Optimal Liquidity Provision
in Limit Order Markets

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Introduction

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Summary
Introduction
Basic Problem

- Various motives for trades on financial markets:
  - Rebalancing of mutual funds.
  - Hedging of derivative positions.
  - Liquidation due to margin calls.
  - ...

- Endogenous motive to trade \( \leadsto \) pay trading costs for consuming liquidity.

- Resulting optimization problems widely studied in Mathematical Finance and Financial Economics.

- But who are the counterparties for these trades? Who provides liquidity and how?
Introduction
Specialist Markets

Who provides liquidity? Classical setting:

- Monopolistic (or oligopolistic) specialists.
- Obliged to match incoming order flow. Compensated by earning the spread between their bid and ask prices.
- Optimization problem: set spread to maximize profits from matching all incoming orders.
- Tradeoff: earning spread vs. inventory risk due to price moves
- Separate literature on adverse selection/information risk (e.g., Glosten & Milgrom (1985)).
Introduction
Limit Order Markets

Who provides liquidity? As stock markets have become automated:

- Monopolistic market makers replaced by electronic limit order books on many trading venues.
- Anybody can post buy and sell orders. Purchases and sales are matched automatically.
- Liquidity provision as an algorithmic trading strategy for hedge funds.
- For small liquidity providers: order book given exogenously. Cannot choose the spread.
- But: not obliged to match all orders. Can choose how much liquidity to provide.

This is the setting we study.
Introduction
Results in a Nutshell

- Optimal policy characterized by upper and lower boundaries for the investor’s position:
  - If a sell order of another market participant allows to buy cheaply, trade to upper boundary.
  - Likewise, jump to lower boundary when the opportunity for a profitable sale arises.

- Between these profitable trades, manage inventory risk by keeping position between boundaries with market orders.

- Kühn and Stroh (2010):
  - Log investor, only holds long positions.
  - Market with constant spread, order flow, and prices following geometric Brownian motion.
  - Boundaries determined by free boundary problem.
Introduction

Results in a Nutshell ct’d

Here: *general* model. *Explicit* asymptotic formulas.

- For tractability:
  - Limiting regime of small spreads and frequent orders by other market participants.
  - Mid price follows a martingale.

- Results:
  - Simple robust formulas for leading-order optimal trading boundaries and their performance.
  - Valid for general dynamics of mid price, spread, and order flow.
  - Preferences of the liquidity provider can be arbitrary, too.
  - Extension that incorporates price impact due to, e.g., adverse selection.
Model
Limit Order Markets

Two types of orders:

- **Market Orders:**
  - Executed immediately.
  - But purchases cost higher exogenous ask price \((1 + \varepsilon_t)S_t\).
  - Sales only earn lower bid price \((1 - \varepsilon_t)S_t\).

- **Limit Orders:**
  - Execution price can be specified freely.
  - But only executed once a matching order of another trader arrives.

- Dealing with arbitrary limit orders is very hard.
- But: for *small* liquidity providers, only orders close to the current best bid-ask prices make sense.
  - Moving into the book delays execution.
  - Narrowing the spread reduces profits.
Our model (cf. Kühn & Stroh (2010), Guilbaud & Pham (2013)):

- Can always trade with market orders at the “bad” side of the bid-ask spread \([1 - \varepsilon_t, 1 + \varepsilon_t]S_t\).
- When buy or sell orders of other traders arrive at the jump times of counting processes \(N^1, N^2\), limit orders in the book are executed at the “good” side of the spread.
- Liquidity provider is small. Orders of any size are executed.
- Limit orders can be placed, updated, or deleted for free.
- Reduces primitives of the model to:
  - Mid price \(S_t\).
  - Spread \(\varepsilon_t\).
  - Arrival rates \(\alpha^1_t, \alpha^2_t\) of incoming buy and sell orders.
Model
Limit Order Markets ct’d

- Mid price $S_t$ is a martingale: $dS_t/S_t = \sigma_t dW_t$
  - Disentangles liquidity provision and directional investment.
  - Leads to long and short positions even in the limit for small spreads.

