0|R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}}\rangle & (4.57)\\ 2\langle\{\xi_{\frac{1}{2}},\xi_0,v_\ell\}|t_{\frac{1}{2}}\rangle &= \langle R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}}|R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\rangle & (4.58)\end{aligned}$$

and

$$2\langle\{\xi_0,\xi_0,v_\ell\}|t_0\rangle = \|R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\|^2. \tag{4.59}$$

Indeed, by the "Peirce calculus" and orthogonality of the Peirce spaces

$$\begin{aligned} 2\langle F(\xi)|t\rangle &= 2\langle\{\xi_{\frac{1}{2}}+\xi_0,\xi_{\frac{1}{2}}+\xi_0,u_\ell+v_\ell\}|t_1+t_{\frac{1}{2}}+t_0\rangle\\ &= 2\langle\{\xi_{\frac{1}{2}},\xi_{\frac{1}{2}},u_\ell\}|t_1\rangle + 2\langle\{\xi_0,\xi_0,v_\ell\}|t_0\rangle\\ &\quad +2\langle\{\xi_0,\xi_{\frac{1}{2}},u_\ell\}|t_{\frac{1}{2}}\rangle + 2\langle\{\xi_{\frac{1}{2}},\xi_0,v_\ell\}|t_{\frac{1}{2}}\rangle.\end{aligned}$$

Using the fact that $R: Z_1(e) \to \mathrm{End}(Z_{\frac{1}{2}}(e))$ is a monomorphism of Jordan-$*$-algebras, (see [Lo75], Lemma 8.1, p. 75), we see that $R|_{Z_1(u_\ell)}: Z_1(u_\ell) \to Z_{\frac{1}{2}}^{(\frac{1}{2})}$ is also a monomorphism of Jordan-$*$-algebras. In particular, for every $\xi_{\frac{1}{2}} \in Z_{\frac{1}{2}}^{(\frac{1}{2})}$,

$$R(t_1^{\frac{1}{2}})R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}} = R(t_1)\xi_{\frac{1}{2}} \quad\text{and}\quad R(t_1^{-\frac{1}{2}})R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}} = R(u_\ell)\xi_{\frac{1}{2}} = \xi_{\frac{1}{2}}.$$

It follows that

$$\begin{aligned} \|R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}}\|^2 &= \langle\xi_{\frac{1}{2}}|R(t_1^{\frac{1}{2}})^*R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}}\rangle = \langle\xi_{\frac{1}{2}}|R(t_1)\xi_{\frac{1}{2}}\rangle\\ &= 2\langle\xi_{\frac{1}{2}}|\{\xi_{\frac{1}{2}},u_\ell,t_1\}\rangle, \text{ since } t_1 \text{ is orthogonal to } v_\ell\\ &= 2\langle\{\xi_{\frac{1}{2}},\xi_{\frac{1}{2}},u_\ell\}|t_1\rangle, \text{ by (4.36)},\end{aligned}$$

and (4.56) is established. Using similar arguments and the fact that $R(t_{\frac{1}{2}})Z_{\frac{1}{2}}^{(0)} \subset Z_{\frac{1}{2}}^{(\frac{1}{2})}$, we obtain

$$\begin{aligned} \langle R(t_1^{\frac{1}{2}})\xi_{\frac{1}{2}}|R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\rangle &= \langle\xi_{\frac{1}{2}}|R(t_1^{\frac{1}{2}})^*R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\rangle = \langle\xi_{\frac{1}{2}}|R(t_{\frac{1}{2}})\xi_0\rangle\\ &= 2\langle\xi_{\frac{1}{2}}|\{\xi_0,v_\ell,t_{\frac{1}{2}}\}\rangle, \text{ since } \xi_0 \text{ is orthogonal to } u_\ell\\ &= 2\langle\{\xi_{\frac{1}{2}},\xi_0,v_\ell\}|t_{\frac{1}{2}}\rangle.\end{aligned}$$

This establishes (4.58). The proof of (4.57) is similar and is therefore omitted. To prove (4.59), notice first that

$$\begin{aligned}\|R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\|^2 &= \langle\xi_0|R(t_{\frac{1}{2}})R(t_1^{-\frac{1}{2}})^*R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\rangle \\ &= \langle\xi_0|R(t_{\frac{1}{2}})R(t_1^{-1})R(t_{\frac{1}{2}})\xi_0\rangle.\end{aligned}$$

Next, since R is a Jordan homomorphism, it preserves the "quadratic representation" operator $P(x) := 2M(x)^2 - M(x^2)$ (where $M(x)y := xy = \{x, e, y\}\ \forall x, y \in Z_1(e)$). Thus

$$\begin{aligned}R(t_{\frac{1}{2}})R(t_1^{-1})R(t_{\frac{1}{2}}) &= R(P(t_{\frac{1}{2}})t_1^{-1}) \\ &= 2R(M(t_{\frac{1}{2}})^2t_1^{-1}) - R(M(t_{\frac{1}{2}}^2)t_1^{-1}).\end{aligned}$$

Now, $M(t_{\frac{1}{2}})^2t_1^{-1} = t_{\frac{1}{2}}(t_{\frac{1}{2}}t_1^{-1}) \in X_1(u_\ell) + X_1(v_\ell)$, hence

$$\begin{aligned}2R(M(t_{\frac{1}{2}})^2t_1^{-1})\xi_0 &= 4\{t_{\frac{1}{2}}(t_{\frac{1}{2}}t_1^{-1}), v_\ell, \xi_0\}, \text{ since } \xi_0 \text{ is orthogonal to } u_\ell \\ &= 4\{v_\ell(t_{\frac{1}{2}}(t_{\frac{1}{2}}t_1^{-1})), v_\ell, \xi_0\} \\ &= 2\{t_0, v_\ell, \xi_0\}, \text{ by (4.54)}.\end{aligned}$$

Next, $t_{\frac{1}{2}}^2 \in X_1(u_\ell) + X_1(v_\ell)$. Hence $M(t_{\frac{1}{2}}^2)t_1^{-1} \in X_1(u_\ell)$, and therefore $R(M(t_{\frac{1}{2}}^2)t_1^{-1})\xi_0 = 0$. It follows that

$$\begin{aligned}\|R(t_1^{-\frac{1}{2}})R(t_{\frac{1}{2}})\xi_0\|^2 &= \langle\xi_0|2R(M(t_{\frac{1}{2}})^2t_1^{-1})\xi_0 - R(M(t_{\frac{1}{2}}^2)t_1^{-1})\xi_0\rangle \\ &= 2\langle\xi_0|\{\xi_0, v_\ell, t_0\}\rangle = 2\langle\{\xi_0, \xi_0, v_\ell\}|t_0\rangle\end{aligned}$$

and (4.59) is established. This completes the proof of Lemma 4.8. ∎

Let us define a measure $\tilde\sigma_\ell$ on the set $N_\Omega A(u_\ell) \times Z_{\frac{1}{2}}^{(\frac{1}{2})}$ via

$$\int_{N_\Omega A(u_\ell)}\int_{Z_{\frac{1}{2}}^{(\frac{1}{2})}} f\,d\tilde\sigma_\ell = \int_{N_\Omega A(u_\ell)}\left(\int_{Z_{\frac{1}{2}}^{(\frac{1}{2})}} f(t+\xi_{\frac{1}{2}})d\xi_{\frac{1}{2}}\right) N_\ell(t_1)^b d\mu_\ell(t). \tag{4.60}$$

Notice the analogy between $\tilde\sigma_\ell$ and σ_ℓ (and the fact that they use the same number of variables. The advantage of $\tilde\sigma_\ell$ is that it uses fixed coordinates $(t_1, t_{1/2}, \xi_{1/2}) \in \Omega_1(u_\ell) \times X_{1/2}(u_\ell) \times Z_{1/2}^{(1/2)}$.

