ON VECTOR AND MATRIX RIEMANN–HILBERT PROBLEMS FOR KDV SHOCK WAVES

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Abstract. This paper discusses some general aspects and techniques associated with the long-time asymptotics of steplike solutions of the Korteweg–de Vries (KdV) equation via vector Riemann–Hilbert problems. We also elaborate on an ill-posedness of the matrix Riemann–Hilbert problems for the KdV case in the class of matrices with $L^2$ integrable singularities. To the best of our knowledge this is the first time such ill-posedness is discussed in applications of Riemann–Hilbert theory. Furthermore, we rigorously justify the asymptotics for the shock wave in the elliptic zone derived previously.

1. Vector and matrix R-H problems associated with the KdV equation

The nonlinear steepest descent (NSD) analysis for oscillatory Riemann–Hilbert (R-H) problems is a versatile tool in asymptotic analysis. This procedure naturally starts from a reformulation of the original scattering problem as a R-H factorization problem. In most cases this will be a matrix R-H problem as these are typically more convenient to analyze. Indeed the fact that a nonsingular solution can be used to cancel jumps on certain parts of the contour is a crucial trick which lies at the heart of the theory. However, for some problems, most prominently the Korteweg–de Vries equation

\[ q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \]

it turned out that a vector R-H is the right choice. This is related to the fact that even in the simplest case of a single soliton there is a nontrivial solution of the associated vanishing problem. However, this is in contradiction to the classical uniqueness result for matrix R-H problems and shows that the matrix problem cannot have a solution in this situation. The remedy, as pointed out in [18], is to work with the vector R-H problem and impose an additional symmetry condition to retain uniqueness.

Next, recall that the asymptotic analysis of such a R-H problem usually consists of three steps: The first step deforms the problem in such a way that the leading asymptotic contribution is revealed. In the second step the parts of the jump which are expected to not contribute to the leading asymptotics are dropped yielding a model problem which then needs to be solved explicitly. Again, in the past it was always possible to find a matrix solution to this model problem and hence the final step, namely showing that the solution of the model problem indeed asymptotically

2000 Mathematics Subject Classification. Primary 37K40, 35Q53; Secondary 37K45, 35Q15.

Key words and phrases. Riemann–Hilbert problem, KdV equation, shock wave.

Research supported by the Austrian Science Fund (FWF) under Grants No. P31651 and W1245.
approximates the solution of the original R-H problem, could be performed using
the well-established tools for matrix problems.

It is the main purpose of the present note to show that this is not always the case
and that there are situations where the model problem does not have a (nonsingular)
matrix solution (at least for certain exceptional values of the parameters \(x\) and \(t\)).
To our best knowledge this observation was not discussed previously. We will
explain the abstract situation in Theorem 1.1 below and then discuss a specific
elementary example in some detail.

As our example for a more detailed discussion we choose the problem of shock
waves, that is the initial value problem for (1.1) with the initial condition \(q(x,0) =
\) satisfying:
\[
\begin{array}{ll}
q(x) \to 0, & \text{as } x \to +\infty, \\
q(x) \to -c^2, & \text{as } x \to -\infty, \quad c > 0.
\end{array}
\]

We recall that the asymptotic behavior of the solutions was first described on a
physical level of rigor in the pioneering work by Gurevich and Pitayevskii [19], [20].
By applying the Whitham approach to the pure step initial data \(q(x) = 0\) for
\(x > 0\) and \(q(x) = -c^2\) for \(x \leq 0\) the authors derived the leading asymptotics in
terms of a modulated elliptic wave. For arbitrary steplike initial data (1.2) the
analogous asymptotic term was calculated in [11], [13] by use of the NSD approach.
In particular, it was shown that in the elliptic zone \(-6c^2t < x < 4c^2t\) the shock
wave is expected to be close, as \(t \to +\infty\), to a modulated one gap solution of
the KdV equation. The same formula appears as the leading asymptotic term for
the so called soliton gas [16]. However, all these asymptotics are not rigorously
justified yet. The goal of this paper is to complete the asymptotic analysis for the
shock wave in the frameworks of the standard NSD method and to clarify which
restrictions on the justification process appear due to the absence of a bounded
(except of the point of discontinuity of the contour where the \(L^2\) singularities are
admissible) invertible matrix model solution. This difficulty is absent, for example,
in the decaying case or for rarefaction waves (\(q(x) \to 0\) as \(x \to +\infty\) and \(q(x) \to c^2\)
as \(x \to -\infty\)).

To describe in more details for which pairs \((x,t)\) the matrix model solution does
not exist, recall the trace formula for a finite gap KdV solution. Namely, denote
by \(\xi = \frac{x}{2t}\) the slowly varying parameter of the problem. In the domain under
consideration we assume that
\[
(1.3) \quad \xi \in \mathcal{I}_\varepsilon := \left[-\frac{c^2}{2} + \varepsilon, \frac{c^2}{3} - \varepsilon\right].
\]
Then, as is shown in [19], [11], there exists a smooth monotonously increasing
positive function \(a = a(\xi)\) such that \(a(-\frac{c^2}{2}) = 0\) and \(a(\frac{c^2}{3}) = c\). This function
characterizes the Whitham zone of the modulated elliptic wave \(q_{\text{mod}}(x,t,\xi)\), which
is on the ray \(\xi = \text{const}\), the periodic one gap solution of the KdV equation, asso-
ciated with the spectrum \([-c^2, -a^2(\xi)]\) \(\cup \mathbb{R}_+\) and with the initial Dirichlet divisor
\((\lambda(0,0,\xi), \pm)\) defined via the scattering data of the potential (1.2) by formulas
(4.13) and (2.27) below.
Let \(\lambda(x,t,\xi) \in [-a^2(\xi), 0]\) be the solution of the Dubrovin equations (24,
Ch. 12) corresponding to the initial value \((\lambda(0,0,\xi), \pm)\). Then the trace formula
reads
\[
(1.4) \quad q_{\text{mod}}(x,t,\xi) = -c^2 - a(\xi)^2 - 2\lambda(x,t,\xi).
\]
We will show that the set of local minima of this solution:
\[ O(\xi) = \{(x, t) : \lambda(x, t, \xi) = 0\}, \]
coincides with the set of points where the associated matrix model problem has no nonsingular solution. This means that for each \( \xi \in I_\varepsilon \) the determinant of any matrix model solution (it does not depend on \( k \)) vanishes as \((x, t)\) tends to \( O(\xi) \). However, using the usual techniques, a rigorous asymptotic analysis can be performed away from \( \cup_{\xi \in I_\varepsilon} O(\xi) \). To this end introduce
\[
D_\varepsilon = \{(x, t) : \xi \in I_\varepsilon, \text{ and } \lambda(x, t, \xi) < -\frac{1}{t^{\frac{1}{2}}}, \}
\]
where \( \frac{1}{2} > \gamma = \gamma(\varepsilon) > 0 \) is an arbitrary positive number. Then taking into account the decay rate with respect to \( t \) of the determinant of the model matrix solution in the domain \( D_\varepsilon \), we will prove (Theorem 7.2) that uniformly with respect to \( \xi \in I_\varepsilon \) the asymptotics
\[
q(x, t) = q^{\text{mod}}(x, t, \xi) + O(t^{-1+\gamma})
\]
is valid, where \( q(x, t) \) is the solution of (1.1)–(1.2). Formula (1.6) is obtained in the framework of a standard approach which consists of constructing the proper matrix model solution and the associated matrix solution of the parametrix problems in vicinities of \( \pm ia(\xi) \). However, when performing this analysis for the KdV steplike case it is essential to take into account some specific features of the vector R-H problems. Note that unlike the matrix R-H problem, the proof of uniqueness for a vector R-H problem is typically more sophisticated and depends on particular properties of the jump matrix and of the contour, as far as on a class of admissible singularities for the solution. That is why it seems important for us to perform NSD deformations and conjugations in a way that does not affect this uniqueness. To this end, in each transformation we follow some additional symmetry restrictions on the contour, on the jump matrix and on the solution itself, including the model problem solution. The initial R-H problem solution is unique (see Theorem 2.1) and possesses all these symmetries. This requirement allows us to provide symmetries for the "error vector". In turns, it allows us to apply a new formula (see formula (1.10) below) for computing the main term of asymptotics which simplifies essentially the final asymptotical analysis.

