Representations of topological groups

Definition 35 A topological group is a group G equipped with a topology such that the group product and the inverse maps are both continuous:

$G \times G \to G$ — the product map in G

$G \to G$ — the inverse map in G

Here, $G \times G$ is viewed with the product topology.

Examples:

$\mathbb{R}$, $\mathbb{R}^+$, $GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$,

any subgroup of a topological group:

e.g. $\mathbb{Z} \subseteq \mathbb{R}$, $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$,

$SL_n(\mathbb{Z}) \subseteq SL_n(\mathbb{R})$,

any discrete group, with the discrete topology (defined by letting every subset be open)

Given a topological group, we study representations with certain continuity properties

↑ with respect to given topology.

Banach space representations

Field $k = \mathbb{C}$ or $\mathbb{R}$ — Banach space,

The representation space E with a norm $\|\cdot\|$

The transformations are continuous linear operators $E \to E$ } $B(E)$

The invertible transformations are bijective linear maps with a continuous inverse } $BGL(E)$

Definition 36 A continuous representation of a topological group G is a homomorphism

$$\rho : G \longrightarrow BGL(E)$$

such that the corresponding action map

$$\begin{cases} G \times E \longrightarrow E \\ (g, v) \longmapsto \rho(g)v \end{cases}$$

is continuous, where $G \times E$ is equiped with the product topology.

If ρ_1 and ρ_2 are continuous representations of G, acting on E_1 and E_2 respectively, an intertwiner is a continuous homomorphism

$$\rho_1 \xrightarrow{\Phi} \rho_2$$

which commutes with the action of G, that is, Φ is a continuous linear map

$$E_1 \xrightarrow{\Phi} E_2 \quad \text{such that}$$

$$\Phi(\rho_1(g)v) = \rho_2(g)\Phi(v) \quad \forall g \in G \quad \forall v \in E_1.$$

Definition 37 Let G be a topological group and

$$\rho: G \to BGL(E)$$

be a representation on a Banach space E.

(1) A representation π of G is a subrepresentation of ρ if π acts on a closed subspace $F \subseteq E$ which is invariant under ρ, and $\pi(g) = \rho(g)|_F$

The quotient representation ρ/π is the induced representation on E/F with the norm

$$\|v\|_{E/F} := \min\{\|w\|_E \mid w = v \bmod F\}$$

(2) ρ is irreducible (or topologically irreducible) if $E \neq \{0\}$ and $\nexists$ non-zero proper closed subspace F of E which is stable under all $\rho(g)$, $g \in G$.

Remarks

- If $\dim(E) < +\infty$, we have $BGL(E) = GL(E)$ and the definition of intertwiner can be taken for the usual euclidean topology on $BGL(E)$
- If $\dim(E) < \infty$, any subspace of E is closed and the definition of the irreducibility is the usual one.

However, $\exists$ ∞-dimensional representations which are topologically irreducible but have many non-zero proper stable subspaces! (these are necessarily dense in E but they are not closed).

We will focus on unitary representations

$k = \mathbb{C}$ or $\mathbb{R}$

$E = \mathcal{H}$ a Hilbert space
equipped with an inner product $\langle \cdot, \cdot \rangle$
which is linear in the first variable
and conjugate-linear in the second one:

$\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$, $\langle v, \alpha w \rangle = \overline{\alpha} \langle v, w \rangle$
$\forall v, w \in \mathcal{H}$, $\forall \alpha \in \mathbb{C}$.

If $T: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a continuous
(synonymously, bounded) linear operator
between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$,
the adjoint of T is the unique linear
operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$\langle T v_1, v_2 \rangle_{\mathcal{H}_2} = \langle v_1, T^* v_2 \rangle_{\mathcal{H}_1}$$

$\forall v_1, v_2$, $v_1 \in \mathcal{H}_1$, $v_2 \in \mathcal{H}_2$.