Lemma 4.9 *Let $0 \le \ell \le r-1$ and $w \in T(\Omega)$. Then*

$$\int_{N_\Omega A(u_\ell)}\int_{Z_{\frac{1}{2}}^{(\frac{1}{2})}} \left|\frac{d\widehat{K_{w,\xi_{\frac{1}{2}}}^{(\ell\frac{a}{2})}}}{d\mu_\ell}(t)\right|^2 d\tilde\sigma_\ell(t, \xi_{\frac{1}{2}}) = \frac{\chi_\ell}{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})} K^{(\ell\frac{a}{2})}(w,w), \tag{4.61}$$

where χ_ℓ is as in Lemma 4.7.

Proof: Write $w = u + iv + \eta$ with $u \in X_1(e)$, $v \in \Omega$ and $\eta \in Z_{\frac{1}{2}}$, and fix $t \in N_\Omega A(u_\ell)$ with Peirce decomposition $t = t_1 + t_{\frac{1}{2}} + t_0$ with $t_\alpha \in X_\alpha(u_\ell)$ and t_0 given by (4.54). Then

$$\left|\frac{d\widehat{K_{w,\xi_{\frac{1}{2}}}^{(\ell\frac{a}{2})}}}{d\mu_\ell}(t)\right|^2 = \frac{(2\pi)^{2d_1}}{\gamma_\ell^2}e^{-\langle\tau(w)|t\rangle}e^{-2\langle F(\xi_{\frac{1}{2}}-\eta)|t\rangle},$$

and in view of Lemma 4.8,

$$\int_{Z_{1/2}^{(1/2)}} e^{-2\langle F(\xi_{1/2}-\eta)|t\rangle}\, d\xi_{1/2} = \int_{Z_{1/2}^{(1/2)}} e^{-\|R(t_1^{1/2})(\xi_{1/2}-\eta_{1/2})-R(t_1^{-1/2})R(t_{1/2})\eta_0\|^2}\, d\xi_{1/2}$$
$$= \int_{Z_{1/2)}^{(1/2)}} e^{-\|z_{1/2}\|^2}\, d(R(t_1^{-1/2})z_{1/2}) = \pi^{\ell b}\, N_\ell(t_1)^{-b}\ .$$

Using this and the knowledge of the Laplace transform of μ_ℓ (see Theorem 4.1), we obtain

$$\int_{N_\ell A(u_\ell)} \int_{Z_{1/2}^{(1/2)}} \left| \frac{d\, \widehat{K_{w,\xi_{1/2}}^{(\ell\frac{a}{2})}}}{d\mu_\ell}(t)\right|^2 d\xi_{1/2}\ N_\ell(t_1)^b\ d\mu_\ell(t) =$$
$$= \frac{(2\pi)^{2d_1+\ell b}}{\gamma_\ell^2} \int_{N_\Omega A(u_\ell)} e^{-\langle\tau(w)|t\rangle}\, d\mu_\ell(t)$$
$$= \frac{(2\pi)^{2d_1+\ell b}}{\gamma_\ell\, 2^{\ell b}}\ N(\tau(w))^{-\ell\frac{a}{2}} = \frac{\chi_\ell}{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})}\ K^{(\ell\frac{a}{2})}(w,w)\ . \quad \blacksquare$$

Theorem 4.5 *Let $T(\Omega)$ be a symmetric Siegel domain of type II. Let $0 \le \ell \le r-1$, and let f be a holomorphic function on $T(\Omega)$. Then the following conditions are equivalent:*

(i) $f \in \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$;

(ii) *The boundary values of f at points of the Shilov boundary H:*

$$f_\xi(x) := f(x+i\,F(\xi)+\xi) \lim_{\Omega\ni y\to 0} f(x+iy+i\,F(\xi)+\xi)$$

exist almost everywhere on H, the Fourier transform $\hat f_\xi(t)$ is a measure with support in $\partial_\ell\,\Omega$ which is absolutely continuous with respect to μ_ℓ, and the Radon-Nikodym derivative $\frac{d\,\hat f_\xi}{d\,\mu_\ell}(t)$ satisfies

$$\int_{\partial_\ell(\Omega)} \int_{Z_{1/2}^{(1/2)}} \left|\frac{d\,\hat f_{\xi_{1/2}}}{d\mu_\ell}(t)\right|^2 d\tilde\sigma_\ell\,(t,\xi_{1/2}) \ <\ \infty\ .$$

Moreover, the operator $V_\ell : \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega)) \to L^2(\partial_\ell\Omega \times Z_{1/2}^{(1/2)}, \tilde\sigma_\ell)$ defined via

$$(Vf)(t,\xi_{1/2}) = \left(\frac{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})}{\chi_\ell}\right)^{\frac{1}{2}} \frac{d\,\hat f_{\xi_{1/2}}}{d\mu_\ell}(t) \qquad (4.62)$$

is a surjective isometry. Thus, for all $f,g \in \mathcal{H}_{\ell\frac{a}{2}}(T(\Omega))$,

$$\langle f,g\rangle_{\ell\frac{a}{2}} = \frac{\Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})}{\chi_\ell} \int_{N_\Omega A(u_\ell)} \int_{Z_{1/2}^{(1/2)}} \frac{d\hat f_{\xi_{1/2}}}{d\mu_\ell}(t)\ \overline{\frac{d\hat g_{\xi_{1/2}}}{d\mu_\ell}}(t)\ d\xi_{1/2}\ N_\ell(t_1)^{\ell\frac{a}{2}+b-\frac{d_1}{r}} dt_1\, dt_{1/2}\ . \qquad (4.63)$$

The proof relies on Lemma 4.9 and standard techniques (as in the proof of earlier Theorems in this section); it is therefore omitted.

Remark: (1) $\partial_\ell(\widehat{(T(\Omega)})$ should not be confused with the boundary orbit $\partial_\ell(T(\Omega))$ of $T(\Omega)$:

$$\partial_\ell(T(\Omega)) \quad = \quad \{z \in \overline{T(\Omega)};\ \tau(z) \in \partial_\ell \Omega\} \tag{4.64}$$

(2) There is a representation $\varphi \mapsto \tilde{\varphi}$ of $\mathrm{GL}(\Omega_\ell)$ on $Z_{1/2}(e)$, defined on the generators of $\mathrm{GL}(\Omega)$ via

$$\widetilde{P(x)} = R(x), \quad x \in \Omega, \quad \text{and} \quad \tilde{\ell} = \ell, \qquad \ell \in L\ .$$

One has

$$\varphi(F(\xi,\eta)) = F(\tilde{\varphi}(\xi), \tilde{\varphi}(\eta)), \quad \varphi \in \mathrm{GL}(\Omega), \xi, \eta \in Z_{1/2}(e).$$

$\mathrm{GL}(\Omega)$ acts also on $\partial_\ell\left(\widehat{T(\Omega)}\right)$ via

$$\varphi.(t,\xi) = (\varphi(t), \tilde{\varphi}(\xi)), \quad \varphi \in \mathrm{GL}(\Omega),\ t \in \partial_\ell(\Omega), \quad \xi \in Z_{1/2}(e) \cap Z_{1/2}(s(t))\ .$$

In particular, $\tilde{\varphi}(\xi) \in Z_{1/2}(e) \cap Z_{1/2}(s(\varphi(t)))$. The proof of Lemma 4.7 yields the transformation formula

$$\sigma_\ell \circ \varphi = (Det\, \varphi)^{(b+\ell\frac{a}{2})/\frac{d_1}{r}}\, \sigma_\ell \qquad \forall \varphi \in \mathrm{GL}(\Omega) \tag{4.65}$$

as well as the Laplace transform formula

$$\int_{\partial_\ell(\widehat{T(\Omega)})} e^{-\langle v+F(\xi)|t\rangle} d\sigma_\ell(t,\xi) = (2\pi)^{b\ell+(r-\ell)\ell\frac{a}{2}}\, \Gamma_{\Omega^{(\ell)}}(\ell\frac{a}{2})\, N(v)^{-\ell\frac{a}{2}} \tag{4.66}$$

for all $v \in \Omega$. These properties are analogous to the corresponding properties of μ_ℓ (see Theorem 4.1).