At the end of this introductory section we single out the circumstances which lead to the non-existence of nonsingular matrix solutions for the R-H problems for certain (arbitrary large) points \((x, t)\) in certain regions of the space-time halfplane. To the best of our knowledge this went unnoticed so far. We establish this property for the case of the shock wave, however, it remains true for the more general case of the initial data which are decaying on the right half axis and which are asymptotically finite gap on the left half axis, with at least one band of the finite band spectrum inside the interval \((-\infty, 0)\). In such a steplike asymptotically finite gap case, the jump contour for R-H problem corresponding to the continuous spectrum of the underlying Schrödinger equation
\[
L(t)y = -\frac{d^2}{dx^2}y + q(x, t)y = k^2y,
\]
and considered in terms of variable \( k \), consists of a few symmetric with respect to the map \( k \mapsto -k \) closed intervals located on \( i\mathbb{R} \), plus the real axis \( \mathbb{R} \). If the operator \( L(t) \) has a discrete spectrum \( \{-\kappa_j^2\}_1^N \), which is supposed to be finite, then
it is located in open gaps of the left background spectrum situated below origin. In this case, as a part of the contour the small nonintersecting circles around points ±ικj, situated in gaps are added. They are also symmetric with respect to the map k → −k. Denote the whole contour by Σ. The real axis is oriented from left to right, and the parts Σ ∩ iR from up to down. The circles are oriented counterclockwise. Suppose for simplicity that there are no resonances on the edges of the continuous spectrum.

The scattering data of operator \( L(t) \) defines then a piecewise continuous \( 2 \times 2 \) matrix-valued function \( v(k) = v(k, x, t) \) on Σ with \( \det v(k) \equiv 1 \), such that

\[
v(-k) = \sigma_1 v(k)^{-1} \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The initial vector R-H problem under consideration has the following statement:

Find a vector-valued function \( m(k, x, t) := m(k) = (m_1(k), m_2(k)) \) holomorphic in the domain \( \mathbb{C} \setminus \Sigma \), which has bounded limits \( m_\pm(k) \) on \( \Sigma \) and satisfies

(i) the jump condition \( m_+(k) = m_-(k)v(k), \ k \in \Sigma \);

(ii) the symmetry condition \( m(-k) = m(k)\sigma_1, \ k \in \mathbb{C} \setminus \Sigma \);

(iii) the normalization condition \( m(k) \to (1, 1) \) as \( k \to \infty \).

Here \( x \) and \( t \) are treated as large parameters. Recall that the traditional formula which connects the potential \( q(x, t) \) with the solution of the initial R-H problem (i)–(iii) is the following one:

\[
\frac{\partial}{\partial x} \lim_{k \to \infty} 2ik(m_1(k) - 1) = q(x, t).
\]

In Section 2 we establish the more convenient formula

\[
q(x, t) = \lim_{k \to \infty} 2k^2 (m_1(k)m_2(k) - 1).
\]

We emphasize that formula (1.10) not only avoids the necessity to justify the differentiation in (1.9) in an asymptotical expansion but also allows to extract the asymptotics from the model vector R-H solution in a shorter way (see Section 4).

Alongside with the vector R-H problem (i)-(iii), we can also give a matrix statement for the R-H problem with the same jump matrix \( v(k) \) given by (1.8). This can be done in two ways, by use of a symmetry condition (see (1.14) below) or by use of the standard normalization by the unit matrix \( I \) at infinity. Simultaneous use of both conditions may seem to be excessive. In fact, we observe the following.

Let Σ ⊂ \( \mathbb{C} \) be a union of finitely many smooth curves (finite or infinite) which intersect in at most a finite number of points and all intersections are transversal (this condition can of course be relaxed, but it is sufficient for the applications we have in mind). We will also require Σ to be symmetric with respect to the reflection \( k \mapsto -k \). Denote by \( \mathcal{G} \) the (finite) set of boundary points \( \partial \Sigma \) together with the node (intersection) points of Σ. We assume that point 0 is an interior point of Σ, that is \( 0 \notin \mathcal{G} \).

Let now \( v(k) \) be a piecewise continuous bounded matrix function on Σ satisfying (1.8) with \( \det v(k) \equiv 1 \). The points of discontinuity of the jump matrix are also treated as node points and belong to \( \mathcal{G} \).

Finally, let \( \mathcal{H} \) be the class of \( 2 \times 2 \) matrix functions \( M(k) \) holomorphic in \( \mathbb{C} \setminus \Sigma \), which have continuous limits up to the boundary \( \Sigma \setminus \mathcal{G} \) and which have bounded
limits as $k \to \infty$ (avoiding $\Sigma$). At points of $\mathcal{G}$ we allow the following singularities:

$$M(k) = O((k - \kappa)^{-1/4}), \quad k \to \kappa \in \mathcal{G}.$$  

(1.11)

Now for an admissible $M \in \mathcal{H}(\Sigma)$ we consider the following R-H factorization problem

$$M_+(k) = M_-(k) v(k), \quad k \in \Sigma,$$

together with the normalization condition

$$M(\infty) := \lim_{k \to \infty} M(k) = I$$

and the symmetry condition

$$M(-k) = \sigma_1 M(k) \sigma_1, \quad k \in \mathbb{C} \setminus \Sigma.$$

**Theorem 1.1.** Suppose $\Sigma \subset \mathbb{C}$ is an admissible contour and $v(k), k \in \Sigma$ an admissible matrix as specified before. Then the following propositions are valid:

(a) If a solution $M \in \mathcal{H}(\Sigma)$ of (1.12) exists for which $M(\infty)$ is nonsingular, that is \(\det M(\infty) \neq 0\), then $M(\infty)\ M(k)$ solves (1.12)–(1.13), and every other solution of (1.12) is given by $M(k) = M(\infty)M(\infty)^{-1}M(k)$ in this case. Moreover, \(\det M(k) = \det M(\infty)\).

(b) If (1.12) has a nonsingular solution, then every solution $M \in \mathcal{H}(\Sigma)$ of (1.12) satisfies the symmetry condition (1.14) provided $M(\infty)$ satisfies the symmetry condition. In this case $M$ is of the form

$$M(k) = \begin{pmatrix} \alpha(k) & \beta(k) \\ \beta(-k) & \alpha(-k) \end{pmatrix}, \quad M(\infty) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with \(\det M(\infty) = a^2 - b^2\). If $M$ is nonsingular then $a + b \neq 0$.

(c) Suppose (1.12) has a nonsingular solution $M$ satisfying (1.14). Then

$$m(k) = \frac{1}{a + b}(1, 1) M(k) = \frac{1}{a + b}(\alpha(k) + \beta(-k), \beta(k) + \alpha(-k)).$$

(1.15)

solves the vector R-H problem (i)–(iii). Moreover, in this case $m$ is the unique solution of the vector R-H problem (i)–(iii) with admissible singularities of the type (1.11).

(d) Suppose the problem (i)–(iii) has a solution $m$ which satisfies the condition $m_+(0) = (0, 0)$. Then there is no invertible solution of the problem (1.12), (1.14) in $\mathcal{H}(\Sigma)$.

**Proof.** (a). This follows similarly as in [6, Theorem 7.18].

(b). Let $M(k) \in \mathcal{H}(\Sigma)$ be the solution of the problem (1.12)–(1.13). By (a) it suffices to show that $M$ satisfies (1.14). To this end set $\tilde{M}(k) = \sigma_1 M(-k) \sigma_1$. Then $\tilde{M}(\infty) = I$ and $\tilde{M}(k) \in \mathcal{H}$. Taking into account symmetry of $\Sigma$ and (1.14) we see that

$$\tilde{M}_+(k) = \sigma_1 M_-(k) \sigma_1 = \sigma_1 M_+(k) v^{-1}(-k) \sigma_1 = \sigma_1 M_+(k) v k.$$

Thus $\tilde{M}(k)$ solves (1.12)–(1.13) and by uniqueness, $\tilde{M}(k) \equiv M(k)$. This proves (1.14). The rest is straightforward.