T is called self-adjoint iff $T^* = T$

T is unitary iff $T T^* = T^* T = id$

A representation ρ is unitary iff the operators $\rho(g)$ are unitary, that is, they preserve the inner product

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$$

$$\forall g \in G, \forall v, w \in \mathcal{H}$$

$U(\mathcal{H})$ = the group of unitary operators

Thus, a unitary representation is a continuous homomorphism

$$\rho : G \to U(\mathcal{H})$$

G can be arbitrary group with the discrete topology

A morphism $\rho_1 \xrightarrow{\Phi} \rho_2$ of unitary representations is a morphism of continuous representations: it is not required that Φ preserve the inner product.

A representation $\rho: G \to BGL(E)$ on a complex vector space E is called unitarizable if $\exists$ an inner product $\langle \cdot, \cdot \rangle$ on E, defining a structure of Hilbert space such that $\rho(g)$ are unitary with respect to $\langle \cdot, \cdot \rangle$ and

$G \to U(E)$ is a unitary representation.

Remarks

- If $\dim E < \infty$, the continuity is automatic.

- Let $\rho_1: G \to U(\mathcal{H}_1)$, $\rho_2: G \to U(\mathcal{H}_2)$ be unitary representations and $\Phi: \mathcal{H}_1 \to \mathcal{H}_2$ is a G-homomorphism. Then the adjoint $\Phi^*: \mathcal{H}_2 \to \mathcal{H}_1$ is also an intertwiner. Indeed,

$$\langle \rho_1(g)(\Phi^*(v)), w \rangle_{\mathcal{H}_1} = \langle \Phi^*(v), \rho_1(g^{-1})w \rangle_{\mathcal{H}_1} =$$
$$= \langle v, \Phi(\rho_1(g^{-1})w) \rangle_{\mathcal{H}_2} = \langle v, \rho_2(g^{-1})\Phi(w) \rangle_{\mathcal{H}_2} =$$
$$= \langle \rho_2(g)v, \Phi(w) \rangle_{\mathcal{H}_2} = \langle \Phi^*(\rho_2(g)v), w \rangle_{\mathcal{H}_1}$$

$$\Rightarrow \Phi^*(\rho_2(g)v) = \rho_1(g)(\Phi^*(v))$$

$\forall w \in \mathcal{H}_1, \forall g \in G, \forall v \in \mathcal{H}_2$

$\Rightarrow \Phi^*$ intertwines ρ_2 and ρ_1.

Proposition 38. Let G be a topological group.

(1) If ρ is a unitary representation of G, then any subrepresentation and any quotient representation of ρ are unitary. The restriction of ρ to any subgroup $H \leq G$ is a unitary representation with the same inner product.

(2) Direct sums are unitary with an inner product

$$\langle v_1 \oplus w_1, v_2 \oplus w_2 \rangle_{\rho_1 \oplus \rho_2} = \langle v_1, v_2 \rangle_{\rho_1} + \langle w_1, w_2 \rangle_{\rho_2}$$

(3) Tensor products of finite-dimensional unitary representations ρ_1 and ρ_2 are unitary, with respect to the inner product defined on pure tensors by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\rho_1 \otimes \rho_2} = \langle v_1, v_2 \rangle_{\rho_1} \langle w_1, w_2 \rangle_{\rho_2}$$

Exercise 39 The regular representation on $L^2(\mathbb{R})$.

$\mathbb{R}$ with usual topology

$\mathcal{H} := L^2(\mathbb{R})$ the space of square-integrable functions on $\mathbb{R}$ with respect to Lebesgue measure.

We define $\rho(t) f(x) := f(x+t)$

Show that it is a homomorphism, it is continuous, it is a unitary representation.