5 Realization of $\mathcal{H}_{\alpha_\ell}(T(\Omega))$ and $\mathcal{H}_{\alpha_\ell}(D)$ by boundary integration

In this section our main concern will be the Wallach points

$$\alpha_\ell = \ell\frac{a}{2} + \frac{d}{r}, \quad 0 \le \ell \le r-1. \tag{5.1}$$

Let D be a Cartan domain and let $T(\Omega)$ be the associated symmetric Siegel domain (as in the previous section). We assume that $T(\Omega)$ is of type II; the analysis in the type I case is easier and will follow from the general case.

For $0 \le \ell \le r-1$ consider the set

$$\partial_\ell(T(\Omega)) = \{z \in \overline{T(\Omega)};\ \tau(z) \in \partial_\ell\Omega\}. \tag{5.2}$$

Thus $\partial_\ell(T(\Omega))$ consists of all points

$$z = x + iy + iF(\xi) + \xi, \quad x \in X_1(e), \xi \in Z_{\frac{1}{2}},\ y \in \partial_\ell\Omega. \tag{5.3}$$

Hence $\partial_\ell(T(\Omega))$ is the direct sum of the Shilov boundary H and $i\,\partial_\ell\Omega$:

$$\partial_\ell(T(\Omega)) = H + i\,\partial_\ell\Omega. \tag{5.4}$$

We equip $\partial_\ell(T(\Omega))$ with the measure

$$dM_\ell^{T(\Omega)}(z) = \varepsilon_\ell^{-1}\, dx\, d\xi\, d\mu_\ell(y), \tag{5.5}$$

where $z = x + iy + iF(\xi) + \xi$ as in (5.3) and

$$\varepsilon_\ell = 2^{d_1 - \ell^2 \frac{a}{2}}\, \pi^{d + \ell(\ell - r)\frac{a}{2}}\, \frac{\Gamma_{\Omega^{(\ell)}}\left(\ell \frac{a}{2}\right)}{\Gamma_{\Omega^{(\ell)}}(\alpha_\ell)}. \tag{5.6}$$

The reason for including the constant ε_ℓ^{-1} in the measure will be clarified by the next lemma. Thus $M_\ell^{T(\Omega)}$ is a constant multiple of the product measure $M_\ell^{T(\Omega)} = m_H \times \mu_\ell$, where

$$dm_H(x + iF(\xi) + \xi) = dx\, d\xi \tag{5.7}$$

is the Haar measure of H.

Lemma 5.1 *Fix $w = u + iv + \eta\ \in T(\Omega)$, with $u \in X_1(e)$, $v \in \Omega$ and $\eta \in Z_{\frac{1}{2}}(e)$. Then*

$$\int\limits_{\partial_\ell(T(\Omega))} \left| K_w^{(\alpha_\ell)} \right|^2 dM_\ell^{T(\Omega)} = K^{(\alpha_\ell)}(w, w). \tag{5.8}$$

Proof: Let $z \in \partial_\ell(T(\Omega))$ have the decomposition (5.3). Then

$$\left| K_w^{(\alpha_\ell)}(z) \right|^2 = \left| N(x - u + 2\,\Im\, F(\xi, \eta) + i\,(y + \frac{1}{2}\tau(w) + F(\xi - \eta))) \right|^{-2\alpha_\ell}.$$

Hence, as in [AU97, Theorem 6.3]

$$\begin{aligned} \int\limits_X \left| K_w^{(\alpha_\ell)}(z) \right|^2 dx &= \int\limits_X \left| N\left(x + i\left(y + \frac{1}{2}\tau(w) + F(\xi - \eta) \right) \right) \right|^{-2\alpha_\ell} dx \\ &= c\, N\left(y + \frac{1}{2}\tau(w) + F(\xi - \eta) \right)^{-2\alpha_\ell + \frac{d_1}{r}}, \end{aligned}$$

where $d_1 = \dim_{\mathbb{R}} X_1(e) = r(r-1)\frac{a}{2} + r$ and

$$c = 4^{d_1 - r\alpha_\ell} \pi^{d_1}\, \frac{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})}{\Gamma_\Omega(\alpha_\ell)^2}.$$

Next, using the formula

$$N(s)^{-2\alpha_\ell + \frac{d_1}{r}} = \frac{1}{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})} \int\limits_\Omega e^{-\langle s | t \rangle} N(t)^{2\alpha_\ell - 2\frac{d_1}{r}}\, dt$$

with $s = y + \frac{1}{2}\tau(w) + F(\xi - \eta)$, we obtain

$$\begin{aligned} \int\limits_{Z_{\frac{1}{2}}(e)} \int\limits_X \left| K_w^{(\alpha_\ell)}(z) \right|^2 dx d\xi &= \frac{c}{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})} \int\limits_\Omega e^{-\langle y + \frac{1}{2}\tau(w) | t \rangle} N(t)^{2\alpha_\ell - 2\frac{d_1}{r}} dt \int\limits_{Z_{\frac{1}{2}}(e)} e^{-\langle F(\xi - \eta) | t \rangle} d\xi \\ &= \frac{(2\pi)^{rb} c}{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})} \int\limits_\Omega e^{-\langle y + \frac{1}{2}\tau(w) | t \rangle} N(t)^{2\alpha_\ell - 2\frac{d_1}{r} - b}\, dt. \end{aligned}$$

Thus,

$$\begin{aligned}
&\int\limits_{\partial_\ell(T(\Omega))} \left|K_w^{(\alpha_\ell)}(z)\right|^2 dM_\ell^{T(\Omega)}(z) = \\
&= \frac{\varepsilon_\ell^{-1}(2\pi)^{rb}c}{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})} \int\limits_\Omega e^{-\langle \frac{1}{2}\tau(w)|t\rangle} N(t)^{2\alpha_\ell - 2\frac{d_1}{r} - b}\, dt \int\limits_{\partial_\ell \Omega} e^{-\langle y|t\rangle}\, d\mu_\ell(y) \\
&= \frac{\varepsilon_\ell^{-1}(2\pi)^{rb}c\,\gamma_\ell}{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})} \int\limits_\Omega e^{-\langle \frac{1}{2}\tau(w)|t\rangle} N(t)^{\alpha_\ell - \frac{d_1}{r}}\, dt \\
&= \frac{\varepsilon_\ell^{-1}(2\pi)^{rb}c\,\gamma_\ell\,\Gamma_\Omega(\alpha_\ell)}{\Gamma_\Omega(2\alpha_\ell - \frac{d_1}{r})} N\left(\frac{1}{2}\tau(w)\right)^{-\alpha_\ell} = K^{(\alpha_\ell)}(w,w). \qquad \blacksquare
\end{aligned}$$

For $0 \le \ell \le r-1$ we consider the Hardy-type space

$$H^2(\partial_\ell(T(\Omega))) = H^2\left(\partial_\ell(T(\Omega)),\, M_\ell^{T(\Omega)}\right)$$

consisting of all holomorphic functions f on $(T(\Omega))$ for which

$$\|f\|^2_{H^2(\partial_\ell(T(\Omega)))} := \sup_{t\in\Omega} \int\limits_{\partial_\ell(T(\Omega))} |f(z+it)|^2\, dM_\ell^{T(\Omega)}(z) \tag{5.9}$$

is finite. Standard arguments show that for $f \in H^2(\partial_\ell(T(\Omega)))$ the boundary values

$$f(z) := \lim_{\Omega \ni t \to 0} f(z+it), \quad z \in \partial_\ell(T(\Omega)) \tag{5.10}$$

exist almost everywhere, and

$$\begin{aligned}
\|f\|^2_{H^2(\partial_\ell(T(\Omega)))} &= \lim_{\Omega\ni t\to 0} \int\limits_{\partial_\ell(T(\Omega))} |f(z+it)|^2\, dM_\ell^{T(\Omega)}(z) \\
&= \int\limits_{\partial_\ell(T(\Omega))} |f(z)|^2\, dM_\ell^{T(\Omega)}(z).
\end{aligned} \tag{5.11}$$

See the proof of Theorem 6.3 in [AU97].