(c). By assumption we have a solution $M$ as in (b) and hence one easily checks that $m$ satisfies (i)–(iii). If $\tilde{m}$ is a second solution, then as in (a) we see that (i) implies that $c = \tilde{m}(k) M^{-1}(k)$ is a constant vector. Hence by (iii) we see $c = \frac{1}{a + b}(1, 1)$. 
(d). Suppose that there exists an invertible symmetric matrix $M(k)$ satisfying (1.12). Without loss of generality we can assume $M(\infty) = I$ and hence by the previous item our assumption implies $m_+(0) = (\alpha_+(0), \beta_+(0), \beta_+(0) + \alpha_-(0)) = (0,0)$. Consequently

$$M_+(0) = \begin{pmatrix} \alpha_+(0) & \beta_+(0) \\ -\alpha_+(0) & -\beta_+(0) \end{pmatrix}$$

implying $\det M(k) = \det M_+(0) = 0$. □

In particular, item (d) implies that any technique relying on existence of a bounded nonsingular matrix solution is bound to fail at all points in the $(x,t)$ plane where $m_+(0) = (0,0)$ holds. In fact, it turns out that for the matrix R-H problem associated with the KdV steplike initial data, one can find arbitrary large pairs $(x,t)$ such that this condition holds (see Remark 2.6 below).

Moreover, even for the one-soliton case this occurs as pointed out in the discussion after Lemma 2.5 in [18]. It may happen also for arbitrary fast decaying KdV solutions with nonempty discrete spectrum $-\kappa_j^2$, $j = 1, \ldots, n$. Indeed, let $\psi_{\pm}(k,x,t)$ be the Jost solutions of the Schrödinger equation $L(t)\psi = k^2 \psi$ with the decaying potential $q(x,t)$ ($q(x,t) \to 0$ as $x \to \pm \infty$ for every $t$), which are normalized according to $\psi_{\pm}(k,x,t)e^{\mp ik_0 t} \to 1$ as $x \to \pm \infty$. Let $W(k)$ be the Wronskian of these solutions. Recall that it does not depend on $x$ and its zeros $\kappa_j$ do not depend on $t$. Suppose for simplicity that $W(0) \neq 0$, i.e. there is no resonance at the edge of the spectrum. Let $T(k) = \frac{2ik}{W(k)}$ be the right transmission coefficient and $R(k) = R(k,0)$ be the right initial reflection coefficient. Consider the vector function

$$m(k) := m(k,x,t) = (T(k)\psi_-(k,x,t)e^{ikx}, \psi_+(k,x,t)e^{-ikx}), \quad k \in \mathbb{C}^+;$$

$$m(k) = m(-k)\sigma_1, \quad k \in \mathbb{C}^-.$$  

Then $m(k)$ is a meromorphic function away from the real axis with simple poles at $\pm i\kappa_j$ and with the jump

$$v(k) = \begin{pmatrix} 1 - |R(k)|^2 & -R(k)e^{-2ikx-ik^3t} \\ R(k)e^{2ikx+ik^3t} & 1 \end{pmatrix}, \quad k \in \mathbb{R},$$

along the real axis oriented from the left to the right. Let $T_j^\pm$ be small nonintersecting circles around points $\pm i\kappa_j$ oriented counterclockwise. The transformation of $m(k)$ inside these circles mentioned above (cf. [7, 18]) make this function holomorphic with additional jumps $v(k)$ along $T_j^\pm$ satisfying (1.8). We preserve for this sectionally holomorphic vector function the same notation $m(k)$. Function (1.16) also satisfies (1.8), and therefore $m(k)$ is the unique solution (see [18]) of the vector R-H problem (i)-(iii) with the jump $v(k)$ on $\Sigma = \mathbb{R} \cup (T_j^+ \cup T_j^-)$. Note that $T(0) = 0$. Thus the respective matrix R-H problem will be ill-posed for all those $x$ and $t$ for which $\psi_+(0,x,t) = 0$ (see also Remarks 2.2 and 2.6 for the steplike case).

Note that in case without discrete spectrum the Jost functions are positive below the spectrum (cf. [15] Corollary 2.4) and hence also at the boundary of the spectrum $k = 0$ by continuity (note that the zeros of a nontrivial solution of a Sturm–Liouville equation must always be simple). Thus a nonsingular matrix solution always exists in this situation (this also follows from [27] Theorem 9.3).
In connection with this observation an additional spectral problem appears: to find conditions which would guarantee that the Jost solution vanishes at the edge of the spectrum for sufficiently large \( x \) and \( t \).

2. Uniqueness of the initial vector R-H problem solution

We start with recalling the statement of the initial R-H vector problem for the KdV shock wave (see [11]). Assume that the initial data (1.2) are sufficiently smooth with \( x^6 q^{(i)}(x) \in L^1(\mathbb{R}) \) and decay exponentially to the background constants:

\[
\int_0^{+\infty} e^{(c+\eta)x} (|q(x)| + |q(-x) + c^2|) dx < \infty,
\]

where \( \eta > 0 \) is arbitrary small. We choose this quite restrictive condition to avoid complications with analytical continuation of the scattering data in the frameworks of NSD. However, this condition also provides the existence of the unique classical solution \( q(x,t) \) for the Cauchy problem (1.1)–(1.2) (cf. [14, 17]) satisfying

\[
\int_0^{+\infty} |x|(|q(x,t)| + |q(-x,t) + c^2|) dx < \infty, \quad t \in \mathbb{R}.
\]

In turn, this means that the use of the inverse scattering transform for the formulation of the respective R-H problem is well grounded.

The spectrum of the associated Schrödinger operator (1.7) consists of an absolutely continuous part \([-c^2, \infty)\) plus a finite number of eigenvalues \(-\kappa_j^2 \in (-\infty, -c^2)\), \(1 \leq j \leq N\). Since the influence of the discrete spectrum in the domain \( x_1 \in I_\varepsilon \) (cf. (1.3)) on the asymptotic formulas and NSD analysis is exponentially small, we assume for simplicity that the discrete spectrum is empty. Let \( \phi(k,x,t) \) and \( \phi_1(k,x,t) \) be the Jost solutions of the equation

\[
L(t)\psi(k,x,t) = k^2\psi(k,x,t), \quad \text{Im}(k) > 0,
\]

which asymptotically look like the free solutions of the background equations

\[
\lim_{x \to +\infty} e^{-ikx} \phi(k,x,t) = 1, \quad \lim_{x \to -\infty} e^{ik_1x} \phi_1(k,x,t) = 1.
\]

Here \( k_1 = \sqrt{k^2 + c^2} \), and \( k_1 > 0 \) for \( k \in [0,ic) \). The last notation means the right side of the cut along the interval \([0,ic]\). Accordingly, \( k_1 < 0 \) for \( k \in [0,ic)_l \), the left side of the cut. Recall that the Jost solutions admit the usual representation via the transformation operators, in particular,

\[
\phi_1(k,x,t) = e^{-ik_1x} + \int_{-\infty}^{x} K_1(x,y,t)e^{-ik_1y} dy,
\]

where \( K_1(x,y,t) \) is a real valued function with

\[
|K_1(x,y,t)| \leq C \int_{-\infty}^{x+y} |q(s,t) + c^2| ds.
\]

Note that the function \( \phi(k,x,t) \) is a holomorphic function of \( k \) in \( \mathbb{C}^+ \) and continuous up to the real axis. It is real-valued as \( k \in [0,ic] \), and does not have a jump on this interval. As for the function \( \phi_1(k,x,t) \), it is holomorphic in the domain \( \mathbb{C}^+ \setminus [0,ic) \) and continuous up to the boundary, where \( [\phi_1(k,x,t)]_r = [\phi_1(k,x,t)]_l \) for \( k \in [0,ic] \).
We observe that condition (2.1) together with (2.3) imply that for $t = 0$ the second left Jost solution:

$$\tilde{\phi}_1(k, x, 0) = e^{ik_1x} + \int_{-\infty}^{x} K_1(x, y, 0) e^{ik_1y} dy,$$

defined for $k_1 \in \mathbb{R}$, where $\tilde{\phi}_1 = \overline{\phi_1}$, admits an analytical continuation into the domain

$$\mathcal{V} = \{ k : 0 < \text{Im} k_1(k) < (c + \eta) \}.$$

Note that $\mathcal{V}$ is a neighbourhood of the contour $\mathbb{R} \cup [ic, 0)$. Then the limiting values satisfy

$$\begin{align*}
\phi_1(k, x, 0)|_r &= [\phi_1(k, x, 0)]_l, \\
\phi_1(k, x, 0)|_r &= [\phi_1(k, x, 0)]_l, \\
as k \in [0, ic].
\end{align*}$$