The contragredient for unitary representations

Given a Hilbert space $\mathcal{H}$, the conjugate space $\overline{\mathcal{H}}$ is the same abelian group but with scalar multiplication and inner products given by

$$\alpha \cdot v := \overline{\alpha} v \,, \quad \langle v, w \rangle_{\overline{\mathcal{H}}} := \langle w, v \rangle_{\mathcal{H}}$$

Claim: $\overline{\mathcal{H}} \underset{\text{isometric}}{\simeq} \mathcal{H}'$ ← the dual of $\mathcal{H}$

The isometry is given by the linear map

$$\Phi = \overline{\mathcal{H}} \to \mathcal{H}'$$

$$w \longmapsto (\tau_w : v \longmapsto \langle v, w \rangle)$$

(use $\tau_{\alpha w} = \bar{\alpha}\tau_w$)

If ρ is a unitary representation of G on $\mathcal{H}$ we can write the matrix coefficients $f_{v,\tau}$ of ρ using inner product on $\mathcal{H}$ only: we have, for $\tau := \tau_w \in \mathcal{H}'$, $v \in \mathcal{H}$,

$$f_{v,\tau}(g) = \tau_w(\rho(g)v) = \langle \rho(g)v, w \rangle$$

Using Φ, we can "transport" the contragredient representation of ρ to an isomorphic representation $\bar{\rho}$ on $\overline{\mathcal{H}}$.

Exercise 40. Let ρ be a unitary representation. Show that $g \longmapsto \overline{\langle \rho(g)v, w \rangle}$

$\forall v, w \in \mathcal{H}$, $g \in G$ are matrix coefficients of $\bar{\rho}$.

Lemma 41 Let ρ be a finite-dimensional unitary representation of G. Then any eigenvalue of $\rho(g)$, $\forall g \in G$, is a complex number of modulus 1.

□ Linear algebra of unitary matrices $\rho(g)$. ■

Example 42 Let G be a finite group.
Let $C(G)$ be the space of complex-valued functions on G, with the regular representation of G.

An inner product on $C(G)$ is given by
↑ this is a vector space

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{|G|} \sum_{x \in G} \varphi_1(x) \overline{\varphi_2(x)}$$

The regular representation is unitary with respect to this inner product:

$$\| \rho(g) \varphi \|^2 = \frac{1}{|G|} \sum_{x \in G} |\varphi(xg)|^2 = \frac{1}{|G|} \sum_{y \in G} |\varphi(y)|^2 =$$
$$= \| \varphi \|^2$$

$\forall g \in G$, using the change of variables $y = xg$.

If G is infinite, $\langle \cdot, \cdot \rangle$ is defined using integration with respect to an invariant measure μ, and $C(G)$ is replaced by $L^2(G, \mu)$.

If $(\rho_i)_{i\in I}$ is any family of unitary representations of G, acting on $\mathcal{H}_i$, then we can define $\rho := \bigoplus_{i\in I} \rho_i$ acting on $\mathcal{H} := \bigoplus_{i\in I} \mathcal{H}_i$

↑ ← infinite orthogonal direct sum

↳ $\ni v := (v_i)_{i\in I}$ such that

$$\|v\|_{\mathcal{H}}^2 := \sum_{i\in I} \|v_i\|_{\mathcal{H}_i}^2 < \infty$$

by $$\rho(g)v := (\rho_i(g)v_i)_{i\in I}$$

- ρ is unitary
- the "standard" direct sum of $\mathcal{H}_i$

($(v_i)_{i\in I}$ with $v_i = 0$ for all but finitely many i) is a dense subspace of $\mathcal{H}$.

It is stable but not closed $\Rightarrow$ not a subrepresentation of ρ.

Proposition 43 (Reducibility of unitary representations)

Let $\rho: G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a topological group G. Then any closed subspace $F \subset \mathcal{H}$ invariant under ρ has a closed stable complement given by $F^\perp \subset \mathcal{H}$. In particular, any finite-dimensional unitary representation is semisimple.

□ Since any finite-dimensional subspace of a Hilbert space is closed, the claim on the finite-dimensional unitary representation follows from the semisimplicity criterion.

