Topics in Algebra: Cryptography

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Cryptography: Overview

Cryptography

- I Past: Diffie–Hellman (1976) and Rivest-Shamir-Adleman (1977)
- II Nowadays: Blockchain ([1991], 2008)
- III Future: Quantum ([1927, 1982], 1983) and Post-quantum cryptography (1994,1996)

RSA cryptosystem

Definition: RSA cryptosystem

Let $n = pq$, where p, q are primes. Let $\mathcal{P} = \mathcal{C} = \mathbb{Z}/n\mathbb{Z}$ and

$$\mathcal{K} = \{(n, p, q, d, e) \ : \ de = 1 \mod \phi(n)\}$$

For $k = (n, p, q, d, e)$, we define

$$E_k(x) = x^e \mod n \ \text{ and } \ D_k(c) = c^d \mod n.$$

Public-key is (n, e) and private-key is (p, q, d).

Here, x is a plaintext.

Euler's function $\phi(n)$ = the number of positive integers less than n and relatively prime to n.

RSA cryptosystem

Encryption and decryption are inverse operations.

$n = pq \Rightarrow \phi(n) = (p-1)(q-1)$
We have that $de = 1 \mod \phi(n)$, i.e. $de = t\phi(n) + 1$ for some $t \in \mathbb{Z}$.

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(1) Suppose that $x \in (\mathbb{Z}/n\mathbb{Z})^\times$, then

$$(x^e)^d = x^{t\phi(n)+1} \mod n = (x^{\phi(n)})^t x \mod n = 1^t x \mod n = x \mod n.$$

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(2) If $x \notin (\mathbb{Z}/n\mathbb{Z})^\times$, then $x = 0 \mod p$ or $x = 0 \mod q$.

If $x = 0 \mod p$, then $(x^e)^d = 0 \mod p$ as well. If the same holds for mod q we are done by the Chinese remainder theorem.

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Otherwise, $x \neq 0 \mod q$. Then, by Fermat's little theorem,
$(x^e)^d = x^{ed-1}x = x^{t(p-1)(q-1)}x = (x^{q-1})^{t(p-1)}x = 1^{t(p-1)}x \mod q = x \mod q$. We conclude by the Chinese remainder theorem.

Reminder: Cryptosystem: basic model for secrecy

Definition: Cryptosystem is a 5 -tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$ satisfying:

- $\mathcal{P}$ is a finite set of possible plaintexts;
- $\mathcal{C}$ is a finite set of possible ciphertexts;
- $\mathcal{K}$, the keyspace, is a finite set of possible keys;
- $\mathcal{E} = \{E_k : k \in \mathcal{K}\}$ consists of encryption functions $E_k : \mathcal{P} \to \mathcal{C}$;
- $\mathcal{D} = \{D_k : k \in \mathcal{K}\}$ consists of decryption functions $D_k : \mathcal{C} \to \mathcal{P}$;
- For all $e \in \mathcal{K}$ there exists $d \in \mathcal{K}$ such that for all plaintexts $p \in \mathcal{P}$ we have:

$$D_d(E_e(p)) = p$$

- Symmetric cryptosystem: $d = e$
- Public-key cryptosystem: d cannot be derived from e in a computationally feasible way

RSA cryptosystem parameters

Algorithm: RSA parameter generation

1. Generate two large primes, p and q, such that $p \neq q$
2. $n \leftarrow pq$ and $\phi(n) \leftarrow (p-1)(q-1)$
3. Choose a random e with $1 < e < \phi(n)$ such that $\gcd(e, \phi(n)) = 1$
4. $d \leftarrow e^{-1} \mod \phi(n)$
5. The public key is (n, e) and the private key is (p, q, d).

Reminder: Breaking encryption algorithms

- A practical method of determining the decryption key is found.

- A weakness in the encryption algorithm leads to a plaintext.

