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On the Initial Value Problem of the Toda and Kac-van Moerbeke Hierarchies

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Abstract. We provide a brief review of the initial value problem associated with the Toda and Kac-van Moerbeke hierarchies. We give a simple proof for the basic (global) existence and uniqueness theorem and provide some additional details for the inverse scattering transform. In addition, we also show how to obtain solutions of the Kac-van Moerbeke hierarchy from solutions of the Toda hierarchy via a Miura type transform.

1. Introduction

In 1967 Gardner et al. ([**10**]) presented a method for solving the Korteweg-de Vries equation which is presently known as inverse scattering transform (IST). Since then, this method has been extended to numerous other completely integrable equations. It consists of three steps. One, find the scattering data of the initial conditions. Two, find the time evolution of scattering data. Three, reconstruct the potential from the (time dependent) scattering data. At first sight this procedure looks relatively simple, but, after a closer look, it turns out that it is highly non-trivial to give a complete and rigorous mathematical justification. In fact, since one has to assume existence of a solution in the outset, there are two additional steps necessary to make the method complete from a mathematical point of view. Firstly, one has to show that the time dependent scattering data give rise to a potential. Secondly, one needs to show that this potential is indeed a solution of the completely integrable system under consideration. These last two steps are often ignored in the literature.

Our first aim is to review the IST for the case of the Toda hierarchy and to show that the situation is much simpler here since we have a global existence and uniqueness theorem at our disposal. Moreover, since rigorous results on scattering theory for Jacobi operators are very rare, we will have a closer look at the actual reconstruction and we will provide a detailed investigation of the Gel'fand-Levitan-Marchenko equations containing some new results. Finally, we will review the connection between the Toda and Kac-van Moerbeke hierarchy and show how to

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obtain solutions of the Kac-van Moerbeke hierarchy from solutions of the Toda hierarchy via a Miura type transform.

2. The Toda hierarchy

In this section we introduce the Toda hierarchy using the standard Lax formalism ([**12**]). We first review some basic facts from [**6**].

We will only consider bounded solutions and hence require

Hypothesis **H.** 2.1. *Suppose $a(t), b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z},\mathbb{R}),\ \ b(t) \in \ell^\infty(\mathbb{Z},\mathbb{R}),\ \ a(n,t) \neq 0\ \ (n,t) \in \mathbb{Z}\times\mathbb{R}, \tag{2.1}$$

and let $t \mapsto (a(t), b(t))$ be differentiable in the Banach space $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t), b(t)$ is a Jacobi operator

$$\begin{array}{rccl} H(t): & \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ & f & \mapsto & a(n,t)f(n+1) + a(n-1,t)f(n-1) + b(n,t)f(n) \end{array}, \tag{2.2}$$

where $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. The scalar product in $\ell^2(\mathbb{Z})$ is denoted by $\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \overline{f(n)} g(n)$ and δ_n will be the canonical basis.

Moreover, choose constants $c_0 = 1$, c_j, $1 \le j \le r$, $c_{r+1} = 0$, set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n\rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n\rangle + c_{j+1} \end{aligned} \tag{2.3}$$

and consider the Lax operator

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t) g_j(t) S^+ - h_j(t)) H(t)^{r-j} + g_{r+1}(t), \tag{2.4}$$

where $S^\pm f(n) = f(n\pm 1)$. Restricting to the two-dimensional nullspace $\mathrm{Ker}(\tau(t) - z)$, $z \in \mathbb{C}$, of $\tau(t) - z$, we have the following representation of $P_{2r+2}(t)$,

$$P_{2r+2}(t)\Big|_{\mathrm{Ker}(\tau(t)-z)} = 2a(t) G_r(z,t) S^+ - H_{r+1}(z,t), \tag{2.5}$$

where $G_r(z,n,t)$ and $H_{r+1}(z,n,t)$ are monic polynomials in z of the type

$$\begin{aligned} G_r(z,n,t) &= \sum_{j=0}^{r} z^j g_{r-j}(n,t), \\ H_{r+1}(z,n,t) &= z^{r+1} + \sum_{j=0}^{r} z^j h_{r-j}(n,t) - g_{r+1}(n,t). \end{aligned} \tag{2.6}$$