Let $F \subset \mathcal{H}$ be a subrepresentation.
Let us check that

$$F^\perp := \{ v \in \mathcal{H} \mid \langle v, w \rangle = 0 \ \forall w \in F \} \subset \mathcal{H}$$

is a closed G-stable complement of F.

The theory of Hilbert spaces $\Rightarrow$

$F^\perp$ is closed and $F \oplus F^\perp = \mathcal{H}$

Since ρ preserves the inner product, we have

$$\langle \rho(g) v, w \rangle = \langle v, \rho^{-1}(g) w \rangle = 0$$

$\forall g \in G \ \forall w \in F \Rightarrow F^\perp$ is G-stable $\Rightarrow$

$F^\perp$ is a subrepresentation of $\mathcal{H}$.

■

Theorem 44 (Schur's lemma for unitary representations)

(1) Let $\rho_1 : G \to \mathcal{U}(\mathcal{H}_1)$, $\rho_2 : G \to \mathcal{U}(\mathcal{H}_2)$ be non-isomorphic irreducible unitary representations of G.

Then $\mathrm{Hom}_G(\rho_1, \rho_2) = 0$

(2) If $\rho : G \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation of G, then

$$\mathrm{Hom}_G(\rho, \rho) = \mathbb{C} \cdot \mathrm{id}_{\mathcal{H}}$$

The proof requires the spectral theorem for self-adjoint operators on a Hilbert space.

Theorem 45 (The spectral theorem)

Let $\mathcal{H}$ be a Hilbert space, A be a continuous self-adjoint operator on $\mathcal{H}$.

Then $\exists$ a measure space (X, μ), an isometric isomorphism $T : \mathcal{H} \to L^2(X, \mu)$, and a real-valued function $f \in L^\infty(X)$ such that

$$T(Av) = M_f(Tv) = f \cdot T(v) \quad \forall v \in \mathcal{H}$$

where M_f is the multiplication operator on $L^2(X, \mu)$ defined by $M_f(\varphi) := f \cdot \varphi \quad \forall \varphi \in L^2(X, \mu)$.

- If $\mathcal{H}$ is separable $\Rightarrow \exists$ such (X, μ) with $\mu(X) < \infty$
- If A is unitary $\Rightarrow \exists$ such f with values in the unit circle $S^1 \subset \mathbb{C}$.

Proof of Schur's lemma:

(2) Let $\Phi \in \mathrm{Hom}_G(\rho, \rho)$ be an intertwiner.

Then Φ^* is also an intertwiner (see page 61).

We write $\Phi = \frac{\Phi + \Phi^*}{2} + i \frac{\Phi - \Phi^*}{2i} = A + iB$

(A, B: self-adjoint intertwiners)

$\Rightarrow$ It suffices to prove the claim for self-adjoint intertwiners.

$\Rightarrow$ To such intertwiners we can apply the spectral theorem

$\Rightarrow \mathcal{H} = L^2(X, \mu)$, Φ is the multiplication operator M_f by some real-valued bounded function f on X.

Claim: f is almost everywhere constant (say $= \lambda$).

$\Rightarrow \Phi = M_f = \lambda \cdot \mathrm{id}$.

Proof of the claim: Let $\nu := f_* \mu$ the image measure on $\mathbb{R}$, defined by $\nu(B) := \mu(f^{-1}(B))$ on measurable subsets $B \subset \mathbb{R}$.

Let $C := \mathrm{supp}\, \nu$ be the support of ν.

It is a non-empty compact (closed and bounded) subset of $\mathbb{R}$ (as it is the spectrum of the multiplication operator).