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  RSA: Find the private key (p, q, d), knowing the public key (n, e)

- A weakness in the encryption algorithm leads to a plaintext.

  RSA: Invert the RSA encryption function

One-way function

A function that is easy to compute on every input, but almost always hard to invert: a polynomial-time interceptor will fail to invert the function, except with negligible probability.

Definition: Properties of an algorithm

An algorithm is deterministic if the output only depends on the input. Otherwise, it is called probabilistic or randomized.

An algorithm is a polynomial algorithm if the number of operations when executed by a multitape Turing machine is $O(n^k)$ for some $k \in \mathbb{N}$ on input of size n.

Complexity classes

A problem instance x lies in the complexity class

- *P* if x is solvable by a polynomial deterministic algorithm.
- *BPP* if x is solvable by a polynomial probabilistic algorithm.
- *BQP* if x is solvable by a polynomial deterministic algorithm on a quantum computer.
- *NP* if x is verifiable by a polynomial deterministic algorithm.

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- NP if x is verifiable by a polynomial deterministic algorithm.

Known: $P \subseteq NP$, $P \subseteq BPP$, Factorisation and Discrete logarithm problem are in $NP \cap BQP$.

Conjectures: P=BPP, Factorisation and Discrete logarithm problem are not in $NP \cap BPP$.

Open Problem: Is there $x \in NP \setminus BQP$?

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Definition: Negligible function

A function $f : \mathbb{N} \to \mathbb{R}$ is negligible if for each positive polynomial p, $\exists n_0 \in \mathbb{N}$ such that $|f(n)| < \frac{1}{p(n)}$ for all $n \geqslant n_0$

Example: $f = 2^{-n}$ Non-example: $f = n^{-4}$

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Notation: $\{0, 1\}^*$ = set of all finite binary strings
$(\{0, 1\}^*,$ concatenation) is a semi-group

One-way function

Definition: One-way function

A function $f\colon \{0,1\}^* \to \{0,1\}^*$ is a one-way function if

1. for all input $x \in \{0,1\}^*$ there is a polynomial deterministic algorithm that outputs $f(x)$;
2. for all polynomial probabilistic algorithm $A\colon \{0,1\}^* \to \{0,1\}^*$ there is a negligible function negl such that

$$\Pr\left[A(f(x)) \in f^{-1}(f(x))\right] \leqslant \mathrm{negl}\,(\mathrm{n}),$$

where the probability is over the choice of x according to the uniform distribution on $\{0,1\}^n$, and the randomness of A.

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Hard to invert when the input is uniformly distributed. In particular, hard to invert in the average-case (not in the worst-case sense =NP-hard).

Hard to invert for long enough inputs.

One-way function

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Open problem: Do one-way functions exist?

Open problem: Is breaking RSA as hard as factoring integers?

RSA keys vs Factoring

Theorem: RSA keys vs Factoring

If the Factoring is not in BPP, then the Asymmetry of RSA is not in BPP.

Asymmetry problem = compute the private key from the public key
Here: compute d (and not, in addition p and q), knowing (n, e).

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Theorem: One-way $\Leftrightarrow$ Pseudorandom

The existence of one-way functions is a minimal assumption that is both necessary and sufficient for constructions of pseudorandom generators and functions.

Cryptanalysis of RSA: Weakness of the RSA primitive

If the interceptor can factor the modulus n in polynomial-time, then the private key can be efficiently calculated.

Integer factorisation methods

- Trial division
- Pollard's $p - 1$ method
- Elliptic curve method
- Quadratic sieve and Number field sieve
- ...

A 768-bit number factored, two years of computations in 2007-2009.
So, 512-bit keys are 'sufficient'.
In practice, RSA keys are typically 1024- to 2048-bits long.

Cryptanalysis of RSA: Factoring

If n is composite, then it has a prime factor $p \leqslant \sqrt{n}$

Trial division

Exhaustive search over all successive primes until $\sqrt{n}$.

