Outer automorphisms of hyperbolic groups with property (T)

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In this expository note, we present a result due to Frédéric Paulin which implies that a finitely generated hyperbolic group with Kazhdan's property (T) has finite outer automorphism group.

Statement and context

For a group G, the *outer automorphism group*, denoted by $\mathrm{Out}(G)$, is the group of automorphisms of G modulo the normal subgroup of inner automorphisms.

The result we aim to present is the following.

Theorem 1 ([Pau91]). *Let G be a finitely generated hyperbolic group. If the outer automorphism group $\mathrm{Out}(G)$ is infinite, there exists a non-trivial (i.e. without a global fixed point) isometric action of G on an $\mathbb{R}$-tree.*

Corollary 2. *A finitely generated hyperbolic group with Kazhdan's property (T) has finite outer automorphism group.*

Examples of hyperbolic groups with property (T) are lattices in $\mathrm{Sp}(1,n)$ (the isometry group of quaternionic hyperbolic space), and some examples of (T)-groups obtained from Żuk's spectral criterion (in particular, see the construction of Ballmann–Świątkowski [BŚ97, Section 4]).

Note that there are hyperbolic groups with infinite outer automorphism group, for instance the free group, and more generally, fundamental groups of compact orientable surfaces.

Also, there exist groups with property (T) and infinite outer automorphism group [Cor07; OW07], an example of which we sketch below.

Example 3 ([Cor07],[BHV08, Exercise 1.8.19]). Let $n \geqslant 3$ and consider the semi-direct product

$$G := (\mathrm{M}_n(\mathbb{Z}), +) \rtimes \mathrm{SL}_n(\mathbb{Z}),$$

where $\mathrm{SL}_n(\mathbb{Z})$ acts by left matrix multiplication on the additive group of $n \times n$-matrices $\mathrm{M}_n(\mathbb{Z})$. Then G has Kazhdan's property (T), cf. [BHV08, Corollary 1.4.16, Exercise 1.8.7].

To see that $\mathrm{Out}(G)$ is infinite, note that $\mathrm{SL}_n(\mathbb{Z})$ also acts by right matrix multiplication on $\mathrm{M}_n(\mathbb{Z})$, and this induces an action of $\mathrm{SL}_n(\mathbb{Z})$ on G. — More precisely, for each $\gamma \in \mathrm{SL}_n(\mathbb{Z})$, let

$$\phi_\gamma \colon G \to G, \quad \phi_\gamma((M,s)) = (M\gamma^{-1}, s),$$

and observe that ϕ_γ is a group homomorphism. Now suppose that ϕ_γ is an inner automorphism, that is $\phi_\gamma(g) = (M_0, s_0)g(M_0, s_0)^{-1}$ for some fixed $(M_0, s_0) \in G$ and all $g \in G$. Then we have,

$$(M\gamma^{-1}, \mathrm{id}_n) = \phi_\gamma((M, \mathrm{id}_n)) = (M_0, s_0)(M, \mathrm{id}_n)(M_0, s_0)^{-1} = (s_0 M, \mathrm{id}_n) \quad \forall M \in \mathrm{M}_n(\mathbb{Z}),$$

where the last equality follows because $(\mathrm{M}_n(\mathbb{Z}), +)$ is abelian. However, this implies that $\gamma^{-1} = s_0 \in \mathrm{SL}_n(\mathbb{Z})$ commutes with all matrices, whence $\gamma \in \{\pm \mathrm{id}_n\}$. We conclude that the homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Aut}(G), \gamma \mapsto \phi_\gamma$ factors to an embedding $\mathrm{PSL}_n(\mathbb{Z}) \hookrightarrow \mathrm{Out}(G)$, which proves that $\mathrm{Out}(G)$ is infinite.

Furthermore, there are hyperbolic groups with finite outer automorphism group but without property (T). — Let us recall the classical *Mostow rigidity* theorem (for an exposition of a proof see [Roe03, Chapter 8]).

Theorem 4 (Mostow rigidity)**.** *Let M and N be closed hyperbolic[1] n-manifolds with $n \geqslant 3$. Then any homotopy equivalence $M \xrightarrow{\simeq} N$ is homotopic to an isometry $M \xrightarrow{\cong} N$.*

Corollary 5. *For a closed hyperbolic manifold M of dimension at least 3, we have that $\mathrm{Out}(\pi_1(M))$ is finite.*

Proof. By the Cartan–Hadamard theorem, M is a $K(\pi_1(M), 1)$ Eilenberg–MacLane space. Thus, $\mathrm{Out}(\pi_1(M))$ is isomorphic to the group of homotopy equivalences $M \xrightarrow{\simeq} M$ modulo free homotopy. If $\mathrm{Out}(\pi_1(M))$ were infinite, by Theorem 4, there would exist an infinite sequence of pairwise non-homotopic isometries $M \xrightarrow{\cong} M$. However, this is impossible on a compact manifold. □

Note that the fundamental group of a closed hyperbolic manifold is a-T-menable (e.g. because real hyperbolic space admits measured walls [CDH10, Example 3.7]), whence it does not have property (T).