For $k \in \mathcal{V}$ introduce two Wronskians:

$$\begin{align*}
W(k) &= \phi_1(k, x, 0) \phi'(k, x, 0) - \phi'(k, x, 0) \phi(k, x, 0); \\
W_1(k) &= \overline{\phi}_1(k, x, 0) \phi'(k, x, 0) - \overline{\phi}'(k, x, 0) \phi(k, x, 0),
\end{align*}$$

where $f' = \frac{\partial}{\partial x} f$. Then by (2.5)

$$\begin{align*}
[W(k)]_r = [W_1(k)]_l = [\overline{W}(k)]_l = [\overline{W}_1(k)]_r.
\end{align*}$$

In $\mathcal{D}$ introduce also the function

$$\chi(k) = \frac{4kk_1}{W(k)\overline{W}_1(k)}.$$

From (2.6) it follows that its limiting values satisfy

$$\begin{align*}
[\chi(k)]_r &= i[\chi(k)], \\
[\chi(k)]_l &= -i[\chi(k)], \\
as k \in [0, ic].
\end{align*}$$

As the discrete spectrum is assumed to be empty, we conclude that $W(k) \neq 0$ as $k \neq ic$. However, unlike the case considered in [11], we admit the possible resonance at the point ic, that is, we do not assume the condition $W(ic) \neq 0$ corresponding to the nonresonant case. In the resonant case the Wronskian has a square root zero at $k = ic$ (12). Hence we conclude that in the nonresonant case

$$\begin{align*}
[\chi(k)]_r &= C(k - ic)^{1/2}(1 + o(1)),
\end{align*}$$

and in the resonant case

$$\begin{align*}
[\chi(k)]_r &= C(k - ic)^{-1/2}(1 + o(1)).
\end{align*}$$

Note that the function $[\chi(k)]$, $k \in [0, ic]$ and the right reflection coefficient $R(k)$, $k \in \mathbb{R}$ of the initial potential satisfying (1.2) and (2.1) constitute the minimal set of scattering data for our problem without the discrete spectrum.

Next, the Jost solutions (2.2) are connected by the scattering relation

$$T(k, t) \phi_1(k, x, t) = \phi(k, x, t) + R(k, t) \phi(k, x, t), \quad k \in \mathbb{R},$$

where $T(k, t) = \frac{e^{ik_1x}}{W(k, t)}$, $R(k, t)$ are the right transmission and reflection coefficients. According to our notations $R(k, 0) = R(k)$, $k \in \mathbb{R}$, and $|T(k, 0)|^2 = \frac{k}{k_1}[\chi(k)]_r$ for $k \in [0, ic]$. 

We define a vector-valued function \( m(k,x,t) = (m_1(k,x,t), m_2(k,x,t)) \), holomorphic in the spectral parameter \( k \in \mathbb{C} \setminus (\mathbb{R} \cup [-ic,ic]) \) where \( x,t \) are fixed parameters, as follows

\[
(2.11) \quad m(k,x,t) = \begin{cases} 
(T(k,t)\phi_1(k,x,t)e^{ikx}, \phi(k,x,t)e^{-ikx}) & , \quad k \in \mathbb{C}^+ \setminus (0,ic], \\
(m(-k,x,t)\sigma_1, & , \quad k \in \mathbb{C}^- \setminus [-ic,0].
\end{cases}
\]

It is known that this function has the following asymptotical behavior as \( k \to \infty \):

\[
m(k,x,t) = (1,1) - \frac{1}{2ik} \left( \int_{x}^{+\infty} q(y,t)dy \right) (-1,1) + O \left( \frac{1}{k^2} \right).
\]

This expansion allows us to restore the shock wave solution by formula (1.9). However, as was mentioned in the introduction, there is more convenient formula:

\[
(2.12) \quad q(x,t) = \lim_{k \to \infty} 2k^2(m_1(k,x,t)m_2(k,x,t) - 1),
\]

which can be computed by use of the well-known asymptotic formulas for the Weyl functions.

Indeed, it is known that for \( k \) large enough both functions \( \phi(k,x,t) \) and \( \phi_1(k,x,t) \) do not vanish for all \( x \) and \( t \). Thus,

\[
m_1(k,x,t)m_2(k,x,t) = T(k,t)\phi(k,x,t)\phi_1(k,x,t)
\]

\[
= \frac{2ik}{\phi_1(k,x,t) - \phi'_1(k,x,t)} = \frac{2ik}{m(k,x,t) - m_1(k,x,t)},
\]

where \( m \) and \( m_1 \) are the right and left Weyl functions corresponding to the potential \( q(x,t) \). For \( k \to \infty \) we have (cf. [4]):

\[
m(k,x,t) = ik + \frac{q(x,t)}{2ik} + \frac{f(x,t)}{4k^2} + O(k^{-3}),
\]

\[
m_1(k,x,t) = -ik - \frac{q(x,t)}{2ik} + \frac{f(x,t)}{4k^2} + O(k^{-3}).
\]

Thus,

\[
m_1(k,x,t)m_2(k,x,t) - 1 = \frac{2ik}{2ik + \frac{q(x,t)}{ik} + O(k^{-3})} - 1 = \frac{q(x,t)}{2k^2} + O(k^{-4}),
\]

which proves (2.12). \( \square \)

Let \( \Sigma \) be the contour consisting of the real axis oriented left to right and the vertical interval \([iic,-ic]\) oriented top-down. We are interested on the jump condition for the vector function \( m(k) = m(k,x,t) \) on this contour. To describe it, first continue the function \( \chi(k) \) into the lower half plane by

\[
(2.13) \quad \chi(k) = \chi(-k), \quad \text{which implies} \quad [\chi(k)]_+ = -i[\chi(k)], \quad k \in [0,-ic].
\]

Also introduce the phase function

\[
(2.14) \quad \Phi(k) = 4ik^3 + ik\frac{x}{t}, \quad k \in \mathbb{C}.
\]

The following uniqueness result is then valid:

**Theorem 2.1.** Let \( L(0) \) be the Schrödinger operator with the potential (1.2), satisfying (2.1). Assume that \( L(0) \) has no discrete spectrum. Let

\[
(2.15) \quad \{R(k), k \in \mathbb{R}; \quad \chi_+(k) = [\chi(k)]_+, \quad k \in [ic,0]\}
\]
be the minimal set of scattering data of the operator $L(0)$. Let $\Sigma = \mathbb{R} \cup [ic, -ic]$ be the contour oriented left-to-right $\cup$ top-down. Let $\Phi(k) = \Phi(k, x, t)$ be defined by formula (2.14). Then $m(k) = m(k, x, t)$ defined in (2.11) is the unique solution of the following vector Riemann–Hilbert problem:

find a vector-valued function $m(k)$ holomorphic away from $\Sigma$ and satisfying:

(i) The jump condition $m_+(k) = m_-(k)v(k)$

\[
v(k) = \begin{cases} 
\left( \frac{1 - |R(k)|^2}{R(k)e^{2\Phi(k)}} - \frac{R(k)e^{-2\Phi(k)}}{1} \right), & k \in \mathbb{R}, \\
\left( \begin{array}{cc} 1 \\
\chi_+(k)e^{2\Phi(k)} \end{array} \right), & k \in [ic, 0], \\
\sigma_1(v(-k))^{-1}\sigma_1, & k \in [0, -ic],
\end{cases}
\]

(2.16)

(ii) the symmetry condition

\[
m(-k) = m(k)\sigma_1, \quad k \in \mathbb{C} \setminus \Sigma,
\]

(iii) and the normalization condition

\[
\lim_{\kappa \to \infty} m(i\kappa) = (1 \ 1).
\]

(iv) In addition, in a vicinity of the point $ic$ the function $m(k)$ has the following behavior. If $\chi_+(k)$ satisfies (2.9) then $m(k)$ has continuous limits as $k$ approaches $ic$ from the domain $\mathbb{C} \setminus \Sigma$. If $\chi_+(k)$ satisfies (2.10) then for $k \to ic$ one has

\[
m(k) = \left( C_1(x, t)(k - ic)^{-1/2}, C_2(x, t) \right) (1 + o(1)) \quad C_1C_2 \neq 0; \ or
\]

\[
m(k) = (C(x, t), 0) (1 + o(1)).
\]

(2.18)

At the point $-ic$ the analog of condition (2.18) holds by symmetry (2.17).

