Optimal Liquidity Provision in Limit Order Markets

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Introduction

Basic Problem

- Various motives for trades on financial markets:
  - Rebalancing of mutual funds.
  - Hedging of derivative positions.
  - Liquidation due to margin calls.
  - ...
- Endogenous motive to trade $\rightsquigarrow$ pay trading costs for consuming liquidity.
- Resulting optimization problems widely studied in Mathematical Finance and Financial Economics.
- But who are the counterparties for these trades? Who provides liquidity and how?

Introduction
Specialist Markets

Who provides liquidity? Classical setting:

- Monopolistic (or oligopolistic) *specialists*.
- Obliged to match incoming order flow. Compensated by earning the spread between their bid and ask prices.
- Optimization problem: set spread to maximize profits from matching all incoming orders.
- Tradeoff: earning spread vs. inventory risk due to price moves
  - Garman (1976). Amihud & Mendelson (1980). Ho & Stoll (1981). Avallaneda & Stoikov (2008). Gueant, Lehalle, & Fernandez-Tapia (2013).
- Separate literature on adverse selection/information risk (e.g., Glosten & Milgrom (1985)).

Introduction

Limit Order Markets

Who provides liquidity? As stock markets have become automated:

- Monopolistic market makers replaced by electronic limit order books on many trading venues.
- Anybody can post buy and sell orders. Purchases and sales are matched automatically.
- Liquidity provision as an algorithmic trading strategy for hedge funds.
- For small liquidity providers: order book given exogenously. Cannot choose the spread.
- But: not obliged to match all orders. Can choose how much liquidity to provide.
- This is the setting we study.

Introduction

Results in a Nutshell

- Optimal policy characterized by upper and lower boundaries for the investor's position:
  - If a sell order of another market participant allows to buy cheaply, trade to upper boundary.
  - Likewise, jump to lower boundary when the opportunity for a profitable sale arises.
- Between these profitable trades, manage inventory risk by keeping position between boundaries with market orders.
- Kühn and Stroh (2010):
  - Log investor, only holds long positions.
  - Market with constant spread, order flow, and prices following geometric Brownian motion.
  - Boundaries determined by free boundary problem.

Introduction

Results in a Nutshell ct'd

Here: *general* model. *Explicit* asymptotic formulas.

- For tractability:
  - Limiting regime of small spreads and frequent orders by other market participants.
  - Mid price follows a martingale.
- Results:
  - Simple robust formulas for leading-order optimal trading boundaries and their performance.
  - Valid for general dynamics of mid price, spread, and order flow.
  - Preferences of the liquidity provider can be arbitrary, too.
  - Extension that incorporates price impact due to, e.g., adverse selection.

Model

Limit Order Markets

Two types of orders:

- *Market Orders*:
  - Executed immediately.
  - But purchases cost higher exogenous ask price $(1+\varepsilon_t)S_t$. Sales only earn lower bid price $(1-\varepsilon_t)S_t$.
- *Limit Orders*:
  - Execution price can be specified freely.
  - But only executed once a matching order of another trader arrives.
- Dealing with arbitrary limit orders is very hard.
- But: for *small* liquidity providers, only orders close to the current best bid-ask prices make sense.
  - Moving into the book delays execution.
  - Narrowing the spread reduces profits.

Model

Limit Order Markets ct'd

Our model (cf. Kühn & Stroh (2010), Guilbaud & Pham (2013)):

- Can always trade with market orders at the "bad' side of the bid-ask spread $[(1-\varepsilon_t)S_t, (1+\varepsilon_t)S_t)]$.
- When buy or sell orders of other traders arrive at the jump times of counting processes N^1, N^2, limit orders in the book are executed at the "good" side of the spread.
- Liquidity provider is small. Orders of any size are executed.
- Limit orders can be placed, updated, or deleted for free.
- Reduces primitives of the model to:
  - Mid price S_t.
  - Spread ε_t.
  - Arrival rates α_t^1, α_t^2 of incoming buy and sell orders.

