Representation theory of groups WS 2016

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Def. 1 Let G be a group and let k be a field. A *linear representation* of G over k is

a group homomorphism $\rho : G \longrightarrow GL(E)$.

E is a vector space over k
$GL(E)$ is the group of invertible k-linear maps on E

The *dimension* of $\rho := \dim_k E$

sometimes it is called the *degree* of ρ or the *rank* of ρ

One thinks of ρ as the map

$$\begin{cases} G \times E \longrightarrow E \\ (g, v) \longmapsto g \cdot v \end{cases}$$

where $g \cdot v := \rho(g) v$ the image of v under the linear transformation $\rho(g)$.

Remark: ρ is *not* injective in general

Main idea of representation theory:

Use tools of linear algebra for $GL(E)$ in order to study G

Examples of representations:

0) Any subgroup G of $GL(E)$ is given by a natural representation $G \hookrightarrow GL(E)$

For any group G and field k we form a vector space
$k(G) :=$ the k-vectorspace with a basis $(e_g)_{g \in G}$
If G is infinite, then $k(G)$ is infinite-dimensional

1) We define $\pi : G \longrightarrow GL(k(G))$
$g \longmapsto \pi(g)$

where $\pi(g) e_h := e_{gh} \quad \forall g, h \in G$.

π is indeed a homomorphism:
$\pi(1) e_h = e_h \quad \forall h \in G \Rightarrow \pi(1)$ is the identity transformation

$\pi(g_1 g_2) e_h = e_{g_1 g_2 h}$ and $\pi(g_1)(\pi(g_2) e_h) = \pi(g_1) e_{g_2 h} = e_{g_1 g_2 h}$
$\Rightarrow \pi$ is a homomorphism into $GL(k(G))$

2) Let k^G be the space of all functions $f : G \to k$ with pointwise addition and scalar multiplication of functions.
We define $\rho : G \longrightarrow GL(k^G)$ right-regular representation
$g \longmapsto \rho(g)$

where $\rho(g) f(x) := f(xg) \quad \forall f \in k^G, g \in G$.
It is a homomorphism:
$\rho(gh) f(x) = f(xgh)$, $\rho(h) f : y \mapsto f(yh)$
$\rho(g)(\rho(h) f)(x) = \rho(h) f(xg) = f(xgh)$

3) The formula $\lambda(g)f(x) := f(g^{-1}x)$ $\forall x \in G$ $\forall g \in G$ $\forall f \in k^G$ also defines a representation of G on k^G

It is called the <u>left-regular representation</u>

4) = Exercise Show that the formula

$$\rho(g,h)f(x) := f(g^{-1}xh) \qquad \forall g,h \in G \ \forall x \in G \ \forall f \in k^G$$

defines a representation of $G \times G$ on k^G

<u>Remark:</u> In the above examples ρ is injective.

<u>Def. 2</u> Let G be a group and let k be a field.

A representation ρ of G over k is <u>faithful</u> if $\ker\rho = \{1\}$

A representation ρ of G over k is <u>trivial</u> if $\ker\rho = G$.

(that is, if $\forall g \in G$ $\rho(g)$ is the identity map of E)

A representation ρ of G over k is <u>1-dimensional</u> and <u>"the" trivial</u> it is denoted by $\mathbb{1}$ if $E = k$ and $\rho(g) = 1 \in k^\times$ $\forall g \in G$

$k^\times$: the multiplicative group of the field.

<u>Examples</u> over $\mathbb{C}$ and $\mathbb{R}$ of 1-dimensional representations

1) $k = \mathbb{C}$ $\quad G = (\mathbb{C}, +) \longrightarrow (\mathbb{C}^\times, \cdot) = GL_1(\mathbb{C}) = GL(\mathbb{C})$

$z \longmapsto e^z$

2) $k = \mathbb{C}$, $G = \mathbb{R}$ or $\mathbb{C}$, then $\forall s \in \mathbb{C}$

$\chi_s : x \longmapsto e^{sx}$ is a 1-dim. representation

3) If $G = \mathbb{R}/\mathbb{Z}$, $\forall m \in \mathbb{Z}$ $\quad x \longmapsto e^{2\pi i m x}$

is a 1-dim. representation.