Theorem 5.1 *For $0 \le \ell \le r-1$ we have $\mathcal{H}_{\alpha_\ell} = H^2(\partial_\ell(T(\Omega)))$, and moreover*

$$\|f\|_{\alpha_\ell} = \|f\|^2_{H^2(\partial_\ell(T(\Omega)))}, \quad \forall f \in \mathcal{H}_{\alpha_\ell}(T(\Omega)). \tag{5.12}$$

Thus, for all $f, g \in \mathcal{H}_{\alpha_\ell}$

$$< f, g >_{\alpha_\ell} = \lim_{\Omega\ni t\to 0} \int\limits_{\partial_\ell(T(\Omega))} f(z+it)\,\overline{g(z+it)}\, dM_\ell^{T(\Omega)}(z). \tag{5.13}$$

Theorem 5.1 is the generalization of Theorem 6.3 of [AU97] to symmetric Siegel domains of type II. The proof uses Lemma 5.1 (which yields (5.12) and (5.13) for functions in $\mathcal{H}_{\alpha_\ell}(T(\Omega))^{(0)} = \mathrm{span}\{K_w^{(\alpha_\ell)};\ w \in T(\Omega)\}$) as well as the standard arguments used in the proofs of the theorems in Section 6 and in the proof of Theorem 6.3 in [AU97].

Notice that, in particular, the reproducing kernel of $H^2(\partial_\ell(T(\Omega)))$ is

$$K^{(\alpha_\ell)}(z,w) = N(\tau(z,w))^{-\alpha_\ell}, \quad z \in \partial_\ell(T(\Omega)),\ w \in T(\Omega). \tag{5.14}$$

Consider the inverse Cayley transform $c^{-1} : T(\Omega) \to D$,

$$c^{-1}(w) := \frac{w_1 - ie}{w_1 + ie} + \sqrt{2}i\, R((w_1 + ie)^{-1})\, w_{\frac{1}{2}} \tag{5.15}$$

(where $w = w_1 + w_{\frac{1}{2}},\ w_1 \in Z_1(e),\ w_{\frac{1}{2}} \in Z_{\frac{1}{2}}(e)$). c^{-1} extends to $\partial(T(\Omega)) = \{w \in \overline{T(\Omega)};\ \tau(w) \in \partial\Omega\}$, and it maps holomorphic boundary components of $T(\Omega)$ to holomorphic boundary components of D, and preserves the rank of the boundary components. But not every holomorphic boundary component $B(v) = v + D_0(v)$ of D is obtained in this way, since $c(B(v)) = \infty$ if $e - v$ is not invertible in $Z_1(e)$. Thus

$$c^{-1}\left(\partial_\ell(T(\Omega))\right) = \bigcup_{\substack{v \in S_{r-\ell} \\ e-v \text{ invertible}}} B(v) \subsetneqq \partial_{r-\ell}(D). \tag{5.16}$$

On the set $c^{-1}\left(\partial_\ell(T(\Omega))\right)$ consider the measure

$$dM_\ell^D(z) := |Jc(z)|^{-\frac{2\alpha_\ell}{p}} dM_\ell^{T(\Omega)}(c(z)). \tag{5.17}$$

Then M_ℓ^D is absolutely continuous with respect to the volume measure on $c^{-1}\left(\partial_\ell(T(\Omega))\right)$. Since $\partial_{r-\ell}(D) \backslash c^{-1}\left(\partial_\ell(T(\Omega))\right)$ is a lower dimensional subset of $\partial_{r-\ell}(D)$, its volume measure is zero. This consideration enables us to consider M_ℓ^D as an absolutely continuous measure on all of $\partial_{r-\ell}(D)$ in a unique way.

The *Hardy space*

$$H^2(\partial_{r-\ell}(D)) = H^2\left(\partial_{r-\ell}(D),\ M_\ell^D\right) \tag{5.18}$$

is the space of all holomorphic functions in D for which

$$\|f\|^2_{H^2(\partial_{r-\ell}(D))} := \sup_{0<t<1} \int\limits_{\partial_{r-\ell}(D)} |f(tz)|^2\ dM_\ell^D(z) \tag{5.19}$$

is finite. By standard arguments, for each $f \in H^2(\partial_{r-\ell}(D))$ the radial limit (here $f_t(z) := f(tz)$)

$$f_1(z) = \lim_{t \to 1-} f_t(z), \quad z \in \partial_{r-\ell}(D) \tag{5.20}$$

exists in $L^2(\partial_{r-\ell}(D))$ and almost everywhere on $\partial_{r-\ell}(D)$. Moreover

$$\|f\|_{H^2(\partial_{r-\ell}(D))} \;=\; \lim_{t \to 1-} \|f_t\|_{L^2(\partial_{r-\ell}(D))} \|f_1\|_{L^2(\partial_{r-\ell}(D))}. \tag{5.21}$$

Recall that the operator

$$f \mapsto (f \circ c)(Jc)^{\alpha_\ell/p}$$

maps $\mathcal{H}_{\alpha_\ell}(T(\Omega))$ isometrically onto $\mathcal{H}_{\alpha_\ell}(D)$. Therefore Theorem 5.1 enables us to obtain the following result.

Theorem 5.2 *For $0 \leq \ell \leq r-1$ we have $\mathcal{H}_{\alpha_\ell}(D) = H^2\left(\partial_{r-\ell}(D),\, M_\ell^D\right)$ and*

$$\|f\|_{\alpha_\ell} = \|f\|_{H^2\left(\partial_{r-\ell}(D), M_\ell^D\right)} \quad \forall f \in \mathcal{H}_{\alpha_\ell}(D). \tag{5.22}$$

Thus, for all $f, g \in \mathcal{H}_{\alpha_\ell}(D)$

$$< f, g >_{\alpha_\ell} = \lim_{t \to 1-} \int\limits_{\partial_{r-\ell}(D)} f(tz)\, \overline{g(tz)}\, dM_\ell^D(z). \tag{5.23}$$

Theorems (3.1) and (5.2) combine to yield the following result.

Theorem 5.3 *Let $0 \leq \ell \leq r-1$ and, as before, let $\alpha_\ell = \ell\frac{a}{2} + \frac{d}{r}$. Then there exists an operator T on $C^\infty(D \cup \partial_{r-\ell}(D))$ which is $\mathrm{GL}(\Omega^{(\ell)})$-invariant, so that*

(i) *For every $f \in \mathcal{H}_{\ell\frac{a}{2}}(D)$ with Peter-Weyl expansion $f = \sum_{\mathbf{m}^{(\ell)}} f_{\mathbf{m}^{(\ell)}}$, one has*

$$Tf = \sum_{\mathbf{m}^{(\ell)}} \frac{(\alpha_\ell)_{\mathbf{m}^{(\ell)}}}{(\ell\frac{a}{2})_{\mathbf{m}^{(\ell)}}} f_{\mathbf{m}^{(\ell)}}. \tag{5.24}$$

(ii) *For all $f, g \in \mathcal{H}_{\ell\frac{a}{2}}(D)$,*

$$< f, g >_{\ell\frac{a}{2}} \;=\; < Tf, g >_{H^2(\partial_{r-\ell}(D))} = \lim_{t \to 1-} \int\limits_{\partial_{r-\ell}(D)} T(f\overline{g})\, dM_\ell^D. \tag{5.25}$$