More generally, the following result due to Thurston and Gromov holds.

Theorem 6 ([Gro87, §5, 5.4.A])**.** *Let M be a closed aspherical manifold of dimension at least 3 with hyperbolic fundamental group. Then $\mathrm{Out}(\pi_1(M))$ is finite.*

[1] A Riemannian manifold is called *hyperbolic* if it has constant sectional curvature $K \equiv -1$.

Non-parabolic actions on hyperbolic spaces

For the rest of this note, we closely follow Paulin [Pau91].

Let (X,d) be some proper geodesic hyperbolic space and $\pi\colon G \to \mathrm{Isom}(X), g \mapsto \pi_g$ some group action by isometries. Let $S = S^{-1}$ be a finite generating subset of G, and define a function,

$$l_{\pi,S}\colon X \to \mathbb{R}_{\geqslant 0}, \quad l_{\pi,S}(x) = \max_{s\in S} d(x, \pi_s(x)).$$

For all $g \in G, x \in X$ we have,

$$d(x, \pi_g(x)) \leqslant l_{\pi,S}(x)\, |g|_S, \tag{1}$$

where $|g|_S$ is the word length of g with respect to S.

For simplicity, we will sometimes write $g \cdot x$ instead of $\pi(g)(x)$, and l_S instead of $l_{\pi,S}$ (if it is clear from context which action we use).

We denote the Gromov boundary of X by $\partial_\infty X$, for a definition and discussion of which we refer to [BH99, III.H, 3]. Every quasi-isometry $f\colon X \to X$ gives rise to an homeomorphism $\partial_\infty f\colon \partial_\infty X \to \partial_\infty X$. In particular, G acts by homeomorphisms on $\partial_\infty X$. We call the action of G on (X,d) *parabolic* if the induced action on $\partial_\infty X$ has a global fixed point. A finitely generated hyperbolic group is called *elementary* if the action on a Cayley graph of itself is parabolic. (In fact, this only the case if the group is virtually cyclic.)

Lemma 7. *If the action is not parabolic, then the function $l_S\colon X \to \mathbb{R}_{\geqslant 0}$ is proper. In this case, the minimum $l_S^{\min} := \min_{x\in X} l_S(x)$ is attained, and $l_S^{\min} = 0$ iff there is a point in X fixed by all of G.*

Proof. Assume by contraposition that l_S is not proper. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ tending to infinity (in the locally compact sense) such that $L := \sup_{n\in\mathbb{N}} l_S(x_n) < \infty$. Since $X \cup \partial_\infty X$ is compact, we may assume — after passing to a subsequence — that $x_n \to x_\infty \in \partial_\infty X$. By (1), we have $d(g\cdot x_n, x_n) \leqslant L|g|_S$ for all $n \in \mathbb{N}$, and hence $g \cdot x_\infty = x_\infty$ for all $g \in G$. Thus, the action is parabolic.

To see the second claim, observe that a non-negative proper function $X \to \mathbb{R}$ always attains its minimum, and clearly, a point $x \in X$ is a global fixed point iff $l_S(x) = 0$. □

Let $\mathrm{Cay}(G,S)$ the Cayley graph of G with respect to S, and consider it as a geodesic metric space by declaring each edge to have length 1. Note that each automorphism $\varphi \in \mathrm{Aut}(G)$ induces an isometric action $\pi\colon G \to \mathrm{Cay}(G,S)$, where $\pi(g)\colon \mathrm{Cay}(G,S) \to \mathrm{Cay}(G,S)$ is defined by left-multiplication with $\varphi(g)$ for each $g \in G$. Note that if G is non-elementary, then the induced action on $\mathrm{Cay}(G,S)$ is not parabolic for each automorphism of G. We write $[\varphi]$ for the element of $\mathrm{Out}(G)$ represented by $\varphi \in \mathrm{Aut}(G)$.

Lemma 8. *Suppose that G is non-elementary hyperbolic, and let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathrm{Aut}(G)$ be a sequence of automorphisms. For each $n \in \mathbb{N}$, let $\pi_n\colon G \to \mathrm{Isom}(X)$ the action on $X := \mathrm{Cay}(G,S)$ induced by the automorphism φ_n, and define $\lambda_n := l^{\min}_{\pi_n,S}$.*

If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded, then the set $\{[\varphi_n] \mid n \in \mathbb{N}\} \subseteq \mathrm{Out}(G)$ is finite.