Model

Limit Order Markets ct'd

- Mid price S_t is a martingale: $dS_t/S_t = \sigma_t dW_t$
  - Disentangles liquidity provision and directional investment.
  - Leads to long and short positions even in the limit for small spreads.
- Small spreads and frequent incoming orders:
  - Spread $\varepsilon_t = \varepsilon \mathcal{E}_t$ for Itô process $\mathcal{E}_t$ and small parameter ε.
  - Arrival rates $\alpha_t^i = \varepsilon^{-\vartheta} \lambda_t^i$ for Itô processes λ_t^i and $\vartheta \in (0,1)$.
  - $\vartheta \in (0,1)$ ensures "continuity" for $\varepsilon \to 0$. Continuous trading and no market-making profits in the frictionless limit.
- Regularity assumptions on σ_t, $\mathcal{E}_t$, λ_t^i:
  - Continuous semimartingales.
  - Bounded and bounded away from zero.
  - Drift and diffusion parts absolutely continuous with bounded rate.

Model
Preferences

- Arbitrary utility function $U : \mathbb{R} \to \mathbb{R}$:
  - Strictly increasing, strictly concave, C^2.
  - Absolute risk aversion $\mathrm{ARA} = -U''/U$ bounded and bounded away from zero.
  - Marginal utility U' bounded by an exponential.
- Investor starts with x_0 in cash, maximizes expected utility from terminal liquidation wealth:

$$E[U(X_T)] \to \max!$$

- Admissibility of a *family* $(X^\varepsilon)_{\varepsilon>0}$ of wealth processes:
  - Bounded risky position, in line with "risk budgets" in practice.
  - Converges to zero uniformly for $\varepsilon \to 0$. In line with small inventories of high-frequency traders.

Main Results

Optimal Policy

- Define position limits

$$\overline{\beta}_t = \frac{2\varepsilon_t \alpha_t^2}{\mathrm{ARA}(x_0)\sigma_t^2}, \quad \underline{\beta}_t = -\frac{2\varepsilon_t \alpha_t^1}{\mathrm{ARA}(x_0)\sigma_t^2}$$

- Keep risky position between $\underline{\beta}$, $\overline{\beta}$ by market orders, trade to boundaries when limit orders are executed:

$$d\beta_{t+}^{\varepsilon} = \beta_t^{\varepsilon}\sigma_t dW_t + (\overline{\beta}_t - \beta_t^{\varepsilon})dN_t^1 + (\underline{\beta} - \beta_t^{\varepsilon})dN_t^2 + d\Psi_t, \quad \beta_0^{\varepsilon} = 0$$

Ψ is minimal finite variation process that ensures $\beta^{\varepsilon} \in [\underline{\beta}, \overline{\beta}]$.

- This strategy *optimal at the leading order* $\varepsilon^{2(1-\vartheta)}$ for small ε.

Main Results

Optimal Policy ct'd

- The position limits

$$\overline{\beta}_t = \frac{2\varepsilon_t \alpha_t^2}{\mathrm{ARA}(x_0)\sigma_t^2}, \quad \underline{\beta}_t = -\frac{2\varepsilon_t \alpha_t^1}{\mathrm{ARA}(x_0)\sigma_t^2}$$

  are:
  - Myopic. Only local dynamics matter. Like for liquidity takers facing proportional transaction costs.
  - Inversely proportional to risk aversion and variance.
  - Proportional to spread earned per trade, and trading rates.
  - Like classical Merton proportion $\mu/\mathrm{ARA}\sigma^2$. Drift rate μ replaced by rates at which revenues accumulate by limit orders.

Main Results

Welfare

- Performance of above strategy can also be quantified. Certainty equivalent:

$$x_0 + \frac{\mathrm{ARA}(x_0)}{2} E\left[\int_0^T (\overline{\beta}_t^2 1_{A_t^1} + \underline{\beta}_t^2 1_{A_t^2})\sigma_t^2 dt\right]$$

  $\omega \in A_t^1$ if the investor's last trade before time t was a purchase and $\omega \in A_t^2$ if it was a sale.
- Certainty equivalent of order $O(\varepsilon^{2(1-\vartheta)})$. Dominates all families of competitors up to terms of order $o(\varepsilon^{2(1-\vartheta)})$.
- Average of future squared target positions. Scaled by risk aversion.