How we relate possibly different representations of G over k

Def. 3 Let G be a group and let k be a field.
A *morphism* between representations ρ_1 and ρ_2 of G, both over k and acting on the vector spaces E_1 and E_2 resp. is a k-linear map

$$\Phi : E_1 \to E_2$$

such that $$\Phi(\rho_1(g)v) = \rho_2(g)(\Phi(v)) \in E_2, \forall g \in G, v \in E_1$$

We say Φ *intertwines* ρ_1 and ρ_2 or Φ is a *intertwiner*
We visualize this definition by saying that $\forall g \in G$ the *square diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \rho_1(g)}\downarrow & & \downarrow{\scriptstyle \rho_2(g)} \\ E_1 & \xrightarrow{\Phi} & E_2 \end{array}$$

of linear maps *commutes*

We can also express this by writing

$$\Phi(g \cdot v) = g \cdot \Phi(v)$$

Observe that the set of morphisms (also called *homomorphisms*) from ρ_1 to ρ_2 is a k-vector space $\underset{\text{sub}}{\text{Hom}_G(\rho_1, \rho_2)}$ of $\text{Hom}(E_1, E_2)$.

Proposition 4 Let G be a group and let k be a field.
(i) For any representation ρ of G over k and a vector space E, the identity map on E is a morphism

$$\rho \to \rho$$

(ii) Given representations ρ_1, ρ_2 and ρ_3 on E_1, E_2 and E_3 and morphisms

$$E_1 \xrightarrow{\Phi_1} E_2 \xrightarrow{\Phi_2} E_3,$$

the composition $E_1 \xrightarrow{\Phi_2 \circ \Phi_1} E_3$ is a morphism between ρ_1 and ρ_3.

Remark: In the language of category theory: the representations of G over k are the objects of a category with morphisms given by intertwining linear maps.

Remark If a morphism Φ is a bijective linear map, then its inverse Φ^{-1} is also a morphism.

Def 5 If a morphism Φ between ρ_1 and ρ_2 is a bijective linear map it is called isomorphism between ρ_1 and ρ_2.

Using the diagram: $\rho_2(g) \circ \Phi = \Phi \circ \rho_1(g)$

or equivalently $\Phi^{-1} \circ \rho_2(g) = \rho_1(g) \circ \Phi^{-1}$

in this case, $\forall g \in G$.

Assume now that for a vector subspace $F \subset E$ we have $\rho(g)(F) \subset F \quad \forall g \in G$, that is F is stable under all transformations $\rho(g)$ $\forall g \in G$, then $\rho(g)|_F$ defines a homomorphism

$\tilde{\rho}: G \longrightarrow GL(F)$, i.e. a k-representation

and the inclusion linear map $i: F \hookrightarrow E$ is a morphism of representations.

In that case we say $\tilde{\rho}$ is a <u>subrepresentation</u> of ρ (or of E itself, if the action is clear).

<u>Example 6</u> <u>Trivial subrepresentations</u> and <u>invariant vectors</u>

Let $F := \{ v \in E \mid g \cdot v = v \ \forall g \in G \} \subseteq E$

Since G acts by linear maps this is a linear subspace.
It is sometimes denoted by E^G

Observe that the representation of G on E^G is trivial (in the sense of our definition above)

If n is the dimension of E^G and $\mathbb{1}^n = k^n$ denotes the k-vector space of dimension n with trivial action of G, we have isomorphism

$$\mathbb{1}^n \xrightarrow{\sim} E^G$$

This space of invariant vectors is the largest subrepresentation of E (for inclusion) which is trivial.

Individually: $\forall \underset{0}{v} \in E$ such that $\rho(g)v = v \ \forall g \in G$ defines a trivial subrepresentation of dimension 1

i.e. injective morphism $\begin{cases} \mathbb{1} \hookrightarrow E \\ t \longmapsto tv \end{cases}$ of representations

This gives a k-linear isomorphism

$$E^G \simeq \mathrm{Hom}_G(\mathbb{1}, E)$$

(the reciprocal map sending $\Phi : \mathbb{1} \to E$ to $\Phi(1)$)

Example 7. The invariant subspace of the regular represent.

If $f \in (k^G)^G$ we have

$$f(x) = \rho(g) f(x) = f(xg) \qquad \forall x, g \in G$$

$\Rightarrow$ taking $x = 1$ we see that f is constant.

$\Rightarrow$ The invariant subspace of the regular representation is the 1-dimensional subspace of constant k-valued functions.