- Small spreads and frequent incoming orders:
  - Spread $\varepsilon_t = \varepsilon \mathcal{E}_t$ for Itô process $\mathcal{E}_t$ and small parameter $\varepsilon$.
  - Arrival rates $\alpha_t^i = \varepsilon^{-\vartheta} \lambda_t^i$ for Itô processes $\lambda_t^i$ and $\vartheta \in (0, 1)$.
  - $\vartheta \in (0, 1)$ ensures “continuity” for $\varepsilon \to 0$. Continuous trading and no market-making profits in the frictionless limit.

- Regularity assumptions on $\sigma_t$, $\mathcal{E}_t$, $\lambda_t^i$:
  - Continuous semimartingales.
  - Bounded and bounded away from zero.
  - Drift and diffusion parts absolutely continuous with bounded rate.
Model
Preferences

- Arbitrary utility function $U : \mathbb{R} \rightarrow \mathbb{R}$:
  - Strictly increasing, strictly concave, $C^2$.
  - Absolute risk aversion $\text{ARA} = -U''/U$ bounded and bounded away from zero.
  - Marginal utility $U'$ bounded by an exponential.

- Investor starts with $x_0$ in cash, maximizes expected utility from terminal liquidation wealth:

  $$E[U(X_T)] \rightarrow \max!$$

- Admissibility of a family $(X^\varepsilon)_{\varepsilon > 0}$ of wealth processes:
  - Bounded risky position, in line with “risk budgets” in practice.
  - Converges to zero uniformly for $\varepsilon \rightarrow 0$. In line with small inventories of high-frequency traders.
Main Results
Optimal Policy

Define position limits

\[ \bar{\beta}_t = \frac{2\varepsilon_t \alpha_t^2}{\text{ARA}(x_0) \sigma_t^2}, \quad \beta_t = -\frac{2\varepsilon_t \alpha_t^1}{\text{ARA}(x_0) \sigma_t^2} \]

Keep risky position between \( \underline{\beta}, \bar{\beta} \) by market orders, trade to boundaries when limit orders are executed:

\[ d\beta_t^\varepsilon = \beta_t^\varepsilon \sigma_t dW_t + (\bar{\beta}_t - \beta_t^\varepsilon) dN_t^1 + (\underline{\beta}_t - \beta_t^\varepsilon) dN_t^2 + d\Psi_t, \quad \beta_0^\varepsilon = 0 \]

\( \Psi \) is minimal finite variation process that ensures \( \beta^\varepsilon \in [\underline{\beta}, \bar{\beta}] \).

This strategy **optimal at the leading order** \( \varepsilon^2(1-\vartheta) \) for small \( \varepsilon \).
The position limits

\[
\bar{\beta}_t = \frac{2\varepsilon_t \alpha^2_t}{\text{ARA}(x_0)\sigma^2_t}, \quad \beta_t = -\frac{2\varepsilon_t \alpha^1_t}{\text{ARA}(x_0)\sigma^2_t}
\]

are:

- Myopic. Only local dynamics matter. Like for liquidity takers facing proportional transaction costs.
- Inversely proportional to risk aversion and variance.
- Proportional to spread earned per trade, and trading rates.
- Like classical Merton proportion \( \mu / \text{ARA} \sigma^2 \). Drift rate \( \mu \) replaced by rates at which revenues accumulate by limit orders.
Main Results
Welfare

- Performance of above strategy can also be quantified. Certainty equivalent:

\[ x_0 + \frac{\text{ARA}(x_0)}{2} E \left[ \int_0^T \left( \beta_t^2 1_{A_t^1} + \beta_t^2 1_{A_t^2} \right)^2 \sigma_t^2 dt \right] \]

\( \omega \in A_t^1 \) if the investor’s last trade before time \( t \) was a purchase and \( \omega \in A_t^2 \) if it was a sale.

- Certainty equivalent of order \( O(\varepsilon^{2(1-\vartheta)}) \). Dominates all families of competitors up to terms of order \( o(\varepsilon^{2(1-\vartheta)}) \).