Proof. The fact that $m$ satisfies the jump condition (2.16) is established in [11]. Note that the jump matrix on $\mathbb{R}$ also satisfies the symmetry $v(k) = \sigma_1(v(-k))^{-1}\sigma_1$. To prove uniqueness, assume first that $\hat{m}(k)$ and $\hat{m}(k)$ are two solutions for R-H problem (i) - (iv). Then $\mu(k) := \hat{m}(k) - \hat{m}(k)$ satisfies (i), (ii), (iv) and instead of (iii) we have

\[
\mu(k) = O(k^{-1}), \quad k \to \infty.
\]

In $\mathbb{C}^+ \setminus (0, ic]$ introduce the holomorphic function

\[
F(k) = \mu_1(k)\overline{\mu_1(k)} + \mu_2(k)\overline{\mu_2(k)}
\]

where $\mu_{1,2}$ are the components of $\mu$. Then $F(k) = O(k^{-2})$ as $k \to \infty$. Note that since the exact values of the constants $C_1, C_2$ and $C$ in (2.18) are not specified, they may be different for $\hat{m}$ and $\hat{m}$. Furthermore, since $-k = k$ for $k \in \mathbb{R}$, it follows from the symmetry condition (ii) that for such $k$, $\mu_i(k) = \mu_j(k)$, $i \neq j$. We thus get $F(k) = O((k - ic)^{-1/2})$ as $k \to ic$ when (2.10) holds. For the case (2.9) the function $F(k)$ has continuous limits everywhere on $\mathbb{R} \cup [ic, 0]$. Let us denote for simplicity $F_r(k)$ and $F_l(k)$ the limiting values of $F$ from the right and left sides of
Re follows that

The jump condition (2.16) implies

The jump condition (2.16) implies

But as

it follows that

Let now \( \rho > c \) be arbitrary large and let \( C_\rho \) be the boundary of the domain \( (\mathbb{C}^+ \cap \{ k : |k| < \rho \}) \setminus (0, ic) \). We treat \( C_\rho \) as a closed contour oriented counterclockwise. By Cauchy’s theorem \( \oint_{C_\rho} F(k)dk = 0 \), and since \( F(k) = O(k^{-2}) \) as \( k \to \infty \), the integral over the upper semicircle will asymptotically vanish as \( \rho \to \infty \) and we get

Taking into account (2.20), the real part of this integral reads

But \( |R(k)| < 1 \) for \( k \in \mathbb{R} \setminus \{0\} \), therefore all summands in the last formula are non-negative, and we obtain

From this and (2.16) it immediately follows that \( \mu_{1,+}(k) = \mu_{2,+}(k) = 0 \) and \( \mu_{2,-}(k) = \mu_{1,-}(k) = 0 \) for \( k \in [ic, 0] \). Thus, the function \( \mu_2(k) \) is a holomorphic function in \( \mathbb{C} \) with \( \mu_2(k) \to 0 \) as \( k \to \infty \). By Liouville’s theorem \( \mu_2(k) \equiv 0 \) in \( \mathbb{C} \). In turn, this identity and formula (2.16) imply \( \mu_1(k) = \mu_2(k) \) for \( k \in [ic, 0] \). Therefore, \( \mu_1(k) \) is also a holomorphic function in \( \mathbb{C} \) vanishing at infinity. This proves uniqueness.

It remains to verify (iv). In the nonresonant case (2.9) implies that the Wronskian \( W(k, t) \) of the Jost solutions \( \phi(k, x, t) \) and \( \phi_1(k, x, t) \) does not vanish at \( k = ic \) for all \( t \). In turn, \( T(k, t) \) is bounded and continuous as \( k \to ic \), and the same is true for the components of the vector \( m \).

In the resonant case (2.10) implies \( W(ic, t) = 0 \). Now if \( \phi(ic, x, t) = C_2(x, t) \neq 0 \) then \( \phi_1(ic, x, t) \neq 0 \) (otherwise the Wronskian would not have zero at \( k = ic \)). This proves the first line of (2.18). Otherwise, if \( \phi(ic, x, t) = 0 \), then also \( \phi_1(ic, x, t) = 0 \). Since \( W(k, t) = \tilde{C}(t)(k^{ic})^{1/2}(1+o(1)) \) and \( \phi_2(k, x, t) = \tilde{C}_1(x, t)(k^{ic})^{1/2}(1+o(1)) \) as \( k \to ic \), this proves the second line of (2.18).
**Remark 2.2.** It may happen that $\phi(0, x^*, t^*) = 0$ for some $x^*$ and $t^*$ arbitrary large. Since $T(0, t) = 0$ then $m(0, x^*, t^*) = (0, 0)$ (cf. (2.11)). However, we can not apply Theorem 2.4 directly, because: (1) the jump matrix $v$ has non-admissible singularities at $\partial\Sigma$ in the resonant case; (2) point $0$ is a point of intersection of two contours in $\Sigma$. Nevertheless, after a few invertible transformations and contour deformations we will get an equivalent vector R-H problem for which Theorem 1.1, item (d) will be applicable (see Remark 2.6).

Now we recall briefly the conjugation and deformation steps which lead to the model problem solution in the domain $-6c^2t < x < 4c^2t$ or, more precisely, when $\frac{x}{2c^2} := \xi \in \mathcal{I}_c$ (see (1.3)). As is shown in [11], (see also [13]) for $\xi \in (-\frac{c^2}{2}, \frac{c^2}{2})$ the equality

\begin{equation}
\int_0^{ic} \left(k^2 + \xi + \frac{c^2 - a^2}{2}\right) \left(\sqrt{\frac{k^2 + a^2}{k^2 + c^2}}\right) dk = 0,
\end{equation}

generates an implicitly given positive function $a(\xi)$, monotonously increasing such that $a(-\frac{c^2}{2}) = 0$, $a(\frac{c^2}{2}) = c$.

In the domain $\mathbb{C} \setminus [ic, -ic]$ we introduce the function

\begin{equation}
g(k) := g(k, x, t) = 12 \int_{ic}^{k} \left(k^2 + \xi + \frac{c^2 - a^2}{2}\right) \sqrt{\frac{k^2 + a^2}{k^2 + c^2}} dk.
\end{equation}

Here we use the standard branch of the square root with the cut along $\mathbb{R}_-$.

**Lemma 2.3.** ([11]). The function $g$ possesses the following properties

(a) $g(k) = -g(-k)$ for $k \in \mathbb{C} \setminus [ic, -ic]$;

(b) $g_-(k) + g_+(k) = 0$ as $k \in [ic, ia] \cup [-ia, -ic]$;

(c) $g_-(k) - g_+(k) = B$ as $k \in [ia, -ia]$, where $B := B(\xi) = -2g_+(ia) > 0$;

(d) the asymptotical behavior

$$
\Phi(k, \xi) - ig(k, \xi) = O \left(\frac{1}{k}\right).
$$

holds as $k \to \infty$.

The signature table for for the imaginary part of $g$ is shown on the following picture:

![Figure 1. Sign of Im(g)](image-url)
STEP 1. Let \( m(k) \) be the vector solution of the R-H problem described in Theorem 2.1 and introduce the vector

\[
m^{(1)}(k) = m(k) e^{i \theta g(k) - i \Phi(k)} = m(k) \begin{pmatrix} e^{i \theta g(k) - i \Phi(k)} & 0 \\ 0 & e^{-i \theta g(k) + i \Phi(k)} \end{pmatrix}.
\]

Then \( m^{(1)}(k) \) is a holomorphic function in \( \mathbb{C} \setminus \Sigma \) which solves the jump problem \( m_+^{(1)} = m_-^{(1)} v^{(1)} \) with

\[
v^{(1)}(k) = \begin{cases} 
\begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)} e^{-2i \theta g(k)} \\
R(k) e^{2i \theta g(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}, \\
\begin{pmatrix} e^{i (g_+ - g_-)} & 0 \\
\chi_+(k) e^{i (g_++g_-)} & e^{-i (g_+ - g_-)} \end{pmatrix}, & k \in [ic, 0], \\
\sigma_1^{(v^{(1)}(-k))}(-k)^{-1} \sigma_1, & k \in [0, -ic];
\end{cases}
\]

and satisfies properties (ii)-(iv) from Theorem 2.1.