The volume measure m on $\partial_{r-\ell}(D)$ is given by

$$\int\limits_{\partial_{r-\ell}(D)} f\, dm = \int\limits_{S_{r-\ell}} d\nu_{r-\ell}(v) \int\limits_{D_0(v)} f_v(z)\, dm_v(z) \tag{5.26}$$

where m_v is the Lebesgue measure on $D_0(v)$. Let us consider the Radon-Nikodym derivative

$$\omega(z) = \frac{dM_\ell^D}{dm}(z), \quad z \in \partial_{r-\ell}(D).$$

Then formula (5.25) can be written in the form

$$\begin{aligned} < f, g >_{\ell\frac{a}{2}} &= \int\limits_{S_{r-\ell}} d\nu_{r-\ell}(v) \int\limits_{D_0(v)} T_v(f_v)(z)\, \overline{g_v(z)}\, \omega(v+z)\, dm_v(z) \\ &= \int\limits_{S_{r-\ell}} d\nu_{r-\ell}(v) \int\limits_{D_0(v)} T_v(f_v\, \overline{g_v})(z)\, \omega(v+z)\, dm_v(z). \end{aligned} \tag{5.27}$$

6 Canonical Realization of Lassalle measure

Analogous to the domain D, the cone Ω has boundary orbits (under $GL(\Omega)$) given by

$$\partial_\ell \Omega = \{x \in \overline{\Omega} : \mathrm{rank}\,(x) = \ell\} \tag{6.1}$$

for $0 \leq \ell \leq r-1$. It is known [RV76], [La87] that the Riesz distribution for parameter $\ell\,\frac{a}{2}$ can be realized as a measure μ_ℓ on $\partial_\ell\Omega$ which is relatively invariant under the action of $GL(\Omega)$. We will show that μ_ℓ has a natural polar decomposition with respect to the subgroup $Aut\,(X) \subset GL(\Omega)$. Along the way we also give an explicit construction of μ_ℓ using elementary Jordan theory.

Let G be a locally compact group with a (left) Haar measure μ, and let $H \subset G$ be a closed subgroup with a left Haar measure β. By [Bou63, p.44] every (Radon) measure ν on G satisfying

$$\delta_h \nu = \Delta_H(h)\,\nu \qquad \forall\ h \in H \tag{6.2}$$

induces a "quotient" measure ν/β on G/H such that

$$\int_G d\nu \cdot f = \int_{G/H} d(\nu/\beta)(gH) \int_H d\beta(h)\ f(gh) \tag{6.3}$$

for all $f \in \mathcal{C}_c(G)$. Here δ_h denotes right translation on G and Δ_H is the modulus function on H.

If $H' \subset H$ is a closed subgroup, with a left Haar measure β', such that for all $h' \in H'$

$$\Delta_{H'}(h') = \Delta_H(h'), \tag{6.4}$$

the quotient measures ν/β' and β/β' also exist, and

$$\int_{G/H'} d(\nu/\beta')\,\varphi = \int_{G/H} d(\nu/\beta)(gH) \int_{H/H'} d(\beta/\beta')(hH') \cdot \varphi(ghH') \tag{6.5}$$

for all $\varphi \in \mathcal{C}_c(G/H')$ [Bou63, p.64]. If $\chi : G \to \mathbb{R}_+^*$ is a character such that for all $h' \in H'$

$$\chi(h') = \frac{\Delta_{H'}(h')}{\Delta_G(h')} \tag{6.6}$$

then the measure $\chi\mu$ on G satisfies

$$\delta_{h'}(\chi\mu) = \delta_{h'}\chi \cdot \delta_{h'}\mu = \frac{\Delta_{H'}(h')}{\Delta_G(h')}\ \chi\,\Delta_G(h')\,\mu = \Delta_{H'}(h')\,\chi\mu \tag{6.7}$$

and the resulting quotient measure $\chi\mu/\beta'$ on G/H' is relatively invariant with multiplier χ [Bou63, p.58].

Now assume G is a Lie group, and consider the left translation action $g \mapsto \gamma_g$ of G on G/H. Let $J(g,s)$ denote the Jacobian of γ_g evaluated at $s \in G/H$. For $h \in H$, $Ad_{\mathfrak{g}}(h)$ leaves $\mathfrak{h} \subset \mathfrak{g}$ invariant, and γ_h has the tangent map $Ad_{\mathfrak{g}/\mathfrak{h}}(h)$ at $H \in G/H$. Therefore

$$J(h,H) = Det\ Ad_{\mathfrak{g}/\mathfrak{h}}(h) = \frac{Det\ Ad_{\mathfrak{g}}(h)}{Det\ Ad_{\mathfrak{h}}(h)}\ . \tag{6.8}$$

With [Dieu74, 19.16.4.3] this implies

$$|J(h,H)| = \frac{\Delta_H(h)}{\Delta_G(h)}\ . \tag{6.9}$$

On the other hand, we have $\gamma_h\, H = H$ and therefore

$$J(gh, H) = J(g, H)\, J(h, H)\ . \tag{6.10}$$

It follows that

$$\rho(g) := |J(g, H)| \tag{6.11}$$

satisfies for all $h \in H$:

$$\rho(gh) = \rho(g)\ \frac{\Delta_H(h)}{\Delta_G(h)} = \rho(g)\, \rho(h)\ . \tag{6.12}$$

Hence

$$\delta_h(\rho\mu) = \delta_h\, \rho \cdot \delta_h\, \mu = \rho\ \frac{\Delta_H(h)}{\Delta_G(g)}\ \Delta_G(h)\, \mu = \Delta_H(h)\, \rho\mu\ , \tag{6.13}$$

so that the quotient measure $\rho\mu/\beta$ exists on G/H. For $h' \in H'$, (6.4), (6.6) and (6.9) imply

$$\chi(h') = \frac{\Delta_{H'}(h')}{\Delta_G(h')} = \frac{\Delta_H(h')}{\Delta_G(h')} = \rho(h') \tag{6.14}$$

and therefore

$$\frac{\chi(gh')}{\rho(gh')} = \frac{\chi(g)\, \chi(h')}{\rho(g)\, \rho(h')} = \frac{\chi(g)}{\rho(g)}\ . \tag{6.15}$$

It follows that $\frac{\chi}{\rho}$ is a function on G/H'. Applying (6.5) we obtain

$$\int\limits_{G/H'} d(\chi\mu/\beta')\, \varphi = \int\limits_{G/H'} d(\rho\mu/\beta')\ \frac{\chi}{\rho}\, \varphi =$$

$$= \int\limits_{G/H} d(\rho\mu/\beta)(gH) \int\limits_{H/H'} d(\beta/\beta')(hH')\ \frac{\chi(gh)}{\rho(gh)}\ \varphi(gh\, H') \tag{6.16}$$

for all $\varphi \in \mathcal{C}_c(G/H')$. The function $\frac{\chi}{\rho}$ can be determined as follows: Suppose for any $g \in G$ there exists $k \in K$ and $h \in H$ such that

$$gH' = khH'\ . \tag{6.17}$$

Here $K \subset G$ is a compact subgroup. Then

$$\frac{\chi}{\rho}\ (gH') = \frac{\chi(kh)}{\rho(kh)} = \frac{\chi(k)\, \chi(h)}{\rho(k)\, \rho(h)} = \frac{\chi(h)}{\Delta_H(h)}\ \Delta_G(h)\ . \tag{6.18}$$

We will now apply these general considerations to the reductive (hence unimodular) Lie group

$$G := GL(\Omega) \tag{6.19}$$

with maximal compact subgroup

$$K := Aut\, (X) \tag{6.20}$$

(Jordan algebra automorphisms). The group G has an involution $g \mapsto g^*$ satisfying

$$g\{x(g^*y)^*z\} = \{(gx)\, y^*(gz)\} \tag{6.21}$$

for all $x, y, z \in X$ [U87]. The corresponding Lie algebra involution $A \mapsto A^*$ of $\mathfrak{g} = \mathfrak{gl}(\Omega)$ satisfies

$$A\{xy^*z\} = \{(Ax)\, y^*z\} - \{x(A^*\, y)^*z\} + \{xy^*(Az)\}\ . \tag{6.22}$$

We have $K = \{k \in G :\ k^* = k^{-1}\}$.