A straightforward computation shows that the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \quad t \in \mathbb{R}, \tag{2.7}$$

is equivalent to

$$\mathrm{TL}_r(a(t),b(t))_1 = \dot{a}(t) - a(t)\Big(g^+_{r+1}(t) - g_{r+1}(t)\Big) = 0, \tag{2.8}$$

$$\mathrm{TL}_r(a(t),b(t))_2 = \dot{b}(t) - \Big(h_{r+1}(t) - h^-_{r+1}(t)\Big) = 0, \tag{2.9}$$

where the dot denotes a derivative with respect to t and $f^{\pm}(n) = f(n\pm 1)$. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy (TL hierarchy)

$$\mathrm{TL}_r(a,b) = (\mathrm{TL}_r(a,b)_1, \mathrm{TL}_r(a,b)_2) = 0, \quad r \in \mathbb{N}_0. \tag{2.10}$$

The Lax equation implies the well-known isospectrality theorem.

THEOREM 2.2. *Let $a(t)$, $b(t)$ satisfy $\mathrm{TL}_r(a,b) = 0$ and (H.2.1). Then the Lax equation (2.7) implies the existence of a unitary propagator $U_r(t,s)$ for $P_{2r+2}(t)$ such that*

$$H(t) = U_r(t,s)H(s)U_r(t,s)^{-1}, \quad (t,s) \in \mathbb{R}^2. \tag{2.11}$$

Thus all operators $H(t)$, $t \in \mathbb{R}$, are unitarily equivalent.

In addition, if $\psi(s) \in \ell^2(\mathbb{Z})$ solves $H(s)\psi(s) = z\psi(s)$, then the function

$$\psi(t) = U_r(t,s)\psi(s), \tag{2.12}$$

fulfills

$$H(t)\psi(t) = z\psi(t), \qquad \frac{d}{dt}\psi(t) = P_{2r+2}(t)\psi(t). \tag{2.13}$$

In addition, we will need the basic existence and uniqueness theorem for the Toda hierarchy ([**17**], Theorem 2.2, see also [**8**], Proposition 1).

THEOREM 2.3. *Suppose $(a_0,b_0) \in M = \ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto (a(t),b(t))$ in $C^\infty(\mathbb{R},M)$ of the Toda equations, that is, $\mathrm{TL}_r(a(t),b(t)) = 0$, such that $(a(0),b(0)) = (a_0,b_0)$.*

PROOF. The Toda equation gives rise to a vector field X_r on the Banach space $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$, that is,

$$\frac{d}{dt}(a(t),b(t)) = X_r(a(t),b(t)) \quad \Leftrightarrow \quad \mathrm{TL}_r(a(t),b(t)) = 0. \tag{2.14}$$

Since this vector field has a simple polynomial dependence in a and b it is clearly smooth. Hence, by standard theory, solutions for the initial value problem exist locally and are unique (cf., e.g. [**1**], Theorem 4.1.5). In addition, since the Toda flow is isospectral, we have $\|a(t)\|_\infty + \|b(t)\|_\infty \le 2\|H(t)\| = 2\|H(0)\|$ (at least locally). Thus any integral curve $(a(t),b(t))$ is bounded on finite t-intervals implying global existence (see e.g., Proposition 4.1.22 of [**1**]). □

3. Inverse scattering transform

Now we want to review the inverse scattering method for solving the initial value problem of the Toda hierarchy.

As a preparation, we first consider the trivial solution of the Toda equations,

$$a_0(n,t) = a_0 = \frac{1}{2}, \qquad b_0(n,t) = b_0 = 0. \tag{3.1}$$

The sequences

$$\psi_\pm(z,n,t) = k^{\pm n} \exp\Big(\frac{\pm\alpha_r(k)t}{2}\Big), \quad z = \frac{k+k^{-1}}{2}, \tag{3.2}$$

where

$$\alpha_r(k) = 2\Big(kG_{0,r}(z) - H_{0,r+1}(z)\Big) = (k-k^{-1})G_{0,r}(z) \tag{3.3}$$

satisfy

$$\begin{aligned} H_0(t)\psi_\pm(z,n,t) &= z\psi_\pm(z,n,t),\\ \frac{d}{dt}\psi_\pm(z,n,t) &= P_{0,2r+2}(t)\psi_\pm(z,n,t)\\ &= 2a_0G_{0,r}(z)\psi_\pm(z,n+1,t) - H_{0,r+1}(z)\psi_\pm(z,n,t) \end{aligned} \tag{3.4}$$