STEP 2. Let \( \rho = \rho(\varepsilon) > 0 \) be a small number,

\[
\rho < \frac{1}{2} \min \left\{ c - a \left( \frac{c^2}{3} - \varepsilon \right), \ a \left( -\frac{c^2}{2} + \varepsilon \right) \right\}.
\]

Denote \( b := a - \rho \) and introduce two domains \( \Omega^U \) and \( \Omega^L \), bounded by \( \mathbb{R} \) and contours \( \Sigma^U \) and \( \Sigma^L \) which are symmetric with respect to the map \( k \mapsto -k \) and oriented left to right (cf. Figure 2). Moreover, their boundaries \( \Sigma^U \) and \( \Sigma^L \) must be contained in the region where \( \text{Im}(g) > \delta \) and \( \text{Im}(g) < -\delta \) respectively, for some \( \delta > 0 \).

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The first deformation step}
\end{figure}

Given condition [2.1] the reflection coefficient \( R(k) \) can be continued analytically in the domain \( \Omega^U \), and \( \overline{R(k)} \) can be continued analytically as

\[
\overline{R(k)} = R(-k), \quad k \in \Omega^L.
\]

Set

\[
m^{(2)}(k) = m^{(1)}(k) \begin{cases} \begin{pmatrix} 1 & 0 \\
-R(k) e^{2i \theta g(k)} & 1 \end{pmatrix}, & k \in \Omega^U, \\
\begin{pmatrix} 1 & -R(-k) e^{-2i \theta g(k)} \\
0 & 1 \end{pmatrix}, & k \in \Omega^L, \\
I, & k \in \mathbb{C} \setminus (\overline{\Omega^U} \cup \overline{\Omega^U}).
\end{cases}
\]

\[
\text{where } \chi_+ := \frac{1}{\varepsilon} \log \left( \frac{b}{a} \right).
\]
This transformation leads to cancelation of the jump on \( \mathbb{R} \) for \( m^{(2)} \) and respects also the symmetry condition \( (2.17) \) and property \( (iv) \) of the initial vetor \( m \). Vector \( m^{(2)} \) satisfies the normalization condition and solves the jump problem \( m_+^{(2)} = m_-^{(2)} v^{(2)} \) with

\[
v^{(2)}(k) = \begin{cases} 
  e^{it(g_+ - g_-)} 
  & k \in [i\beta, 0], \\
  (R_- - R_+ + \chi_+) e^{t(g_+ - g_-)} e^{-it(g_+ - g_-)} 
  & k \in \Sigma^U, \\
  \frac{1}{R(k)} e^{2itg(k)} 
  & k \in [ic, ib], \\
  v^{(1)}(k), \\
  \sigma_1(v^{(2)}(-k))^{-1}\sigma_1 
  & k \in [0, -ic] \cup \Sigma^L.
\end{cases}
\]

As it is shown in \([11]\), the following equality is valid:

\[
R_-(k) - R_+(k) + \chi_+(k) = 0, \quad k \in [i\beta, 0].
\]

Together with property \( (c) \) of the \( g \)-function this gives a more simple formula for \( v^{(2)} \):

\[
v^{(2)}(k) = \left( \begin{array}{cc}
  e^{-itB} & 0 \\
  0 & e^{itB}
\end{array} \right), \quad k \in [i\beta, -i\beta] .
\]

STEP 3. Our next conjugation step deals with a factorization of the jump matrix on the set \([ic, ia] \cup [-ia, -ic] \). To this end consider the following function \( F(k) = F(k, \xi), k \in \mathbb{C} \setminus [ic, -ic] \):

\[
(2.25) \quad F(k) = \exp \left\{ \frac{w(k)}{2\pi i} \left( \int_{ic}^{ia} f(s) ds + \int_{ic}^{-ia} f(s) ds - i\Delta \int_{-ia}^{ia} \frac{ds}{w(s)(s - k)} \right) \right\},
\]

where

\[
w(k) = \sqrt{(k^2 + c^2)(k^2 + a^2)}, \quad k \in \mathbb{C} \setminus ([ic, ia] \cup [-ia, -ic]), \quad w(0) > 0,
\]

(2.26) \quad f(k) := \log \frac{\vert \chi(k) \vert}{w_+(k)}, \quad k \in [ic, -ic],

(2.27) \quad \Delta = \Delta(\xi) = 2i \int_{ic}^{ia} \frac{\log \vert \chi(s) \vert}{w_+(s)} ds \left( \int_{-ia}^{ia} \frac{ds}{w(s)} \right)^{-1} \in \mathbb{R}.

**Lemma 2.4** \([III]\). The function \( F(k) \) possesses the following properties \( (26) \):

1. \( F_+(k)F_-(k) = \vert \chi(k) \vert \) for \( k \in [ic, ia] \);
2. \( F_+(k)F_-(k) = \vert \chi(k) \vert^{-1} \) for \( k \in [-ia, -ic] \);
3. \( F_+(k) = F_-(k) e^{i\Delta} \) for \( k \in [ic, -ic] \);
4. \( F(k) \to 1 \) as \( k \to \infty \);
5. \( F(-k) = F^{(-1)}(k) \) for \( k \in \mathbb{C} \setminus [ic, -ic] \);
6. If \( \vert \chi(k) \vert \) satisfies \( (2.9) \) then \( F(k) = C(k - ic)^{1/4}(1 + o(1)) \) as \( k \to ic \);
7. If \( \vert \chi(k) \vert \) satisfies \( (2.10) \) then \( F(k) = C(k - ic)^{-1/4}(1 + o(1)) \) as \( k \to ic \).

Taking into account these properties and property \( (2.8) \) we observe then that the matrix \( v^{(2)}(k) \) can be factorized on \([ic, ia] \cup [-ia, -ic] \) as follows:
where (cf. (2.4)) depicted in Figure 3.

\[ \text{symmetric domains } \Omega \]

Moreover as (2.28)

Recall that in (2.29) \( m \in D \)

Lemma 2.3 and property (3) of Lemma 2.4 we see that (2.21)

Define \( m(3)(k) \) as

\[ m^{(3)}(k) = m^{(2)}(k) \begin{cases} G(k), & k \in \Omega^U_1, \\ \sigma_1 G(-k) \sigma_1, & k \in \Omega_1^L, \\ (F(k))^{-\sigma_3} & k \in \mathbb{C} \setminus (\Omega^U_1 \cup \Omega^L_1). \end{cases} \]

Since \( F(k) \to 1 \) as \( k \to \infty \), the normalization condition is preserved for \( m^{(3)} \). Since \( F^{-1}(-k) = F(k), \) (2.17) is also preserved. Taking into account property (c) of Lemma 2.3 and property (3) of Lemma 2.4 we see that

\[ F_-(k) e^{i \pi (\chi(k) - \chi^*(k))} = e^{-i \pi B - i \Delta}, \quad k \in [ia, -ia], \]
and therefore the jump matrix for $m^{(3)}$ looks as follows

\begin{equation}
v^{(3)}(k) = \begin{cases}
\begin{pmatrix}
0 & i \\
i & 0 \\
F_-(k) & 0 \\
F_+(k) & 0 \\
F_+(k) & e^{it(g_+(k) - g_-(k))} \\
F_+(k) & 0 \\
F_-(k) & e^{it(g_+(k) + g_-(k))} \\
F_-(k) & 0 \\
e^{-itB-i\Delta} & 0 \\
0 & e^{itB+i\Delta} \\
1 - \frac{p^2(k)}{k} & e^{-2itg(k)} \\
0 & 1 \\
R(k)F^{-2}(k)e^{2itg(k)} & 0 \\
1 & 1 \\
\sigma_1[v^{(3)}(-k)]^{-1}\sigma_1, & k \in [\iota, \iota a], \\
\end{pmatrix}
k \in [\iota a, \iota b], \\
k \in [\iota b, -\iota b], \\
k \in \Sigma_1^U, \\
k \in \Sigma_1^L, \\
k \in \Sigma_1^L \cup \Sigma_1^L \cup [\iota b, -\iota c].
\end{cases}
\end{equation}

**Remark 2.5.** As is shown in (2.29),

\[
F_-(k)e^{it(g_+(k) - g_-(k))} = e^{-itB-i\Delta}, \quad k \in [\iota a, \iota b].
\]

We preserve the form (2.30) of the jump matrix on $[\iota a, \iota b] \cup [-\iota b, -\iota a]$ because it simplifies further considerations of the local parametrix problem.