Fix $0 \leq \ell < r$, and consider the boundary orbit

$$\partial_\ell\, \Omega = G \cdot u_\ell\ . \tag{6.23}$$

The closed subgroup

$$H' := \{h' \in G :\ h'\, u_\ell = u_\ell\}\ , \tag{6.24}$$

with Lie algebra

$$\mathfrak{h}' := \{A' \in \mathfrak{g} :\ A'\, u_\ell = 0\}\ , \tag{6.25}$$

induces a diffeomorphism

$$G/H' \ni gH' \mapsto g\, u_\ell \in \partial_\ell\, \Omega\ . \tag{6.26}$$

Via this identification, we have a commuting diagram

$$\begin{array}{ccc} \partial_\ell\, \Omega & \stackrel{g}{\longleftarrow} & \partial_\ell\, \Omega \\ \uparrow & & \uparrow \\ G/H' & \underset{\gamma_g}{\longleftarrow} & G/H'\ , \end{array} \tag{6.27}$$

where γ_g denotes left translation on G/H'.

Let P_ℓ denote the compact space of all rank ℓ projections in X. Consider the fibration

$$\partial_\ell\, \Omega = \dot{\bigcup_{p \in P_\ell}}\ \Omega_1(p) \qquad \text{(disjoint union)} \tag{6.28}$$

into boundary components $\Omega_1(p)$ (the strictly positive cone in $X_1(p)$) [Lo77]. Since G permutes the fibers of (6.28), there exists an action $g \mapsto \widetilde{g}$ of G on P_ℓ satisfying

$$g(\Omega_1(p)) = \Omega_1(\widetilde{g}(p)) \tag{6.29}$$

for all $g \in G$ and $p \in P_\ell$. Equivalently, the diagram

$$\begin{array}{ccc} \partial_\ell\, \Omega & \stackrel{g}{\longleftarrow} & \partial_\ell\, \Omega \\ \pi \downarrow & & \downarrow \pi \\ P_\ell & \underset{\widetilde{g}}{\longleftarrow} & P_\ell \end{array} \tag{6.30}$$

commutes, where π is the canonical projection. The action of G on P_ℓ is transitive, so that there exists a diffeomorphism

$$G/H \ni gH \mapsto \widetilde{g}(u_\ell) \in P_\ell\ , \tag{6.31}$$

where

$$H \ := \ \{h \in G :\ \widetilde{h}(u_\ell) = u_\ell\} \tag{6.32}$$

$$\ = \ \{h \in G :\ h(u_\ell) \in \Omega_1(u_\ell)\} \tag{6.33}$$

is a closed subgroup containing H', with Lie algebra

$$\mathfrak{h} := \{A \in \mathfrak{g} : \; A\, u_\ell \in X_1(u_\ell)\} \; . \tag{6.34}$$

Via this identification, we have a commuting diagram

$$\begin{array}{ccc} P_\ell & \overset{\widetilde{g}}{\longleftarrow} & P_\ell \\ \uparrow & & \uparrow \\ G/H & \underset{\gamma_g}{\longleftarrow} & G/H \end{array} , \tag{6.35}$$

where γ_g denotes left translation on G/H.

For $h \in H$, we have $h\, u_\ell \in \Omega_1(u_\ell)$ and $h^*(e - u_\ell)) \in \Omega_0(u_\ell)$. Thus we obtain characters

$$h \mapsto N(h\, u_\ell + e - u_\ell) \tag{6.36}$$

and

$$h \mapsto N(u_\ell + h^*(e - u_\ell)) \tag{6.37}$$

of H. The associated infinitesimal characters on $\mathfrak{h}$ are given by

$$A \mapsto \tau(A\, u_\ell) \tag{6.38}$$

and

$$A \mapsto \tau(A^*(e - u_\ell)) \; , \tag{6.39}$$

respectively. Here $\tau : X \to \mathbb{R}$ is the Jordan algebra trace, normalized by $\tau\, e_1 = 1$.

Proposition 6.1 *For $A \in \mathfrak{h}$ we have*

$$tr\ ad_{\mathfrak{g}/\mathfrak{h}}(A) = \frac{a}{2}\, [\ell\, \tau(A^*(e - u_\ell)) - (r - \ell)\, \tau(A\, u_\ell)] \tag{6.40}$$

and

$$\tau(A\, u_\ell) + \tau(A^*(e - u_\ell)) = \tau(Ae) \; . \tag{6.41}$$

Proof: We will use the (restricted) root decomposition

$$\mathfrak{g} = \sum_\alpha^{\oplus} \mathfrak{g}_\alpha \tag{6.42}$$

of a reductive Lie algebra $\mathfrak{g}$ with respect to a Cartan subspace $\mathfrak{a} \subset \mathfrak{g}$. For $\alpha \in \mathfrak{a}^\sharp$ (= linear dual space of $\mathfrak{a}$), we put

$$\mathfrak{a}_\alpha := \{B \in \mathfrak{g} : \; [A, B] = \alpha(A)\, B \quad \forall\ A \in \mathfrak{a}\} \; . \tag{6.43}$$

Then $\mathfrak{g}_0 \supset \mathfrak{a}$ and

$$[\mathfrak{g}_\alpha,\, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{6.44}$$

for all $\alpha, \beta \in \mathfrak{a}^\sharp$. In order to describe the root decomposition of $\mathfrak{g} = \mathfrak{gl}(\Omega)$, fix a frame $\{e_1, \ldots, e_r\}$ of projections in X and define the Cartan subspace

$$\mathfrak{a} := \left\{ \sum_{k=1}^{r} \lambda_k\, e_k \,\Box\, e_k^* : \; \lambda_1, \ldots, \lambda_r \in \mathbb{R} \right\} \; . \tag{6.45}$$

Using the associated Peirce spaces

$$X_{ij} = X_{ji} := \left\{ x \in X : \ \{e_k \, e_k^* \, x\} = \frac{\delta_i^k + \delta_j^k}{2} \, x \quad \forall \ k \right\} \tag{6.46}$$

for $1 \le i, j \le r$, define

$$\mathfrak{g}_{ij} := \{ e_i \ \Box \ x^* = x \ \Box \ e_j^* : \ x \in X_{ij} \} \ . \tag{6.47}$$

Here we use the "triple operator"

$$(x \ \Box \ y^*) \, z := \{ x y^* z \} \tag{6.48}$$

for all $x, y, z \in X$. The Jordan triple identity [U87] implies, for $i \neq j$ and $x \in X_{ij}$

$$\begin{aligned} [e_k \ \Box \ e_k^*, \, e_i \ \Box \ x^*] &= \{e_k \, e_k^* \, e_i\} \ \Box \ x^* - e_i \ \Box \ \{x \, e_k^* \, e_k\}^* = \\ &= \delta_i^k \, e_i \ \Box \ x^* - \frac{\delta_i^k + \delta_j^k}{2} \ e_i \ \Box \ x^* = \frac{\delta_i^k - \delta_j^k}{2} \ e_i \ \Box \ x^* \ . \end{aligned} \tag{6.49}$$

Therefore $\mathfrak{g}_{ij}$ is the root space for

$$\alpha \left(\sum_{k=1}^r \ \lambda_k \, e_k \ \Box \ e_k^* \right) := \sum_{k=1}^r \ \lambda_k \ \frac{\delta_i^k - \delta_j^k}{2} = \frac{\lambda_i - \lambda_j}{2} \ . \tag{6.50}$$