- Average of future squared target positions. Scaled by risk aversion.
For a symmetric order flow $\alpha^1_t = \alpha^2_t$:

- Certainty equivalent:
  \[
  x_0 + \frac{\text{ARA}(x_0)}{2} E \left[ \int_0^T \left( \frac{\beta_t}{S_t} \right)^2 d\langle S \rangle_t \right]
  \]

- Squared trading boundaries in numbers of shares.

- Averaged with respect to business time $d\langle S \rangle_t$.

- Physical probability coincides with frictionless dual pricing measure here.

- Like for liquidity takers with proportional transaction costs.
Main Results
Welfare ct’d

If all model parameters \((\sigma, \varepsilon, \alpha_1, \alpha_2)\) are constant:

★ Explicit formula for certainty equivalent:

\[
x_0 + \frac{(2\varepsilon \alpha_1)(2\varepsilon \alpha_2)}{2 \text{ARA}(x_0)\sigma^2} T.
\]

★ Liquidity provision equivalent to an annuity:

★ Inversely proportional to risk aversion and variance.
★ Proportional to the rate at which revenues are earned from the spread.

★ For a symmetric order flow \(\alpha_t^1 = \alpha_t^2 = \alpha\):

★ Like classical squared Sharpe ratio \(\mu^2/2 \text{ARA} \sigma^2\).
★ Drift \(\mu\) again replaced by \(2\varepsilon \alpha\).
Adverse Selection and Price Impact

Motivation

So far:

▶ Incoming orders do not affect bid-ask prices.
▶ Justified if these are small and uninformed. Small noise traders.

But:

▶ Larger trades eat into order book:
  ▶ Purchases increase prices.
  ▶ Sales decrease them.
▶ Adverse selection of counterparties with superior information:
  ▶ Prices increase after insider purchases.
  ▶ Decrease after they sell.

In both cases:

▶ Price impact systematically works against liquidity provider.
Adverse Selection and Price Impact

Extension of the Model

- Prices rise after exogenous purchases, drop after sales.
- Captured by simple reduced form model:

\[ \frac{dS_t}{S_t} = \sigma_t \, dW_t - \kappa \epsilon_t \, dN_t^1 + \kappa \epsilon_t \, dN_t^2 \]

- Limit orders executed at \( S_{t-} \). Adverse price move immediately after execution.
- \( \kappa \in [0, 1] \) measures (relative) price impact.
- \( \kappa = 0 \): baseline model without price impact.
- \( \kappa \approx 1 \): model à là Madhavan et al. (1997). Market makers do not earn the spread but only small exogenous compensation.
Adverse Selection and Price Impact

Results

- Model remains tractable.
- Target positions of a similar form:

\[
\beta_t = \frac{2\epsilon_t ((1 - \frac{\kappa}{2})\alpha_t^2 - \frac{\kappa}{2}\alpha_t^1)}{\text{ARA}(x_0)\sigma_t^2}, \quad \beta_t = -\frac{2\epsilon_t ((1 - \frac{\kappa}{2})\alpha_t^1 - \frac{\kappa}{2}\alpha_t^2)}{\text{ARA}(x_0)\sigma_t^2},
\]

Liquidity provision reduced by adverse price impact. Inventory management changed as well.

- For a symmetric order flow \((\alpha_t^1 = \alpha_t^2 = \alpha)\):

\[
\overline{\beta}_t = \frac{2\epsilon_t (1 - \kappa)\alpha_t}{\text{ARA}(x_0)\sigma_t^2}, \quad \beta_t = -\frac{2\epsilon_t (1 - \kappa)\alpha_t}{\text{ARA}(x_0)\sigma_t^2},
\]

Liquidity provision simply reduced by factor \(1 - \kappa\).

- Formula for certainty equivalent remains valid.
Summary

- Small liquidity provider trading in a limit order market.
- General dynamics for mid-price, spread, and order flow. Arbitrary preferences.
- Explicit formulas for almost optimal trading boundaries, associated welfare.
- Extension of the model to account for adverse selection/price impact.

For more information (and proofs):