Proof. Assume that $R := \sup_{n\in\mathbb{N}} \lambda_n < \infty$. If $\{[\varphi_n] \mid n \in \mathbb{N}\}$ were infinite, we can assume that $[\varphi_n] \neq [\varphi_m]$ for $n \neq m$. For each $n \in \mathbb{N}$, we choose $\bullet_n \in X$ such that $\lambda_n = l_{\pi_n,S}(\bullet_n)$, and $g_n \in G \subseteq X$ such that $d(g_n, \bullet_n) < 1$. For each $s \in S$, we have,

$$d(g_n, \varphi_n(s)g_n) < d(\bullet_n, \pi_n(s)(\bullet_n)) + 2 \leqslant R + 2,$$

and so $d\left(1_G, g_n^{-1}\varphi_n(s)g_n\right) < R + 2$ for all $s \in S$, $n \in \mathbb{N}$. Since S and $B_{R+2}(1_G) \cap G$ are finite sets, exist $m \neq n$ such that,

$$g_m^{-1}\varphi_m(s)g_m = g_n^{-1}\varphi_n(s)g_n, \qquad \forall s \in S.$$

As S is a generating set, we conclude $[\varphi_m] = [\varphi_n] \in \mathrm{Out}(G)$, which contradicts our assumption. □

Asymptotic cones

Unlike our source, we will not employ equivariant Gromov–Hausdorff topologies for the proof. Instead we use ultralimits and asymptotic cones, but otherwise it is exactly the same as in [Pau91].

Let ω be a non-principal ultrafilter on $\mathbb{N}$ and $(X_n, d_n, \bullet_n)_{n\in\mathbb{N}}$ a sequence of pointed metric spaces. We define the *ultraproduct* by

$$X_\omega = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \;\middle|\; \sup_{n\in\mathbb{N}} d_n(x_n, \bullet_n) < \infty \right\} \Big/ \sim,$$

where $(x_n)_n \sim (y_n)_n$ if $d_\omega((x_n)_n, (y_n)_n) := \lim_\omega d_n(x_n, y_n) = 0$. Then d_ω induces a metric on X_ω. If each (X_n, d_n) is a δ_n-hyperbolic geodesic space with $\lim_{n\to\infty} \delta_n = 0$, then X_ω is a 0-hyperbolic geodesic space, i.e. an $\mathbb{R}$-tree.

For every $n \in \mathbb{N}$, let $\pi_n \colon G \to \mathrm{Isom}(X_n)$ be an isometric group action, and assume that $\sup_{n\in\mathbb{N}} d_n(\bullet_n, \pi_n(g)(\bullet_n)) < \infty$ for every $g \in G$. Then there is an isometric group action $\pi_\omega \colon G \to \mathrm{Isom}(X_\omega)$ defined by $\pi_\omega(g)((x_n)_n) = (\pi_n(g)(x_n))_n$.

If we apply this construction to a sequence of the form $(X, \frac{1}{\lambda_n}d, \bullet_n)$, where (X, d) is some fixed metric space with a sequence $(\bullet_n)_n \subset X$, and $(\lambda_n)_n \subset \mathbb{R}_{\geqslant 0}$ some sequence with $\lambda_n \to \infty$ as $n \to \infty$, then we call the resulting space X_ω an *asymptotic cone* of (X, d).

Proof of Theorem 1. We consider the case that G is non-elementary and choose a finite symmetric generating set $S \subseteq G$. We assume that there is an infinite sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathrm{Aut}(G)$ such that $[\varphi_n] \neq [\varphi_m] \in \mathrm{Out}(G)$ for all $n \neq m$. For the rest of this proof, we use the notation from Lemma 8. We conclude that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is unbounded, whence we may assume that $\lambda_n \to \infty$ as $n \to \infty$. We construct an asymptotic cone X_ω by applying the above construction to the sequence $(X, \frac{1}{\lambda_n}d, \bullet_n)$, where $\bullet_n \in X$ is a point which minimizes $l_{\pi_n,S}$. Since G is hyperbolic, it follows that X_ω is an $\mathbb{R}$-tree. The sequence of actions $(\pi_n)_n$ satisfies $d(\bullet_n, \pi_n(g)(\bullet_n)) \leqslant l_{\pi_n,S}(\bullet_n)|g|_S = \lambda_n|g|_S$ for all $n \in \mathbb{N}$, $g \in G$. Therefore, the sequence $\left(\frac{1}{\lambda_n}d(\bullet_n, \pi_n(g)(\bullet_n))\right)_{n\in\mathbb{N}}$ is bounded for all $g \in G$, and there is an induced action $\pi_\omega \colon G \to \mathrm{Isom}(X_\omega)$.

Now suppose that the action π_ω has a global fixed point $\mathbf{x} \in X_\omega$. — For a sequence $(x_n)_{n\in\mathbb{N}}$ representing $\mathbf{x}$, this means,

$$\lim_\omega \frac{d\,(x_n, \pi_n(g)(x_n))}{\lambda_n} = 0, \qquad \forall g \in G.$$

In particular, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, \pi_{n_0}(s)(x_{n_0})) < \frac{\lambda_{n_0}}{2}$ for all $s \in S$. Hence it follows that $l_{\pi_{n_0},S}(x_{n_0}) < \frac{\lambda_{n_0}}{2}$, but on the other hand, we have by definition $\lambda_{n_0} \leqslant l_{\pi_{n_0},S}(x_{n_0})$, a contradiction. □

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