One can show [UU94] that

$$\mathfrak{g} = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^- \oplus \sum_{i \neq j} \ \mathfrak{g}_{ij} \ , \tag{6.51}$$

where $\mathfrak{g}_0^- = \mathfrak{a}$ and

$$\mathfrak{g}_0^1 = \{ A \in \mathfrak{g} : \ A \, e_k = 0 \ \ \forall \ k \} \tag{6.52}$$

belong to $\alpha = 0$. Now consider the sub-algebra

$$\mathfrak{h} = \left\{ A \in \mathfrak{g} : \ A \, u_\ell \in X_1(u_\ell) = \sum_{1 \le i \le j \le \ell}^{\oplus} \ X_{ij} \right\} \ . \tag{6.53}$$

For $A \in \mathfrak{g}_0^1$, we have $A \, u_\ell = A \, e_1 + \ldots + A \, e_\ell = 0$. Therefore $\mathfrak{g}_0^1 \subset \mathfrak{h}$. Since

$$(e_k \ \Box e_k^*) \, u_\ell = \begin{cases} e_k & k \le \ell \\ 0 & k > \ell \end{cases} \tag{6.54}$$

we also have $\mathfrak{a} \subset \mathfrak{h}$. Now let $A = x \ \Box \ e_j^*$ for some $x \in X_{ij}$ with $i \neq j$. If $j > \ell$, then $A \, u_\ell = \{x \, e_j^* \, e_1\} + \ldots + \{x \, e_j^* \, e_\ell\} = 0$. If $j \le \ell$, then

$$A \, u_\ell = \{ x \, e_j^* \, e_j \} = \frac{x}{2} \tag{6.55}$$

belongs to $X_1(u_\ell)$ if and only if $i \le \ell$. Hence

$$\mathfrak{h} = \mathfrak{g}_0 \ \oplus \sum_{\substack{1 \le i,j \le \ell \\ i \neq j}} \ \mathfrak{g}_{ij} \ \oplus \ \sum_{\substack{j > \ell \\ i \neq j}} \ \mathfrak{g}_{ij} \tag{6.56}$$

and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{j \le \ell < i}^{\oplus} \mathfrak{g}_{ij} \; . \tag{6.57}$$

It suffices to prove (6.40) and (6.41) for $A \in \mathfrak{h}$ belonging to the various root subspaces. Suppose first that

$$A = e_i \,\Box\, x^* = x \,\Box e_j^* \tag{6.58}$$

for some $x \in X_{ij}$ with $i \neq j$ and $i \le \ell$ or $j > \ell$. Since A belongs to a non-zero root (6.50), we have $tr\ ad_{\mathfrak{g}/\mathfrak{h}}(A) = 0$ by (6.44). On the other hand, $A\,u_\ell = \{x\,e_j^*\,u_\ell\} \in X_{ji}$ and $A^*(e-u_\ell)) = \{x\,e_i^*(e-u_\ell)\} \in X_{ij}$ are both traceless since $i \neq j$. Similarly, $A\,e = \{x\,e_j^*\,e\} = \frac{x}{2} \in X_{ij}$ has trace 0. Therefore (6.40) and (6.41) hold in this case. Next, assume $A \in \mathfrak{g}_0^1 \subset aut\,(X) = \mathfrak{k}$. Then $A^* = -A$ and

$$A\,u_\ell = A^*(e-u_\ell) = A\,e = 0 \tag{6.59}$$

whereas $tr\ ad_{\mathfrak{g}/\mathfrak{h}}\,(A) = 0$ since K is compact. This proves (6.40) and (6.41).

Now let

$$A = \sum_{k=1}^{r} \lambda_k\ e_k \,\Box\, e_k^* \in \mathfrak{a} \; . \tag{6.60}$$

Since dim $\mathfrak{g}_{ij} =$ dim $X_{ij} = a$ for $i \neq j$, we have

$$tr\ ad_{\mathfrak{g}_{ij}}(A) = a\ \frac{\lambda_i - \lambda_j}{2} \tag{6.61}$$

and hence, by (6.57)

$$tr\ ad_{\mathfrak{g}/\mathfrak{h}}(A) = \frac{a}{2} \sum_{j \le \ell < i} (\lambda_i - \lambda_j) = \frac{a}{2}[(\lambda_{\ell+1} + \ldots + \lambda_r)\,\ell - (\lambda_1 + \ldots + \lambda_\ell)(r-\ell)] \; . \tag{6.62}$$

On the other hand, $A^* = A$ and therefore $\tau(A\,u_\ell) = \sum_{k=1}^{\ell} \lambda_k$ and $\tau(A^*(e-u_\ell)) = \sum_{k=\ell+1}^{r} \lambda_k = \tau(A(e-u_\ell))$. This proves (6.40) and (6.41) in the remaining case. ∎

Proposition 6.2 *For $h \in H$, we have*

$$\Delta_H(h) = Det\ Ad_{\mathfrak{g}/\mathfrak{h}}(h) = \frac{N(u_\ell + h^*(e-u_\ell))^{\ell\,a/2}}{N(h\,u_\ell + e - u_\ell)^{(r-\ell)a/2}} \tag{6.63}$$

and

$$N(h\,u_\ell + e - u_\ell)\ N(u_\ell + h^*(e-u_\ell)) = N(h\,e) \; . \tag{6.64}$$

Proof: Since

$$\mathfrak{g}_0^- \oplus \sum_{1 \le i < j \le r} \mathfrak{g}_{ij} \subset \mathfrak{h} \tag{6.65}$$

it follows that $\{Ae : \ A \in \mathfrak{h}\} = X$. Therefore the identity component H° of H acts transitively on Ω [ ]. Thus for each $h \in H$ there exists $h_1 \in H^\circ$ satisfying $he = h_1\,e$. Hence

$$k := h_1^{-1}\,h \in H \cap Aut\,(X) \; . \tag{6.66}$$

Moreover, $h_1 = exp\,(A_1)\cdots exp\,(A_n)$ for suitable $A_1,\ldots,A_n \in \mathfrak{h}$, implying

$$h = exp\,(A_1)\cdots exp\,(A_n)\,k\ . \tag{6.67}$$

Since both sides of (6.63) and (6.64) define characters of H, it suffices to consider the factors of (6.67) separately.

For $h = exp\,A$, $A \in \mathfrak{h}$, the identities (6.63) and (6.64) follow from (6.40) and (6.41), respectively, by differentiation. For $k \in H \cap Aut\,(X)$, $k\,u_\ell \in X_1(u_\ell)$ is a rank ℓ projection, hence $k\,u_\ell = u_\ell$. Also, $k^* = k^{-1}$ and hence

$$k^*(e - u_\ell) = k^{-1}(e - u_\ell) = k^{-1}\,e - k^{-1}\,u_\ell = e - u_\ell\ . \tag{6.68}$$

It follows that

$$N(u_\ell + k^*(e - u_\ell)) = N(k\,u_\ell + e - u_\ell) = N(k\,e) = 1\ . \tag{6.69}$$

Since $H \cap Aut\,(X)$ is compact, we also have

$$Det\ Ad_{\mathfrak{g}/\mathfrak{h}}(k) = 1\ . \tag{6.70}$$

Thus (6.63) and (6.64) hold for k as well. ∎

Since H acts transitively on $\Omega_1(u_\ell)$, with stabilizer subgroup H', there is a diffeomorphism

$$H/H' \ni h\,H' \mapsto h\,u_\ell \in \Omega_1(u_\ell)\ . \tag{6.71}$$

Now $\Omega_1(u_\ell)$ has the invariant measure

$$N(x + e - u_\ell)^{-n'/\ell}dx \tag{6.72}$$

where $n' = \dim\ X_1(u_\ell)$. It follows that (6.4) is satisfied and β/β' corresponds to (6.72) under the identification (6.71).