According to the signature table of $\text{Im}g(k)$ (see Figure 1), the off-diagonal elements of matrix $v^{(3)}$ are exponentially small with respect to $t$ on the parts $[\iota a, -\iota a] \cup \Sigma_1^L \cup \Sigma_1^L \cup \Sigma_1^L$ outside of small, symmetric with respect to reflection $k \to -k$, vicinities $B^U(\varepsilon)$ and $B^L(\varepsilon)$ of the points $\pm \iota a$. We will define the precise shape of these vicinities later, however we require that the points $\pm \iota b$ belong to their boundaries, that is $\iota b \in \partial B^U(\varepsilon) \cup [\iota c, 0]$. Taking into account (2.29) we conclude that outside of these vicinities the matrix $v^{(3)}(k)$ is asymptotically close as $t \to \infty$ to a piecewise constant matrix on $[\iota c, -\iota c]$:

\[
v^{\text{mod}}(k) := \begin{cases}
\iota \sigma_1, & k \in [\iota c, \iota a] \\
e^{-\iota \varepsilon_3} & k \in [\iota a, 0] \\
\sigma_1[v^{\text{mod}}(-k)]^{-1}\sigma_1, & k \in (0, -\iota c].
\end{cases}
\]

Here we put

\begin{equation}
\Lambda := tB + \Delta \in \mathbb{R}.
\end{equation}

The solution of the vector R-H problem

\[
m^{\text{mod}}_+(k) = m^{\text{mod}}_-(k)v^{\text{mod}}(k), \quad m^{\text{mod}}(-k) = m^{\text{mod}}(k)\sigma_1, \quad m^{\text{mod}}(\infty) = (1, 1),
\]

which is usually called the model solution, is found (see [11], [13]) in the class of functions with singularities of order $(k-\kappa)^{-1/4}$ in the points $\kappa$ of discontinuity of the jump matrix. In our case $\kappa = \pm \iota a$. However uniqueness of the model solution is not proved yet. In view of the discussion given in the introduction, it seems important to show it. We also explain below how to construct the symmetric matrix model solution and how to describe the points $(x, t)$ where this solution does not exist. The next three sections are devoted to these problems.

At the end of this section we discuss the influence of the last transformation (2.28) on the behavior of the vector function $m^{(3)}(k)$ in the neighborhood of the points...
with the respective symmetric behavior at \(-i\epsilon\). Then the values of \(m(2)(k)\) are bounded near \(\pm i\epsilon\). This includes also the cases of possible zeros

\[ m(2)(k) \sim (C_1(k - i\epsilon)^{1/2}, C_2), \quad \text{or} \quad m(2)(k) \sim (C_1, C_2(k - i\epsilon)), \quad \text{as} \quad k \to i\epsilon, \]

with the respective symmetric behavior at \(-i\epsilon\). Moreover, from (2.25)–(2.26) it follows that (see [26])

\[ F(k) = C_\pm (k \mp i\epsilon)^{\pm 1/4}(1 + o(1)), \quad \text{as} \quad k \to \pm i\epsilon, \quad \text{where} \quad C_+ C_- \neq 0. \]

Since \(i\epsilon \in \Omega'_i\), \(m(3) = m(2)G\) in a vicinity of this point, that is

\[ m(3)(k) = \left( F^{-1}(k)m_1(2)(k), \frac{F(k)}{\chi(k)m_1(2)(k)}m_2(2)(k)e^{-2i\vartheta_0(k)} + F(k)m_2(2)(k) \right). \]

Thus

- if \(\chi(k)\) satisfies (2.9) and \(m_1(2)(i\epsilon) \neq 0\) then
  \[ m(3)(k) \sim (C_1, C_2)(k - i\epsilon)^{-1/4}, \quad k \to i\epsilon, \quad C_1C_2 \neq 0; \]

- if \(\chi(k)\) satisfies (2.9) and \(m_1(2)(i\epsilon) = 0\) then
  \[ m(3)(k) \sim (C_1, C_2)(k - i\epsilon)^{1/4}, \quad k \to i\epsilon, \quad C_1C_2 \neq 0. \]

Analogously, if \(\chi(k)\) satisfies (2.10) then (since \(W(i\epsilon) = 0\)) the possible behaviour of \(m(2)(k)\) and \(F(k)\) as \(k \to i\epsilon\) is the following:

\[ m(2)(k) \sim (C_1(k - i\epsilon)^{-1/2}, C_2), \quad \text{or} \quad m(2)(k) \sim (C_1, C_2(k - i\epsilon)), \quad C_1C_2 \neq 0, \]

\[ F(k) = C(k - i\epsilon)^{-1/4}(1 + o(1)). \]

By use of (2.32) we conclude that for \(k \to i\epsilon\):

\[ m(3)(k) \sim (C_1, C_2)(k - i\epsilon)^{-1/4}, \quad \text{or} \quad m(3)(k) \sim (C_1, C_2)(k - i\epsilon)^{1/4}, \]

where \(C_1C_2 \neq 0\) for the second case.

**Remark 2.6.** The R-H problem for \(m(3)\) is equivalent to the initial R-H problem (i)-(iv) from Theorem 2.1. Therefore, it has also a unique solution. Moreover, the jump matrix \(v(3)(k)\) is bounded, and point 0 is an interior point for the respective jump contour (see Fig. 3). Thus, Theorem 1.1 is applicable for this case. Moreover, the property (2.33) is also in agreement with condition (1.11) on the singularities of matrix solutions, although for the initial problem it was not fulfilled (see (2.18)). Let now points \((x^*, t^*)\) be as in Remark 2.2. We observe from (2.23) and (2.24) that for such \((x^*, t^*)\) \(m(3)_{\pm}(0) = m_{\pm}(0) = (0, 0)\). We conclude that for such points there is no an invertible matrix solution and thus the initial R-H problem does not admit a well - posed matrix analog.

3. Uniqueness for the Vector Model Solution

**Lemma 3.1.** The following R-H problem has a unique solution: find a vector-valued function \(m^{\text{mod}}(k) = (m_1^{\text{mod}}(k), m_2^{\text{mod}}(k))\) holomorphic in the domain \(\mathbb{C} \setminus \{i\epsilon, i\alpha, -i\alpha, -i\epsilon\}\) that for such points there is no an invertible matrix solution and thus the initial R-H problem does not admit a well - posed matrix analog.

\[ m^{\text{mod}}(k) = m^{\text{mod}}(-k)v^{\text{mod}}(k), \]

\[ m^{\text{mod}}_+(k) = m^{\text{mod}}_-(k)v^{\text{mod}}(k), \]

\[ m^{\text{mod}}(k) \sim (C_1(k - i\epsilon)^{-1/2}, C_2), \quad \text{or} \quad m^{\text{mod}}(k) \sim (C_1, C_2(k - i\epsilon)), \quad C_1C_2 \neq 0, \]

\[ F(k) = C(k - i\epsilon)^{-1/4}(1 + o(1)). \]
Moreover, the symmetry condition
\begin{equation}
\label{symmetry_condition}
m^\text{mod}(-k) = m^\text{mod}(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{equation}
and the normalization condition
\begin{equation}
\label{normalization_condition}
\lim_{k \to i\infty} m^\text{mod}(k) = (1 1).
\end{equation}

At any point $\kappa \in \mathcal{G}$ the vector function $m^\text{mod}(k)$ can have at most a fourth root singularity: $m^\text{mod}(k) = O((k - \kappa)^{-1/4})$, $k \to \kappa$.

**Proof.** Let $m$ and $\tilde{m}$ be two solutions of the RH problem. Their difference $\tilde{m} = m - \tilde{m}$ is a holomorphic vector in $\mathbb{C} \setminus [ic, -ic]$ which satisfies conditions \eqref{symmetry_condition} and \eqref{normalization_condition} and has the following behavior
\begin{equation}
\tilde{m}(k) = (1 - 1) \frac{\hat{h}}{k}(1 + o(1)) \text{ as } k \to i\infty.
\end{equation}

Moreover, $\tilde{m}(k) = O((k - \kappa)^{-1/4})$ as $k \to \kappa$ for $\kappa \in \mathcal{G}$.