(we omit n,t in the arguments of $G_{0,r}$, $H_{0,r+1}$ since these quantities do not depend on n, t). Note $\alpha_r(k) = -\alpha_r(k^{-1})$. Explicitly we have

$$\begin{aligned} \alpha_0(k) &= k-k^{-1},\\ \alpha_1(k) &= \frac{k^2-k^{-2}}{2} + c_1(k-k^{-1}),\\ &\text{etc. }. \end{aligned} \tag{3.5}$$

Now we turn to scattering theory for $H(t)$ (cf. [**7**], [**11**], [**19**]). That is, we will assume that $H(t)$ looks asymptotically like H_0 (the operator associated with $a_0 = 1/2$ and $b_0 = 0$). More precisely, we will require $a(n,t) > 0$ and

$$\sum_{n\in\mathbb{Z}} |n|(|1-2a(n)| + |b(n)|) < \infty. \tag{3.6}$$

This implies

$$\sigma_{ess}(H) = \sigma_{ac}(H) = [-1,1], \quad \sigma_p(H) = \{\lambda_j\}_{j=1}^N \subseteq \mathbb{R}\backslash[-1,1], \tag{3.7}$$

where $N \in \mathbb{N}$ is finite, and the existence of the so called Jost solutions $f_\pm(k,n,t)$,

$$\Big(\tau - \frac{k+k^{-1}}{2}\Big)f_\pm(k,n,t) = 0, \qquad \lim_{n\to\pm\infty} k^{\mp n} f_\pm(k,n,t) = 1, \quad |k|\le 1. \tag{3.8}$$

See (e.g.) [**15**] where a more general result for periodic rather than constant background operator H_0 is proven.

Transmission $T(k,t)$ and reflection $R_\pm(k,t)$ coefficients are then defined via

$$T(k,t)f_\mp(k,n,t) = f_\pm(k^{-1},n,t) + R_\pm(k,t)f_\pm(k,n,t), \quad |k|=1, \tag{3.9}$$

and the norming constants $\gamma_{\pm,j}(t)$ corresponding to $\lambda_j \in \sigma_p(H)$ are given by

$$\gamma_{\pm,j}(t)^{-1} = \sum_{n\in\mathbb{Z}} |f_\pm(k_j,n,t)|^2, \quad k_j = \lambda_j - \sqrt{\lambda_j^2-1} \in (-1,0)\cup(0,1). \tag{3.10}$$

Clearly we are interested how the scattering data vary with respect to t. But first we ensure that it suffices to check (3.6) for the initial condition.

Lemma 3.1. *Suppose $a(n,t)$, $b(n,t)$ is a solution of the Toda system satisfying (3.6) for one $t_0 \in \mathbb{R}$, then (3.6) holds for all $t \in \mathbb{R}$.*

For a proof see [**17**], Lemma 3.1 (for the semi-infinite Toda chain ($r=0$) see also [**8**], Proposition 4).

Theorem 3.2. *Suppose $a(n,t)$, $b(n,t)$ is a solution of the Toda system satisfying (3.6) for one (and hence for all) $t_0 \in \mathbb{R}$. The functions*

$$\exp(\pm\alpha_r(k)t)f_\pm(k,n,t) \tag{3.11}$$

satisfy (2.13) weakly (i.e., they are not in $\ell^2(\mathbb{Z})$*) with* $z = (k + k^{-1})/2$*. Here* $f_\pm(k, n, t)$ *are the Jost solutions and* $\alpha_r(k)$ *is defined in (3.3). In addition, we have*

$$\begin{aligned} T(k,t) &= T(k,0), \\ R_\pm(k,t) &= R_\pm(k,0)\exp(\pm\alpha_r(k)t), \\ \gamma_{\pm,\ell}(t) &= \gamma_{\pm,\ell}(0)\exp(\mp 2\alpha_r(k_\ell)t), \quad 1 \le \ell \le N. \end{aligned} \tag{3.12}$$

