Representation Theory of Groups - Blatt 7

11:30-12:15, Seminarraum 9, Oskar-Morgenstern-Platz 1, 2.Stock

`http://www.mat.univie.ac.at/~gagt/rep_theory2016`

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On this exercise sheet, we will describe the *Gelfand-Naimark-Segal construction* in the context of group representations. The following generalises the notion of positive matrix into bounded operators on Hilbert space:

Definition. A function $f : G \to \mathbb{C}$ is *positive definite* if for every finite subset $F \subset G$ the matrix:

$$(f(g^{-1}h))_{g,h\in F} \in M_F(\mathbb{C})$$

is positive.

Theorem. *Let $f : G \to \mathbb{C}$ be a positive definite function. Then there is a Hilbert space $\mathcal{H}$, a representation $\pi : G \to \mathfrak{B}(\mathcal{H})$ and a vector $v \in \mathcal{H}$ such that $f(s) = \langle \pi(s)v, v\rangle$.*

(Note, such representations are called *cyclic* by operator algebraists).

Question 1. Let $\phi, \psi \in C_c(G;\mathbb{C})$, the algebra of compactly supported functions from G to $\mathbb{C}$.

a) Show that the form:
$$\langle \phi, \psi\rangle_f = \sum_{g,h\in G} f(g^{-1}h)\phi(g)\overline{\psi(h)}.$$
is a positive semidefinite bilinear form on $C_c(G;\mathbb{C})$. Construct a Hilbert space $\mathcal{H}_f$ from this data.

b) Let $\lambda^f : G \to \mathfrak{B}(\mathcal{H}_f)$ be the continuous extension of the left regular representation λ of G on $C_c(G;\mathbb{C})$ by convolution. Show that λ^f defines a unitary representation of G on $\mathcal{H}_f$.

c) Find a suitable vector $v \in C_c(G;\mathbb{C})$ such that:
$$\langle \lambda^f(g)v, v\rangle_f = f(g)$$
for every $g \in G$. (Hint: To find v, consider what happens with a basis for $C_c(G;\mathbb{C})$ as a vector space).