Corollary 6.1 *There exists a measure μ_ℓ on $\partial_\ell\,\Omega \approx G/H'$ which is relatively invariant under G with character*

$$\chi(g) := N(ge)^{\ell\,a/2}\ . \tag{6.73}$$

Proof: For $h' \in H'$ we have $h'\,u_\ell = u_\ell$ and hence

$$\Delta_H(h') = N(u_\ell + (h')^*(e - u_\ell))^{\ell\,a/2} = N(h'e)^{\ell\,a/2} \tag{6.74}$$

by (6.63) and (6.64). Since G is unimodular, (6.4) implies that (6.6) is satisfied, and $\chi\mu/\beta'$ is a relatively invariant measure (for χ) on G/H' which, under the identification (6.26), gives the Lassalle measure μ_ℓ on $\partial_\ell\,\Omega$. ∎

Theorem 6.1 *The Lassalle measure μ_ℓ on $\partial_\ell\,\Omega$ has the polar decomposition*

$$\int_{\partial_\ell\,\Omega} d\mu_\ell \cdot \varphi = \int_{P_\ell} dp \int_{\Omega_1(p)} N(y + e - p)^{\frac{ra}{2} - \frac{n'}{\ell}}\varphi(y)\ . \tag{6.75}$$

Here dp is the K-invariant probability measure on P_ℓ.

Proof: The measure $\rho\mu/\beta$ on G/H is invariant under the left translation action of K since

$$\rho(kg) = |J(kg,H)| = |J(k,gH)||J(g,H)| = |J(g,H)| = \rho(g) \ . \tag{6.76}$$

It follows that $\rho\mu/\beta$ is (proportional to) the normalized K-invariant measure dp on $P_\ell \approx G/H$.

Now let $g \in G$. Then

$$p := \widetilde{g}(u_\ell) = k\, u_\ell \tag{6.77}$$

for some $k \in Aut\,(X)$. Since $k = \widetilde{k}$, $\ h := k^{-1}\, g$ satisfies

$$\widetilde{h}(u_\ell) = \widetilde{k}^{-1}(\widetilde{g}(u_\ell)) = u_\ell \ . \tag{6.78}$$

Therefore $h \in H$. Put

$$y := g\, u_\ell = k\, h\, u_\ell \in \partial_\ell\, \Omega \approx G/H' \ . \tag{6.79}$$

Computing $\frac{\chi}{\rho}$ as in (6.18), (6.63) and (6.64) imply

$$\frac{\chi}{\rho}(y) = \frac{\chi(h)}{\Delta_H(h)} = \frac{N(he)^{\ell\, a/2}\, N(h\, u_\ell + e - u_\ell)^{(r-\ell)\, a/2}}{N(u_\ell + h^*(e - u_\ell))^{\ell\, a/2}}$$

$$= N(h\, u_\ell + e - u_\ell)^{r\, a/2} = N(y + e - p)^{r\, a/2} \tag{6.80}$$

since $y = k\, h\, u_\ell$ and $p = k\, u_\ell$. Now the assertion follows with (6.16), since the measure $N(y+e-p)^{-n'/\ell}dy$ on $\Omega_1(p)$ is the image under k of the measure $N(x+e-u_\ell)^{-n'/\ell}dx$ on $\Omega_1(u_\ell)$, which is identified with β/β'. ∎

References

[A92-1] J. Arazy, *Realization of the invariant inner products on the highest quotients of the composition series*, Arkiv Mat. 30 (1992), 1-24.

[A92-2] J. Arazy, *Integral formulas for the invariant inner products in spaces of analytic functions in the unit ball*, in Function Spaces, Lecture Notes in Pure and Applied Mathematics Vol.136, Marcel Dekker (1992), 9-23.

[A95] J. Arazy, *A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains*, Contemp. Math. 185 (1995), 7-65.

[A96] J. Arazy, *Boundedness and compactness of generalized Hankel operators on bounded symmetric domains* J. Funct. Anal. 137 (1996), 97-151

[AU97] J. Arazy and H. Upmeier, *Invariant Inner Products in Spaces of Holomorphic Functions on Bounded Symmetric Domains*, Documenta Mathematica Journal der DMV, 2 (1997), 213-261.

[AU98] J. Arazy and H. Upmeier, *Discrete Series Representations and Integration over Boundary Orbits of Symmetric Domains*, Perspectives on Quantization (Eds. L. A. Coburn and M. A. Rieffel) Contemporary Mathematics Vol. 214 (1998), 1-22

[Be75] F. Berezin, *Quantization in complex symmetric spaces*, Math. USSR-Izv. 9 (1975), 341-379.

[Bou63] N. Bourbaki, *Éléments de Mathématique, Livre VI, Intégration*, Herman (1963).

[Dieu74] J. Dieudonné, *Treatise on Analysis*, vol. IV, Academic Press, 1974.

[FK90] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. 88 (1990), 64-89.

[FK94] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford (1994).

[Gi64] S. Gindikin, *Analysis on homogeneous domains*, Russ. Math. Surv. 19 (1964), 1-89.

[Gi75] S. Gindikin, *Invariant generalized functions in homogeneous domains*, Functional Analysis and its Applications, 9 (1975), 50-52.

[Ga47] L. Gårding, *The solution of Cauchy's problem for two totally hyperbolic linear differential equations by means of Riesz integrals*, Ann. Math. 48 (1947), 785-826.

[He84] S. Helgason, *Groups and Geometric Analysis*, Academic Press (1984).

[He78] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric spaces*, Academic Press (1978).

[Hu63] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Translations of Mathematical Monographs, Amer. Math. Soc. (1963).

[La86] M. Lassalle, *Noyau de Szegö, K-types et algèbres de Jordan*, C. R. Acad. Sci. Paris 303 (1986), 1-4.

[La87] M. Lassalle, *Algèbres de Jordan et ensemble de Wallach*, Invent. Math. 89 (1987), 375-393.

[Lo75] O. Loos, *Jordan Pairs*, Lecture Notes in Mathematics No. 460, Springer-Verlag (1975).

[Lo77] O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, University of California at Irvine (1977).

[M87] I. G. Macdonald, *Commuting differential operators and zonal spherical functions*, In: Algebraic Groups, Utrecht 1986, Lecture Notes in Math., Vol. 1271 (1987), pp.189-200.

[M95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford (1995).

[N89] T. Nomura, *Algebraically independent generators of invariant differential operators on a symmetric cone*, J. reine angew. Math., 400 (1989), 122-133.

[O80] B. Orsted, *Composition series for analytic continuations of holomorphic discrete series representations of $SU(n,n)$*, Trans. Amer. Math. Soc. 260 (1980), 563-573.

[Ri49] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. 81 (1949), 1-223.

[RV76] H. Rossi and M. Vergne, *Analytic continuation of holomorphic discrete series of a semi-simple Lie group*, Acta Mathematica 136 (1975), 1-59.

[Sch69] W. Schmid, *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Invent. Math. 8 (1969), 61-80.

[St89] R. P. Stanley, *Some combinatorial properties of the Jack symmetric functions*, Adv. in Math. 77 (1989), 76-115.

[UU94] A. Unterberger and H. Upmeier, *Berezin Transform and Invariant Differential Operators*, Comm. Math. Phys., 164 (1994), 563-597.

[U83] H. Upmeier, *Toeplitz operators on bounded symmetric domains*, Trans. J. Amer. Math. Soc., 280 (1983), 221-237.

[U86] H. Upmeier, *Jordan algebras and harmonic analysis on symmetric spaces*, Amer. J. Math. 108 (1986), 1-25.

[U87] H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics*, CBMS Series in Math. 67, Amer. Math. Soc., (1987).

[W79] N. Wallach, *The analytic continuation of the discrete series, I, II*, Trans. Amer. Math. Soc. 251 (1979), 1-17 and 19-37.

[Y93] Z. Yan, *Differential operators and function spaces*, Contemp. Math. 142 (1993), 121-142.

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