Complexity of such attacks allow to derive lower bounds on RSA parameters (key size etc.)

Test questions

Question 6

What is the complexity of the RSA parameter generation?

Question 7

Let f be a one-way function. Is $f(f(x))$ necessarily a one-way function?

Question 8

What is the worst-case / average-case complexities of trial division?

Question 9

Design an algorithm computing the square root of a positive integer. What about its complexity? What about its modular variant and its complexity?

Cryptanalysis of RSA: The RSA parameters

Attacks on the RSA function

- Low e or d attack
- Partial d exposure attack

Cryptanalysis of RSA: Implementation attacks

Side-channel attaks

- Time analysis: a correlation between e and the runtime of the cryptographic operation (Solution: delay / blinding)
- Power analysis: monitoring power consumption (Solution: engineering)
- Fault analysis: exploiting errors in cryptographic operations (Solution: verify with e)
- ...

Remainder: RSA cryptosystem parameters

Algorithm: RSA parameter generation

1. Generate two large primes, p and q, such that $p \neq q$
2. $n \leftarrow pq$ and $\phi(n) \leftarrow (p-1)(q-1)$
3. Choose a random e with $1 < e < \phi(n)$ such that $\gcd(e, \phi(n)) = 1$
4. $d \leftarrow e^{-1} \mod \phi(n)$
5. The public key is (n, e) and the private key is (p, q, d).

Randomness of the encryption key is to resist an informed exhaustive plaintext search attack. This contrasts the symmetric key encryption.

Cryptanalysis of RSA: practice

Key choice

1024- to 2048-bits long

Strong primes vs Random primes

Multi-prime RSA

More than two primes p and q, hence, primes are smaller for a big n, the encryption is faster.

Cryptanalysis of RSA: practice

Encoding of plaintext = Padding schemes

Plaintext is preprocessed using a probabilistic encoding: same plaintext with the same e gives a different ciphertext.

Goal: resist to the informed exhaustive plaintext search attack.

Definition: Hash function

A one-way function $h\colon \{0,1\}^* \to \{0,1\}^k$ for some $k \in \mathbb{N}$.

Hash: data of arbitrary size is mapped to data of fixed size.

Optimal asymmetric encryption padding (OAEP)

A simplified variant: ignoring the lengths of input / outputs.

Given: x and (n, e), two hashes h_1 and h_2, and a random number r.

RSA-OAEP encoding

1. Hash r using h_1 and XOR to x:

$$A = h_1(r) \oplus x$$

2. Hash A using h_2 and XOR to r:

$$B = h_2(A) \oplus r$$

3. Apply RSA to the concatenation $A \| B$:

$$c = (A \| B)^e \mod n$$

XOR = the exclusive disjunction = $\oplus = + \mod 2$

Optimal asymmetric encryption padding (OAEP)

Bob can decrypt without knowing r.

RSA-OAEP decoding

1. Decrypt c using d, get $A \| B$
2. Hash A using h_2 and XOR to B, get r:

$$h_2(A) \oplus B = h_2(A) \oplus (h_2(A) \oplus r) = r$$

3. Hash r using h_1 and XOR to A, get x:

$$h_1(r) \oplus A = h_1(r) \oplus (h_1(r) \oplus x) = x$$

Hashes: with trapdoor, given by a polynomial algorithm.

Discrete Logarithm problem

Let G be a finite group, $g \in G$ an element, $\langle g \rangle \leqslant G$ a cyclic subgroup it generates, n its order.

Discrete Logarithm Problem = DLP

Given n, g and $y \in \langle g \rangle$, find the unique integer d, $0 \leqslant d \leqslant n-1$, such that

$$g^d = y.$$

$d := \log_g y$ is called the discrete logarithm of y to base g.

Example: $G = (\mathbb{Z}/p\mathbb{Z})^\times$, g is a primitive element $\mod p$, $n = p - 1$

A primitive element $\mod p$ is an element of $(\mathbb{Z}/p\mathbb{Z})^\times$ of order $p - 1$.