Proof. As in the proof of [**15**], Theorem 5.1, one shows that $f_\pm(k, n, t)$ is continuously differentiable with respect to t and that $\lim_{n\to\pm\infty} k^{\mp n}\dot{f}_\pm(k, n, t) \to 0$. Now let $(k + k^{-1})/2 \in \rho(H(t))$, then Lemmas 4.1 and 4.2 of [**16**] imply that the solution of (2.13) with initial condition $f_\pm(k, n, 0)$ is of the form $C_\pm(t)f_\pm(k, n, t)$. Inserting this into (2.13), multiplying with $k^{\mp n}$ and evaluating as $n \to \pm\infty$ yields $C_\pm(t) = \exp(\pm\alpha_r(k)t)$. The general result for all $|k| < 1$ now follows from continuity. This immediately implies the formulas for $T(k, t), R_\pm(k, t)$. Finally, let $k = k_\ell$, then we have

$$\exp(\pm\alpha_r(k_\ell)t)f_\pm(k_\ell, n, t) = U_r(t, 0)f_\pm(k_\ell, n, 0), \tag{3.13}$$

which implies

$$\frac{d}{dt}\frac{\exp(\mp 2\alpha_r(k_\ell)t)}{\gamma_{\pm,\ell}(t)} = \frac{d}{dt}\|U_r(t, 0)f_\pm(k_\ell, ., 0)\| = 0 \tag{3.14}$$

and concludes the proof. □

Thus the scattering data of $H(t)$ can be expressed in terms of those for $H(0)$. Now we need to know how to reconstruct $H(t)$ from its scattering data. We drop the dependence on t for notational convenience.

Expanding $f_+(k, n)$ with respect to k we obtain

$$f_+(k, n) = \frac{k^n}{A_+(n)}\Big(1 + \sum_{j=1}^{\infty} K_{+,j}(n)k^j\Big), \quad |k| \le 1, \tag{3.15}$$

where

$$A_+(n) = \prod_{m=n}^{\infty} 2a(m), \quad K_{+,1}(n) = -\sum_{m=n+1}^{\infty} 2b(m), \quad \text{etc. .} \tag{3.16}$$

Integrating (3.9) (for the upper sign) around the unit circle we obtain the Gel'fand-Levitan-Marchenko equation

$$(\mathbb{1} + \mathcal{F}_n^+)K_+(n) = A_+(n)^2\delta_0, \tag{3.17}$$

where

$$\mathcal{F}_n^+ f(j) = \sum_{m=0}^{\infty} F^+(2n + m + j)f(m), \quad f \in \ell^2(\mathbb{N}_0), \tag{3.18}$$

is the Gel'fand-Levitan-Marchenko operator. Here

$$F^+(n) = \tilde{F}^+(n) + \sum_{\ell=1}^{N} \gamma_{+,\ell}k_\ell^n \tag{3.19}$$

and

$$\tilde{F}^+(n) = \frac{1}{2\pi\mathrm{i}}\int_{|k|=1} R_+(k)k^n\frac{dk}{k} \in \ell^2(\mathbb{Z}, \mathbb{R}) \tag{3.20}$$

are the Fourier coefficients of $R_+(k^{-1})$. The following theorem collects some properties of the operator $\mathcal{F}_n^+$.

THEOREM 3.3. *Fix $n \in \mathbb{Z}$ and consider $\mathcal{F}_n^+ : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$. Then $\mathcal{F}_n^+$ is a self-adjoint trace class operator satisfying*

$$\mathbb{1} + \mathcal{F}_n^+ \geq \varepsilon_n > 0, \quad \lim_{n\to\infty} \varepsilon_n = 1. \tag{3.21}$$

The trace of $\mathcal{F}_n^+$ is given by

$$\mathrm{tr}(\mathcal{F}_n^+) = \sum_{j=0}^{\infty} F^+(2n+2j) + \sum_{\ell=1}^{N} \gamma_{+,\ell} \frac{k_\ell^{2n}}{1-k_\ell}. \tag{3.22}$$