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Main Results

Welfare ct'd

For a symmetric order flow $\alpha_t^1 = \alpha_t^2$:

- Certainty equivalent:

$$x_0 + \frac{\mathrm{ARA}(x_0)}{2} E\left[\int_0^T \left(\frac{\overline{\beta}_t}{S_t}\right)^2 d\langle S\rangle_t\right]$$

- Squared trading boundaries in numbers of shares.
- Averaged with respect to business time $d\langle S\rangle_t$.
- Physical probability coincides with frictionless dual pricing measure here.
- Like for liquidity takers with proportional transaction costs.

Main Results

Welfare ct'd

If all model parameters $(\sigma, \varepsilon, \alpha^1, \alpha^2)$ are constant:

- Explicit formula for certainty equivalent:

$$x_0 + \frac{(2\varepsilon\alpha^1)(2\varepsilon\alpha^2)}{2\mathrm{ARA}(x_0)\sigma^2} T.$$

- Liquidity provision equivalent to an annuity:
  - Inversely proportional to risk aversion and variance.
  - Proportional to the rate at which revenues are earned from the spread.
- For a symmetric order flow $\alpha_t^1 = \alpha_t^2 = \alpha$:
  - Like classical squared Sharpe ratio $\mu^2/2\mathrm{ARA}\sigma^2$.
  - Drift μ again replaced by $2\varepsilon\alpha$.

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Adverse Selection and Price Impact

Motivation

So far:

- Incoming orders do not affect bid-ask prices.
- Justified if these are small and uninformed. Small noise traders.

But:

- Larger trades eat into order book:
  - Purchases increase prices.
  - Sales decrease them.
- Adverse selection of counterparties with superior information:
  - Prices increase after insider purchases.
  - Decrease after they sell.

In both cases:

- Price impact systematically works against liquidity provider.

Adverse Selection and Price Impact

Extension of the Model

- Prices rise after exogenous purchases, drop after sales.
- Captured by simple reduced form model:

$$dS_t/S_{t-} = \sigma_t \, dW_t - \kappa\varepsilon_t \, dN_t^1 + \kappa\varepsilon_t \, dN_t^2$$

- Limit orders executed at S_{t-}. Adverse price move immediately *after* execution.
- $\kappa \in [0,1]$ measures (relative) price impact.
- $\kappa = 0$: baseline model without price impact.
- $\kappa \approx 1$: model à là Madhavan et al. (1997). Market makers do not earn the spread but only small exogenous compensation.

Adverse Selection and Price Impact

Results

- Model remains tractable.
- Target positions of a similar form:

$$\overline{\beta}_t = \frac{2\varepsilon_t((1-\frac{\kappa}{2})\alpha_t^2 - \frac{\kappa}{2}\alpha_t^1)}{\mathrm{ARA}(x_0)\sigma_t^2}, \quad \underline{\beta}_t = -\frac{2\varepsilon_t((1-\frac{\kappa}{2})\alpha_t^1 - \frac{\kappa}{2}\alpha_t^2)}{\mathrm{ARA}(x_0)\sigma_t^2},$$

  Liquidity provision reduced by adverse price impact.
  Inventory management changed as well.
- For a symmetric order flow ($\alpha_t^1 = \alpha_t^2 = \alpha$):

$$\overline{\beta}_t = \frac{2\varepsilon_t(1-\kappa)\alpha_t}{\mathrm{ARA}(x_0)\sigma_t^2}, \quad \underline{\beta}_t = -\frac{2\varepsilon_t(1-\kappa)\alpha_t}{\mathrm{ARA}(x_0)\sigma_t^2}$$

  Liquidity provision simply reduced by factor $1-\kappa$.
- Formula for certainty equivalent remains valid.

Summary

- Small liquidity provider trading in a limit order market.
- General dynamics for mid-price, spread, and order flow. Arbitrary preferences.
- Explicit formulas for almost optimal trading boundaries, associated welfare.
- Extension of the model to account for adverse selection/price impact.

For more information (and proofs):

- Kühn, C. and Muhle-Karbe, J. Optimal liquidity provision in limit order markets. (Hopefully) available soon.