Cryptosystems based on the DLP: ElGamal cryptosystem

ElGamal'1985 cryptosystem is a cryptosystem based on the Discrete Logarithm problem in $(\mathbb{Z}/p\mathbb{Z})^\times$

DLP assumption

1. The DLP in $(\mathbb{Z}/p\mathbb{Z})^\times$ is not in BPP.
2. The Factoring is not in BPP.

ElGamal cryptosystem: Parameter generation

ElGamal'1985 cryptosystem is a cryptosystem based on the Discrete Logarithm problem in $(\mathbb{Z}/p\mathbb{Z})^\times$

Algorithm: ElGamal parameter generation

1. Generate a large prime p.
2. Choose a primitive element $g \in (\mathbb{Z}/p\mathbb{Z})^\times$.
3. Choose a random d with $1 < d < p - 1$.
4. $y \leftarrow g^d \mod p$
5. The public key is (p, g, y) and the private key is d.

ElGamal cryptosystem: informally

The encryption of a plaintext x is randomised using a random value r chosen by Alice in $\mathbb{Z}/(p-1)\mathbb{Z}$:
There are $p-1$ ciphertexts c that are encryptions of the same x.

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Randomization of x:
the plaintext x is 'masked' by multiplying it by y^r yielding c_2.

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Bob, knowing the private key d, can compute y^r from g^r.

Then he can remove the 'mask' by dividing c_2 by y^r to obtain x.

ElGamal cryptosystem

Definition: ElGamal cryptosystem

Let p be a prime and g a primitive element $\mod p$.
Let $\mathcal{P} = (\mathbb{Z}/p\mathbb{Z})^\times, \mathcal{C} = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times$ and define

$$\mathcal{K} = \{(p, g, d, y) : y = g^d \mod p\}.$$

For $k = (p, g, d, y)$, and for a secrete random number $r \in \mathbb{Z}/(p-1)\mathbb{Z}$, define

$$E_k(x; r) = (c_1, c_2), \text{ where}$$

$$c_1 = g^r \mod p, \quad \text{and} \quad c_2 = xy^r \mod p.$$

For $c_1, c_2 \in (\mathbb{Z}/p\mathbb{Z})^\times$, define

$$D_k(c_1, c_2) = c_2(c_1^d)^{-1} \mod p$$

Public key is (p, g, y) and private key is d.

ElGamal cryptosystem

Encryption and decryption are inverse operations

$$c_2(c_1^d)^{-1} = xy^r \cdot ((g^r)^d)^{-1} \mod p = x \cdot (g^d)^r \cdot ((g^r)^d)^{-1} \mod p = x \mod p$$

Theorem: ElGamal keys vs DLP

If the DLP in $(\mathbb{Z}/p\mathbb{Z})^\times$ is not in BPP, then the Asymmetry of ElGamal is not in BPP.

ElGamal Cryptosystem: finite fields etc.

ElGamal is for an arbitrary finite group G, we had $(\mathbb{Z}/p\mathbb{Z})^\times$. Other groups:

1. The multiplicative group of the finite field $\mathbb{F}_{p^k}$
2. The group of an elliptic curve defined over a finite field.

The group has to satisfy the DLP assumption.

Weierstrass equation

Let **k** be a field.

Weierstrass equations

The affine Weierstrass equation:

$$E\colon y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,\ a_i \in \mathbf{k}.$$

The homogeneous Weierstrass equation:

$$E^*\colon y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,\ a_i \in \mathbf{k}.$$

The vanishing set:

$$E(\mathbf{k}) = \{(x:y:z) \in \mathbb{P}^2 \text{ so that } x,y,z \in \mathbf{k} \text{ is a solution of } E^*\} \subseteq \mathbb{P}^2$$

Singular points and curves

The defining polynomial:

$$F^*: y^2z + a_1xyz + a_3yz^2 - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3), a_i \in \mathbf{k}.$$

Definition: Singular points and curves

Let $P = (x_0 : y_0 : z_0) \in E(\mathbf{k})$.