PROOF. Let $f \in \ell^2(\mathbb{N}_0)$ and abbreviate $\hat{f}(k) = \sum_{j=0}^{\infty} f(j)k^j$. Setting $f(j) = 0$ for $j < 0$ we obtain

$$\sum_{j=0}^{\infty} \overline{f(j)} \mathcal{F}_n^+ f(j) = \frac{1}{2\pi \mathrm{i}} \int_{|k|=1} R_+(k) k^{2n} |\hat{f}(k)|^2 \frac{dk}{k} + \sum_{\ell=1}^{N} \gamma_{+,\ell} k_\ell^{2n} |\hat{f}(k_\ell)|^2 \tag{3.23}$$

from the convolution formula. Since $\overline{R_+(k)} = R_+(\overline{k})$ the integral over the imaginary part vanishes and the real part can be replaced by

$$\begin{aligned} \mathrm{Re}(R_+(k)k^{2n}) &= \frac{1}{2}\left(|1 + R_+(k)k^{2n}|^2 - 1 - |R_+(k)k^{2n}|^2\right) \\ &= \frac{1}{2}\left(|1 + R_+(k)k^{2n}|^2 + |T(k)|^2\right) - 1 \end{aligned} \tag{3.24}$$

(remember $|T(k)|^2 + |R_+(k)k^{2n}|^2 = 1$). This eventually yields the identity

$$\begin{aligned} \sum_{j=0}^{\infty} \overline{f(j)} (\mathbb{1} + \mathcal{F}_n^+) f(j) &= \sum_{\ell=1}^{N} \gamma_{+,\ell} k_\ell^{2n} |\hat{f}(k_\ell)|^2 \\ &+ \frac{1}{4\pi \mathrm{i}} \int_{|k|=1} \Big(|1 + R_+(k)k^{2n}|^2 + |T(k)|^2\Big) |\hat{f}(k)|^2 \frac{dk}{k}, \end{aligned} \tag{3.25}$$

which establishes $\mathbb{1} + \mathcal{F}_n^+ \geq 0$. In addition, by virtue of $|1 + R_+(k)k^{2n}|^2 + |T(k)|^2 > 0$ (a.e.), -1 is no eigenvalue and thus $\mathbb{1} + \mathcal{F}_n^+ \geq \varepsilon_n$ for some $\varepsilon_n > 0$. That $\varepsilon_n \to 1$ follows from $\|\mathcal{F}_n^+\| \to 0$.

To see that $\mathcal{F}_n^+$ is trace class we use the splitting $\mathcal{F}_n^+ = \tilde{\mathcal{F}}_n^+ + \sum_{\ell=1}^{N} \tilde{\mathcal{F}}_n^{+,\ell}$ according to (3.19). The operators $\tilde{\mathcal{F}}_n^{+,\ell}$ are positive and trace class. The operator $\tilde{\mathcal{F}}_n^+$ is given by multiplication with $k^{2n} R_+(k)$ in Fourier space and hence is trace class since $|R_+(k)| \leq 1$ is integrable. $\square$

Now we are able to explicitly invert the process of scattering theory. Clearly, if the scattering data (and thus $\mathcal{F}_n^+$) are given, we can use the Gel'fand-Levitan-Marchenko equation (3.17) to reconstruct $a(n)$, $b(n)$ from $\mathcal{F}_n^+$

$$\begin{aligned} a(n)^2 &= \frac{1}{4} \frac{\langle \delta_0, (\mathbb{1} + \mathcal{F}_n^+)^{-1} \delta_0 \rangle}{\langle \delta_0, (\mathbb{1} + \mathcal{F}_{n+1}^+)^{-1} \delta_0 \rangle}, \\ b(n) &= \frac{1}{2} \Big(\frac{\langle \delta_1, (\mathbb{1} + \mathcal{F}_n^+)^{-1} \delta_0 \rangle}{\langle \delta_0, (\mathbb{1} + \mathcal{F}_n^+)^{-1} \delta_0 \rangle} - \frac{\langle \delta_1, (\mathbb{1} + \mathcal{F}_{n-1}^+)^{-1} \delta_0 \rangle}{\langle \delta_0, (\mathbb{1} + \mathcal{F}_{n-1}^+)^{-1} \delta_0 \rangle} \Big). \end{aligned} \tag{3.26}$$

In other words, the scattering data of $H(t)$ uniquely determine $a(t)$, $b(t)$. Since $\mathcal{F}_n^+$ is trace class, we can use Kramer's rule to express the above scalar products.