Reminder from the spectral theory:

- The spectrum of a bounded operator on a Hilbert space is non-empty.
- If A is a bounded linear operator on $\mathcal{H}$, then

The point spectrum $(A) := \{ \lambda \in \sigma(A) \mid A - \lambda \cdot id$ is not one-to-one $\}$

The continuous spectrum $(A) := \{ \lambda \in \sigma(A) \mid A - \lambda \cdot id$ is one-to-one but not onto; $\operatorname{ran}(A - \lambda Id)$ is dense in $\mathcal{H}\}$

The residual spectrum $(A) := \{ \lambda \in \sigma(A) \mid A - \lambda \cdot id$ is one-to-one but not onto; $\operatorname{ran}(A - \lambda \cdot id)$ is not dense in $\mathcal{H}\}$

The resolvent set $(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda \cdot id$ is one-to-one and onto $\}$

The spectrum $(A) = \sigma(A) = \mathbb{C} \setminus$ The resolvent set (A)

- The resolvent set is an open subset of $\mathbb{C}$ that contains the exterior disc $\{ \lambda \in \mathbb{C} \mid |\lambda| > \|A\| \}$.

Back to the proof of Schur's lemma.

Let $a := \min(C)$, $b := \max(C)$.

If $a = b \Rightarrow C = \{a\} \Rightarrow f$ is almost everywhere equal to $a \Rightarrow$ we are done with the Claim.

If $a \neq b$, let $c := \frac{a+b}{2}$, $U := [a, c[$, $V := [c, b]$.

By definition of the support of the measure,
$\nu(U) > 0$ and $\nu(V) > 0$.

Let A be the multiplication operator, by the characteristic function of $f^{-1}(U)$, and let $\mathcal{H}_1 := \operatorname{Ker} A$.

$\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$;

Since $\nu(U) > 0$, $\nu(V) > 0 \Rightarrow 0 \neq \mathcal{H}_1 \neq \mathcal{H}$.

Fact: A is an intertwiner of $\rho \Rightarrow$

$\Rightarrow \mathcal{H}_1$ is a non-trivial subrepresentation of $\mathcal{H}$.

$\Rightarrow$ contradiction with the irreducibility assumption.

$\Rightarrow$ (2) of Schur's lemma is proved.

We use (2) to show (1).

(1) Let $\Phi: \mathcal{H}_1 \to \mathcal{H}_2$ be a non-zero intertwiner between irreducible representations ρ_1 and ρ_2.

$\Phi^* \circ \Phi: \mathcal{H}_1 \to \mathcal{H}_1$ is a self-intertwiner.

(2) $\Rightarrow$ $\Phi^* \circ \Phi = \lambda \cdot id$ for some $\lambda \in \mathbb{C}$.

We have $\lambda \langle v, w \rangle = \langle \Phi^* \circ \Phi v, w \rangle = \langle \Phi v, \Phi w \rangle$ $\forall v, w \in \mathcal{H}_1$. $\Rightarrow \lambda \neq 0 \Rightarrow \frac{1}{\sqrt{\lambda}} \Phi$ is an isometric intertwiner from $\mathcal{H}_1$ to $\mathcal{H}_2$.

But $\operatorname{Ker}\Phi$ is a subrepresentation of $\mathcal{H}_1$ and $\operatorname{Ker}\Phi \neq \mathcal{H}_1 \Rightarrow \operatorname{Ker}\Phi = \{0\} \Rightarrow$ Φ is injective. Similarly, $\operatorname{Im}\Phi$ is dense in $\mathcal{H}_2$ but $\frac{1}{\sqrt{\lambda}}\Phi$ is isometric $\Rightarrow \operatorname{Im}\Phi$ is closed.

$\Rightarrow$ Φ is an isomorphism. $\Rightarrow$ (1) follows.

■

"Fact" follows from the following

Exercise 46 (X, μ) a measure space, $\mathcal{H} = L^2(X, \mu)$ $A \in BGL(\mathcal{H})$ be the multiplication operator by a real-valued function $f \in L^\infty(X, \mu)$.

$\rho(g)$ is as above, in particular, $A \cdot \rho(g) = \rho(g) \cdot A$ $\forall g \in$

Let $I \subset \mathbb{R}$ be a bounded interval, C be the multiplicatio operator by $\mathbb{1}_I \circ f = \mathbb{1}_{f^{-1}(I)}$.