Example 8. $k(G)^G = \{0\}$ if G is infinite

$k(G)^G$ is 1-dimensional and generated by

$$\sum_{g \in G} e_g \quad \in k(G)$$

if G is finite.

Example 9. 1-dimensional representations

$\chi : G \to k^*$ a homomorphism

For an arbitrary k-representation $\rho : G \to GL(V)$ a non-zero intertwiner $\chi \xrightarrow{\Phi} \rho$ corresponds to the data of a non-zero vector $v \in V$ such that

$$\rho(g) v = \chi(g) v \qquad \forall g \in G$$

$\Rightarrow$ v is a common eigenvector for all operators $\rho(g)$.

Terminology: χ is called a character of G

Example 10. Let $N \trianglelefteq G$ be a normal subgroup of G.

$$E^N = \{ v \in E \mid \rho(g)v = v \ \ \forall g \in N \}$$

It is a subrepresentation of E but not trivial in general.

□ Let $v \in E^N$ and $g \in G$.

We want to check that ~~$\rho(g)v$~~ $\rho(h)v \in E^N \quad \forall h \in G$.

Pick $n \in N$, we have $\overbrace{w}^{N}$

$$\rho(n)\rho(h)v = \rho(nh)v = \rho(h)\rho(\overbrace{h^{-1}nh}^{w})v =$$

$$= \rho(h)v$$

This is $\forall n \in N$, $\Rightarrow \rho(h)v \in E^N$

▨

- What about the case when N is not normal?
- Is it true that E^N is always stable under the action of G?
- Give examples where E^N is not trivial.

Example 11.

$$\pi: \begin{array}{ccc} G & \longrightarrow & GL(k(G)) \\ \cup & & \\ g & \longmapsto & \pi(g) \end{array} \qquad \pi(g)e_h := e_{gh}$$

$$\rho: \begin{array}{ccc} G & \longrightarrow & GL(k^G) \\ \cup & & \\ g & \longmapsto & \rho(g) \end{array} \qquad \rho(g)f(x) := f(xg)$$

Claim:
π is isomorphic to a subrepresentation of ρ

□ Define $\Phi: \begin{array}{ccc} k(G) & \longrightarrow & k^G \\ \cup & & \\ e_g & \longmapsto & \text{the characteristic function of the single point } g^{-1} \end{array}$

that is $$\Phi(e_g)(x) = \begin{cases} 1 & \text{if } x = g^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Φ is injective : $\Phi(v)$ maps g to the coefficient of $e_{g^{-1}}$ in v

Φ is a morphism of representations :

in $k(G)$: $g \circ e_h = e_{gh}$

in k^G : $g \circ \Phi(e_h) = \rho(g)\Phi(e_h)$ maps x to

$$\Phi(e_h)(xg) = \begin{cases} 1 & \text{if } xg = h^{-1}, \text{ i.e. if } x = (gh)^{-1} \\ 0 & \text{otherwise} \end{cases}$$

This is precisely :

$$\Phi(g \circ e_h) = g \circ \Phi(e_h)$$

Is Φ isomorphism?

The map $\tilde{\Phi}$ is an isomorphism if G is finite, but not otherwise;

Image $(\tilde{\Phi})$ = the subspace of functions which are zero except at finitely many points.

The following basic operations can be defined in a natural way:

- Quotients of representations
- Sum and intersection of representations
- The kernel and image of a morphism of representations
- Exact sequences and, in particular, short exact sequences of representations
- The direct sum of representations
- The tensor product of representations
- Given a representation ρ acting on E, the dual of ρ acting on the linear dual space $E' = \mathrm{Hom}_k(E, k)$, and the associated representation of G acting on the space of k-linear maps $\mathrm{End}_k(E) = \mathrm{Hom}_k(E, E)$

Quotients

Given G, ρ and a <u>subrepresentation</u> $F \subset E$ we consider the quotient vector space

$$V = E/F$$

and a natural linear action of G on V:

$$g \longmapsto g \circ v \qquad \text{for } g \in G, v \in V$$

where $g \circ v$ is the image in V of $\rho(g)\tilde{v}$ for any $\tilde{v}$ pre-image of v under the canonical projection map $E \to E/F$

- This is well-defined:

  if $\tilde{v}_1 \neq \tilde{v} \Rightarrow \tilde{v}_1 = \tilde{v} + w$ for some $w \in F$ with image v