If we delete the first row and first column in the matrix representation of $\mathbb{1} + \mathcal{F}_n^+$ we obtain $\mathbb{1} + \mathcal{F}_{n+1}^+$. If we delete the first row and second column in the matrix representation of $\mathbb{1} + \mathcal{F}_n^+$ we obtain an operator $\mathbb{1} + \mathcal{G}_n^+$. By Kramer's rule we have

$$\begin{aligned} \langle \delta_0, (\mathbb{1} + \mathcal{F}_n^+)^{-1} \delta_0 \rangle &= \frac{\det(\mathbb{1} + \mathcal{F}_{n+1}^+)}{\det(\mathbb{1} + \mathcal{F}_n^+)}, \\ \langle \delta_1, (\mathbb{1} + \mathcal{F}_n^+)^{-1} \delta_0 \rangle &= \frac{\det(\mathbb{1} + \mathcal{G}_n^+)}{\det(\mathbb{1} + \mathcal{F}_n^+)}, \end{aligned} \tag{3.27}$$

where the determinants have to be interpreted as Fredholm determinants.

In summary, we have the following procedure:

(1) Compute the Jost solutions $f_\pm(k, n, 0)$ (e.g.) by iterating the corresponding Volterra sum equation used to prove existence of the Jost solutions in [**15**]. This gives the scattering data for $H(0)$.
(2) Read off the scattering data of $H(t)$ from Theorem 3.2.
(3) Compute the Fourier coefficients of $R_+(k, t)$ and use (3.27) to construct $a(n, t)$, $b(n, t)$.

Since we have ensured existence of a solution in the outset (Theorem 2.3 and Lemma 3.1), the sequences constructed by this procedure satisfy the Toda equations.

In the case $r = 0$ the inverse scattering procedure was first established by Flaschka [**9**]. In addition, Flaschka also worked out the inverse procedure in the reflection-less case (i.e., $R_\pm(k, t) = 0$). His formulas clearly apply to the entire Toda hierarchy upon using the t dependence of the norming constants given in (3.12). In addition, these formulas are the same as the ones obtained using the double commutation method (cf. [**16**]).

In the case of the semi-infinite Toda chain an alternative method based on the moment problem is presented in [**2**], [**3**]. This method can also be generalized to solve some semi-infinite non-isospectral flows related to the Toda system [**4**], [**5**]. By choosing $a(n_0, 0) = 0$ for one fixed $n_0 \in \mathbb{Z}$ (implying $a(n_0, t) = 0$), the Toda chain splits into two semi-infinite Toda chains. Hence the results presented here apply to the semi-infinite Toda chain as well.

4. The Kac-van Moerbeke hierarchy and its relation to the Toda hierarchy

In this section we review some basic properties of the Kac-van Moerbeke hierarchy and its connection with the Toda hierarchy.

Suppose $\rho(t)$ satisfies

Hypothesis **H.** 4.1. *Let*

$$\rho(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \quad \rho(n, t) \neq 0, \ (n, t) \in \mathbb{Z} \times \mathbb{R} \tag{4.1}$$

and let $t \mapsto \rho(t)$ be Fréchet differentiable in the Banach space $\ell^\infty(\mathbb{Z})$.