Show that $\rho(g) C = C \rho(g)$ $\forall g \in G$

About continuity

Let $\mathcal{H}$ be a separable Hilbert space
↳ it admits a countable orthonormal basis.

Let $B(\mathcal{H})$ be the space of all continuous linear operators (= the space of all bounded linear operators)

Various topologies can be defined on $B(\mathcal{H})$

- The norm (or uniform) topology on $B(\mathcal{H})$:

$T_n \to T$ if $\|T_n - T\| \to 0$ as $n \to \infty$.

The open neighbourhood base at 0 is given by

$$\{T : \|T\| < \varepsilon\}$$

- The strong topology on $B(\mathcal{H})$:

$T_n \to T$ if $T_n v \to Tv$ $\forall v \in \mathcal{H}$.

Each $v \in \mathcal{H}$ defines open neigbourhood base at 0:

$$\{T : \|Tv\| < \varepsilon\}$$

- The weak topology on $B(\mathcal{H})$:

$T_n \to T$ if $\langle T_n v, w\rangle \to \langle Tv, w\rangle$ $\forall v, w \in \mathcal{H}$

Each pair $v, w \in \mathcal{H}$ gives $\{T : |\langle Tv, w\rangle| < \varepsilon\}$

The group $\mathcal{U}(\mathcal{H})$ of unitary operators inherits these 3 topologies.

$\Rightarrow$ We can define 3 types of continuous unitary representation:

weakly, strongly, and uniformly continuous.

Proposition 47 Let $\rho: G \to \mathcal{U}(\mathcal{H})$ be a homomorphism of a topological group G.

ρ is a unitary representation

$\hookrightarrow$ $\begin{cases} G \times \mathcal{H} \to \mathcal{H} \\ (g, v) \mapsto \rho(g)v \end{cases}$ is continuous

$\Leftrightarrow$ ρ is strongly continuous

$\hookrightarrow$ $\forall v \in \mathcal{H}$

$\begin{cases} G \to \mathcal{H} \\ g \mapsto \rho(g)v \end{cases}$ is continuous

□ $(\Rightarrow)$ is obvious: 2-variable continuity $\Rightarrow$ 1-variable continuity.

$(\Leftarrow)$ Let $(g,v) \in G \times \mathcal{H}$. $\forall (h,w) \in G \times \mathcal{H}$:

$$\|\rho(g)v - \rho(h)w\| = \|\rho(h)(\rho(h^{-1}g)v - w)\| =$$

$$= \|\rho(h^{-1}g)v - w\| \quad \text{by unitarity.}$$

the triangle inequality

$$\|\rho(g)v - \rho(h)w\| \leq \|\rho(h^{-1}g)v - v\| + \|v - w\|$$

Given $\varepsilon>0$, $\exists\, \mathcal{U}_v \ni v$; $v \in \mathcal{H}$: $\|v-w\|<\varepsilon$
open neighborhood for $w \in \mathcal{U}_v$

The continuity assumption $\Longrightarrow$
$\exists\, \mathcal{U}_1 \ni 1$, $1 \in G$ such that $\|\rho(x)v-v\|<\varepsilon$
open neighborhood for $x \in U_1$.

Then, when $(h,w) \in g\mathcal{U}_1^{-1} \times \mathcal{U}_v$ we have

$$\|\rho(g)v-\rho(h)w\|<2\varepsilon.$$

■

Proposition 48 A unitary representation is strongly continuous $\Longleftrightarrow$ weakly continuous.
If ρ is uniformly continuous, then $\mathrm{Im}\rho$ is finite-dimensional.

□ $\Rightarrow$ obvious.

$\Leftarrow$ $\|\rho(g)v-v\|^2 = 2\|v\|^2 - 2\langle \rho(g)v, v\rangle$

$\Rightarrow$ if $g \to 1$ weakly, then $\|\rho(g)v-v\| \to 0$

The result on the uniform continuity is due to I.M. Singer, Annals of Math., (1952).

■.