  $\Rightarrow \rho(g)\tilde{v}_1 - \rho(g)\tilde{v} = \rho(g)w \in F \Rightarrow$

  $\rho(g)\tilde{v}_1 \equiv \rho(g)\tilde{v} \mod F$

- A global description of this quotient action is that it is such that the projection map $E \to E/F$ is a morphism of representations

  (just like the inclusion map $F \hookrightarrow E$ is one)

Given a morphism $\Phi : E_1 \to E_2$

- $\operatorname{Ker}\Phi \subset E_1$ is a subrepresentation of E_1
- $\operatorname{Im}\Phi \subset E_2$ — " — of E_2
- $E_1/\operatorname{Ker}\Phi \simeq \operatorname{Im}\Phi$ is an isomorphism of representations
- $E_2/\operatorname{Im}\Phi$ is a representation as quotient of two representations

Use the <u>linearity</u> of actions of G.

<u>Direct sum</u>:

- Given G, ρ_1 on E_1, ρ_2 on E_2

we define $\rho = \rho_1 \oplus \rho_2 : G \longrightarrow GL(E_1 \oplus E_2)$ by $\rho(g)(v_1+v_2) := \rho_1(g)v_1 + \rho_2(g)v_2$

E_1, E_2, $E_1 \oplus E_2$ are subrepresentations and

$$(\rho_1 \oplus \rho_2)/\rho_1 \simeq \rho_2 \quad \text{given by}$$

$$v_1 + v_2 \longmapsto v_2$$

- One can consider more than two factors
- $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$

Consider any representation ρ of G acting on E and an injection $\Phi: \rho_1 \hookrightarrow \rho$ such that

$$\rho/\rho_1 = \rho/\mathrm{Im}\Phi \simeq \rho_2$$

as k-representations.

Fact $\exists$ a subspace $E_2 \subset E$ such that

$$E = \mathrm{Im}\Phi \oplus E_2 \simeq E_1 \oplus E_2$$

as k-vector spaces.

⚠ It is not always the case that E_2 can be found as a subrepresentation of ρ

If it is the case then this subrepresentation, say π_2 is necessarily isomorphic to ρ_2:

$$\pi_2 \simeq (\rho_1 \oplus \pi_2)/\rho_1 \simeq \rho/\rho_1 \simeq \rho_2$$

Lemma 12. Let G be a group, k a field, $\rho: G \to GL(E)$ a representation.

If $E = E_1 \oplus E_2$, E_1 is a subrepresentation of ρ

then E_2 is also a subrepresentation of ρ $\iff$

$$\Phi: \begin{cases} E & \longrightarrow & E \\ v = v_1 + v_2 & \longmapsto & v_1 \end{cases} \qquad v_1 \in E_1,\ v_2 \in E_2$$

is an intertwiner, i.e. if $\Phi \in \mathrm{Hom}_G(E, E)$.

Ex. 13. $G = \mathbb{R}$, $k = \mathbb{R}$ $\quad \rho: \begin{cases} G & \longrightarrow & GL_2(\mathbb{R}) \\ x & \longmapsto & \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \end{cases}$

$E_1 = \mathbb{R}e_1$

There is no subrepresentation E_2 which is complementary to E_1

Def 14 Let G be a group and k be a field.

(i) $\rho : G \to GL(E)$ is **irreducible** $\Leftrightarrow$

$E \neq 0$ and there is no subspace of E stable under ρ (i.e. invariant under ρ) except $\{0\}$ and E itself.

In other words: if there is no subrepresentation of ρ except 0 and ρ itself.

(ii) $\rho : G \to GL(E)$ is **semisimple** if it can be written as a direct sum of subrepresentations, each of which is irreducible:

$$\rho \simeq \bigoplus_{i \in I} \rho_i$$

Examples 15

- The trivial representation $\mathbb{1}$ is irreducible.
- Any trivial representation on E is semisimple as
$$E \simeq \bigoplus_{i \in I} k e_i$$
after choosing a basis $(e_i)_{i \in I}$ of E
- The decomposition of a semisimple representation is not unique.
- 2-dimensional representation of Example 13 is not semisimple.
- Any character is irreducible.
- If ρ is finite-dimensional there exists at least one irreducible subrepresentation of G contained in ρ. (What is it in Example 13?)