1. P is a singular point of E if

$$\frac{\partial F^*}{\partial x}(x_0, y_0, z_0) = \frac{\partial F^*}{\partial y}(x_0, y_0, z_0) = \frac{\partial F^*}{\partial z}(x_0, y_0, z_0) = 0.$$

2. E is singular if there is a singular point $P \in E(\mathbf{k})$, otherwise E is nonsingular or smooth.

Example: $(0 : 1 : 0)$ is the only point of E at infinity, i.e. at $z = 0$. It has multiplicity 3 as $E^*(x, y, 0): x^3 = 0$.
It is not singular: $\frac{\partial F^*}{\partial z}(0, 1, 0) = 1 \neq 0$.

Elliptic curves

Definition: Elliptic curve

E is elliptic if E is smooth.

Normal forms

1. If $\operatorname{char} \mathbf{k} \neq 2$ then in E substitute $y \mapsto y - \frac{a_1 x + a_3}{2}$ obtaining

$$y^2 = x^3 + a_2' x^2 + a_4' x + a_6'$$

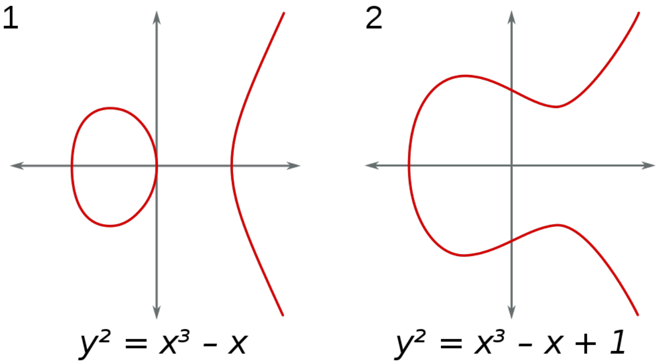
2. If $\operatorname{char} \mathbf{k} \neq 2, 3$ then substitute $x \mapsto x - \frac{1}{3} a_2',\ a_2' = a_2 + \frac{a_1^2}{4}$ obtaining

$$y^2 = x^3 + ax + b$$

$\operatorname{char} \mathbf{k} \neq 2, 3 : \operatorname{disc}(x^3 + ax + b) = -16(4a^3 + 27b^2)$

$\operatorname{char} \mathbf{k} \neq 2,\ y^2 = f(x) = x^3 + a_2' x^2 + a_4' x + a_6'$ is singular $\iff \operatorname{disc} f = 0$

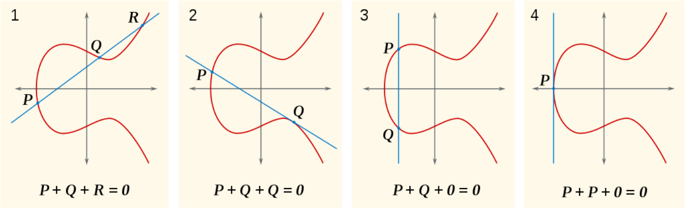
Elliptic curves



Elliptic curves in normal form [image: Wikipedia]

Elliptic curve: The group structure $(E(\bar{\mathbf{k}}), +)$

$\mathbf{k}$ a field, $\bar{\mathbf{k}}$ its algebraic closure



Group structure [image: Wikipedia]

Test questions

Question 10

Which of the following statements are true?

1. If the RSA cryptosystem is breakable, then large numbers can be factored.
2. Breaking the ECC cryptosystem is equivalent to solving the discrete logarithm problem.
3. There is no message expansion in the ECC cryptosystem.

Question 11

Why in practice public-key cryptosystems have longer key lengths than symmetric cryptosystems?