Define the "even" and "odd" parts of $\rho(t)$ by

$$\rho_e(n, t) = \rho(2n, t), \ \rho_o(n, t) = \rho(2n + 1, t), \quad (n, t) \in \mathbb{Z} \times \mathbb{R}, \tag{4.2}$$

and consider the bounded operators (in $\ell^2(\mathbb{Z})$)

$$A(t) = \rho_o(t) S^+ + \rho_e(t), \ A(t)^* = \rho_o^-(t) S^- + \rho_e(t). \tag{4.3}$$

In addition, we set

$$H_1(t) = A(t)^* A(t), \quad H_2(t) = A(t)A(t)^*, \tag{4.4}$$

with

$$H_k(t) = a_k(t)S^+ + a_k^-(t)S^- + b_k(t), \qquad k = 1, 2, \tag{4.5}$$

and

$$\begin{aligned} a_1(t) &= \rho_e(t)\rho_o(t), & b_1(t) &= \rho_e(t)^2 + \rho_o^-(t)^2, \\ a_2(t) &= \rho_e^+(t)\rho_o(t), & b_2(t) &= \rho_e(t)^2 + \rho_o(t)^2. \end{aligned} \tag{4.6}$$

Now we define operators $D(t)$, $Q_{2r+2}(t)$ (the Lax pair) in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ as follows,

$$\begin{aligned} D(t) &= \begin{pmatrix} 0 & A(t)^* \\ A(t) & 0 \end{pmatrix}, \\ Q_{2r+2}(t) &= \begin{pmatrix} P_{1,2r+2}(t) & 0 \\ 0 & P_{2,2r+2}(t) \end{pmatrix}, \end{aligned} \tag{4.7}$$

$r \in \mathbb{N}_0$. Here $P_{k,2r+2}(t)$, $k = 1, 2$, are defined as in (2.4), that is,

$$P_{k,2r+2}(t) = -H_k(t)^{r+1} + \sum_{j=0}^{r} (2a_k(t)g_{k,j}(t)S^+ - h_{k,j}(t))H_k(t)^j + g_{k,r+1},$$

$$P_{k,2r+2}(t)\Big|_{\mathrm{Ker}(\tau_k(t)-z)} = 2a_k(t)G_{k,r}(z,t)S^+ - H_{k,r+1}(z,t), \tag{4.8}$$

where $(g_{k,j}(n,t))_{0\le j\le r}$, $(h_{k,j}(n,t))_{0\le j\le r+1}$, are defined as in (2.3), and the polynomials $G_{k,r}(z,n,t)$, $H_{k,r+1}(z,n,t)$ are defined as in (2.6). Moreover, we choose the same integration constants in $P_{1,2r+2}(t)$ and $P_{2,2r+2}(t)$ (i.e., $c_{1,\ell} = c_{2,\ell} \equiv c_\ell$, $1 \le \ell \le r$).

Analogous to equation (2.7) one obtains that

$$\frac{d}{dt} D(t) - [Q_{2r+2}(t), D(t)] = 0 \tag{4.9}$$

is equivalent to

$$\begin{aligned} \underline{\mathrm{KM}}_r(\rho) &= (\mathrm{KM}_r(\rho)_e, \, \mathrm{KM}_r(\rho)_o) \\ &= \begin{pmatrix} \dot\rho_e - \rho_e(g_{2,r+1} - g_{1,r+1}) \\ \dot\rho_o + \rho_o(g_{2,r+1} - g_{1,r+1}^+) \end{pmatrix} = 0. \end{aligned} \tag{4.10}$$

As in the Toda context (2.10), varying $r \in \mathbb{N}_0$ yields the Kac-van Moerbeke hierarchy (KM hierarchy) which we denote by

$$\mathrm{KM}_r(\rho) = 0, \quad r \in \mathbb{N}_0. \tag{4.11}$$

Again the Lax equation (4.9) implies ([**16**], Theorem 3.2)

THEOREM 4.2. *Let ρ satisfy (H.4.1) and* $\mathrm{KM}(\rho) = 0$. *Then the Lax equation (4.9) implies the existence of a unitary propagator $V_r(t,s)$ such that we have*

$$D(t) = V_r(t,s)D(s)V_r(t,s)^{-1}, \qquad (t,s) \in \mathbb{R}^2. \tag{4.12}$$

Thus all operators $D(t)$, $t \in \mathbb{R}$, are unitarily equivalent.

And as in Theorem 2.3 we infer ([**16**], Theorem 3.3)

THEOREM 4.3. *Suppose $\rho_0 \in \ell^\infty(\mathbb{Z})$. Then there exists a unique integral curve $t \mapsto \rho(t)$ in $C^\infty(\mathbb{R}, \ell^\infty(\mathbb{Z}))$ of the Kac-van Moerbeke equations, that is,* $\mathrm{KM}_r(\rho) = 0$, *such that $\rho(0) = \rho_0$.*

As a simple consequence of Theorem 4.2 we have

$$\frac{d}{dt}D(t)^2 - [Q_{2r+2}(t), D(t)^2] = 0 \tag{4.13}$$

and observing

$$D(t)^2 = \begin{pmatrix} H_1(t) & 0 \\ 0 & H_2(t) \end{pmatrix} \tag{4.14}$$

yields the implication

$$\mathrm{KM}_r(\rho) = 0 \Rightarrow \mathrm{TL}_r(a_k, b_k) = 0, \quad k = 1, 2. \tag{4.15}$$

That is, given a solution ρ of the KM_r equation (4.11), one obtains two solutions, (a_1, b_1) and (a_2, b_2), of the TL_r equations (2.10) related to each other by the Miura-type ([**14**]) transformations (4.6). Note that due to (H.4.1), (a_1, b_1) and (a_2, b_2) both fulfill (H.2.1).

Since we already know how to solve the initial value problem for the Toda equation, it would be nice if one could use this knowledge to solve the initial value problem for the Kac-van Moerbeke equation. To do this we need to invert the above transformation. This is our next goal.

Suppose $\rho(n, t)$ is a solutions of the KM_r equation and let

$$a(n,t) = \rho_e(n,t)\rho_o(n,t), \qquad b(n,t) = \rho_e(n,t)^2 + \rho_o^-(n,t)^2 - \lambda \tag{4.16}$$

be a corresponding solution of the TL_r equation. Here, $\lambda \in \mathbb{R}$ is arbitrary. Then one can verify ([**16**], Theorem 3.4) that

$$\begin{aligned} u(\lambda, n, t) \quad = \quad & \exp\Big(\int_0^t (-2a(0,x)\frac{g_r(0,x)\rho_e(0,x)}{\rho_o(0,x)} - h_r(0,x) \\ & + g_{r+1}(0,x))dx\Big) \begin{cases} \prod\limits_{m=0}^{n-1} \frac{-\rho_e(m,t)}{\rho_o(m,t)} & \text{for } n > 0 \\ 1 & \text{for } n = 0 \\ \prod\limits_{m=n}^{-1} \frac{\rho_o(m,t)}{-\rho_e(m,t)} & \text{for } n < 0 \end{cases} \end{aligned} \tag{4.17}$$

is a solutions of (2.13) for $z = \lambda$. Conversely, let $\rho(n, 0)$ be given such that $a(n, 0)$, $b(n, 0)$ defined as in (4.16) satisfy (3.6). Solving the TL_r equation with this initial condition via the IST gives $a(n, t)$, $b(n, t)$. Moreover, since $f_\pm(k, n, 0)$ are linearly independent, we can write $u(\lambda, n, 0) = C_- f_-(k, n, 0) + C_+ f_+(k, n, 0)$. Hence we infer by Theorem 3.2 that

$$u(\lambda, n, 0) = C_- \exp(-\alpha_r(k)t) f_-(k, n, t) + C_+ \exp(\alpha_r(k)t) f_+(k, n, t), \tag{4.18}$$

where $f_+(k, n, t)$ is given by (3.15) with

$$K_{+,j}(n,t) = A_+(n,t)^2 \langle \delta_j, (\mathbb{1} + \mathcal{F}_n^+(t))^{-1}\delta_0 \rangle \tag{4.19}$$

and a similar expression for $f_-(k, n, t)$. Then $\rho(n, t)$ defined by

$$\rho_o(n,t) = -\sqrt{-\frac{a(n,t)u(\lambda,n,t)}{u(\lambda,n+1,t)}}, \quad \rho_e(n,t) = \sqrt{-\frac{a(n,t)u(\lambda,n+1,t)}{u(\lambda,n,t)}} \tag{4.20}$$

is the solution of the KM_r equation corresponding to the initial condition $\rho(n, 0)$.

For a more detailed investigation of the connection between the TL_r and KM_r hierarchies we refer to [**16**] and the references therein.

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