In $\mathbb{C} \setminus [ic, -ic]$, introduce a holomorphic function
\begin{equation}
\label{holomorphic_function}
f(k) := \tilde{m}_1(k)\tilde{m}_1(k) + \tilde{m}_2(k)\tilde{m}_2(k).
\end{equation}

Due to \eqref{symmetry_condition} this function is even: $f(-k) = f(k)$ and satisfies
\begin{equation}
\label{asymptotic_behavior}
f(k) = \frac{2|\hat{h}|^2}{k^2} (1 + O(k^{-2})) \text{ as } k \to i\infty;
\end{equation}
\begin{equation}
\label{asymptotic_behavior_2}
f(k) = O((k - \kappa)^{-1/2}) \text{ as } k \to \kappa, \text{ for } \kappa \in \mathcal{G}.
\end{equation}

Since $\overline{k} = k$ for $k \in i\mathbb{R}$ and taking into account \eqref{symmetry_condition}, for $k \in [ic, -ic]$ we get
\begin{align*}
f_+(k) &= \tilde{m}_{1, +}(k)\overline{\tilde{m}_{2,-}(k)} + \tilde{m}_{2, +}(k)\overline{\tilde{m}_{1,-}(k)}, \\
f_-(k) &= \tilde{m}_{1, -}(k)\overline{\tilde{m}_{2,+}(k)} + \tilde{m}_{2, -}(k)\overline{\tilde{m}_{1,+}(k)},
\end{align*}
for $k \in [ic, -ic]$.

By use of \eqref{symmetry_condition}
\begin{align*}
f_+(k) &= \pm i \left( |\tilde{m}_{2,-}(k)|^2 + |\tilde{m}_{1,-}(k)|^2 \right) = -f_-(k) \in i\mathbb{R}, \quad k \in [\pm ic, \pm ia], \\
f_+(k) &= e^{-i\lambda} \tilde{m}_{1,-}(k)\overline{\tilde{m}_{2,-}(k)} + e^{i\lambda} \tilde{m}_{2,-}(k)\overline{\tilde{m}_{1,-}(k)} = f_-(k) \in \mathbb{R}, \quad k \in [ia, -ia].
\end{align*}

Thus the function $f(k)$ has no jump on $[ia, -ia]$ and is the solution of the following jump problem
\begin{equation}
\label{jump_problem}
f_+(k) = -f_-(k), \quad k \in [ic, ia] \cup [-ia, -ic],
\end{equation}
for $k \in [ic, ia] \cup [-ia, -ic]$. 

which satisfies (3.6) and (3.7). The unique solution of this problem is given by the formula
\[ f(k) = -\frac{2|\bar{h}|^2}{\sqrt{(k^2 + c^2)(k^2 + a^2)}}. \]
Therefore, if \( \tilde{h} \neq 0 \) then \( f(0) < 0 \). But according to (3.5) and (3.8) we have \( f_+(0) = f_-(0) \geq 0 \). Thus, \( \tilde{h} = 0 \) and hence
\[ \tilde{m}_{1,-}(k) = \tilde{m}_{1,+}(k) = \tilde{m}_{2,+}(k) = \tilde{m}_{2,-}(k) = 0, \quad k \in [ic, ia] \cup [-ia, -ic]. \]
In particular, we see that the jump along \([ic, ia] \cup [-ia, -ic]\) is removable and the only solution of this problem is trivial: \( \tilde{m}(k) \equiv 0 \).

4. Solution of the vector model RH problem

In this section we recall briefly how to solve problem (3.1)–(3.4) (cf. [11]).

Consider the two-sheeted Riemann surface \( X = X(\xi) \) associated with the function
\[ w(k) = \sqrt{(k^2 + c^2)(k^2 + a^2)}, \]
defined on \( \mathbb{C} \setminus ([ic, ia] \cup [ia, ic]) \) with \( w(0) > 0 \). The sheets of \( X \) are glued along the cuts \([ic, ia]\) and \([-ia, -ic]\). Points on this surface are denoted by \( p = (k, \pm) \).

To simplify notations we keep the notation \( k = (k, \pm) \) for the upper sheet of \( X \).

The canonical homology basis of cycles \( \{a, b\} \) is chosen as follows: The \( a \)-cycle surrounds the points \(-ia, ia\) starting on the upper sheet from the left side of the cut \([ic, ia]\) and continues on the upper sheet to the left part of \([-ia, -ic]\) and returns after changing sheets. The cycle \( b \) surrounds the points \( ia, ic \) counterclockwise on the upper sheet. Consider the normalized holomorphic differential
\[
d\omega = \Gamma \frac{d\zeta}{w(\zeta)}, \quad \text{where } \Gamma := \left( \int_a \frac{d\zeta}{w(\zeta)} \right)^{-1} \in i\mathbb{R}_-, \]
then \( \int_a d\omega = 1 \) and
\[
\tau = \tau(\xi) = \int_b d\omega \in i\mathbb{R}_+. \]
Let
\[ \theta_3(z | \tau) = \sum_{n \in \mathbb{Z}} \exp \left\{ (n^2 \tau + 2nz)\pi i \right\}, \quad z \in \mathbb{C}, \]
be the Jacobi theta function. Recall that \( \theta_3 \) is an even function, \( \theta_3(-z | \tau) = \theta_3(z | \tau) \), and satisfies
\[ \theta_3(z + n + \tau(\xi)\ell | \tau) = \theta_3(z | \tau) \exp \left\{ -\pi i\ell^2 - 2\pi i\ell z \right\} \text{ for } l, n \in \mathbb{Z}. \]
Furthermore, let \( A(p) = \int_a^p d\omega \) be the Abel map on \( X \). We identify the upper sheet of \( X \) with the complex plane \( \mathbb{C} \setminus ([ic, ia] \cup [-ia, -ic]) \) with cuts, and put \( (k, +) = k \). Allowing only paths of integration in \( \mathbb{C} \setminus [ic, -ic] \) we observe that \( A(k) \) is a holomorphic function in that given domain with the following properties:
- \( A_+(k) = -A_-(k) \) (mod 1) for \( k \in [ic, ia] \cup [-ia, -ic] \);
- \( A_+(k) - A_-(k) = -\tau \) as \( k \in [ia, -ia] \);
- \( A(-k) = -A(k) + \frac{1}{2} \) (mod 1) as \( k \in \mathbb{C} \setminus [ic, -ic] \),
- \( A_+(ia) = -\frac{\tau}{2} = -A_-(ia), \quad A_+(-ia) = -\frac{\tau}{2} + \frac{1}{2}, \quad A_-(ia) = \frac{\tau}{2} + \frac{1}{2} \),
- \( A((\infty, +)) = \frac{1}{4}; \quad A(k) - A((\infty, +)) = -\Gamma k^{-\frac{1}{2}} + O(k^{-3}) \text{ as } k \to \infty. \)
On $C \setminus [ic, -ic]$ introduce two functions

$$\alpha^A(k) = \theta_3 \left( A(k) - \frac{1}{2} \frac{\tilde{\Lambda}}{2} |\tau| \right) \theta_3 \left( A(k) - \frac{\tilde{\Lambda}}{2} |\tau| \right),$$

$$\beta^A(k) = \theta_3 \left( -A(k) - \frac{1}{2} \frac{\tilde{\Lambda}}{2} |\tau| \right) \theta_3 \left( -A(k) - \frac{\tilde{\Lambda}}{2} |\tau| \right),$$

where $\tilde{\Lambda} = \frac{A}{2\pi} \in \mathbb{R}$ and $A(k) = A((k, +))$ for $k \in \mathbb{C}$. The properties of the Abel integrals listed above imply that the functions $\alpha^0(k)$ and $\beta^0(k)$ have square root singularities at the points $\pm ia$. Using the formula (cf. [10])

$$\theta_3 (u | \tau) \theta_3 \left( u - \frac{1}{2} |\tau| \right) = \theta_3 \left( 2u - \frac{1}{2} |2\tau| \right) \theta_3 \left( \frac{1}{2} |2\tau| \right),$$

we can represent functions $\alpha^A(k)$ and $\beta^A(k)$ as

$$\alpha^A(k) = \theta_3 \left( 2A(k) - \frac{1}{2} \frac{\tilde{\Lambda}}{2} |2\tau| \right) \theta_3 \left( \frac{1}{2} |2\tau| \right),$$

$$\beta^A(k) = \theta_3 \left( -2A(k) + \frac{1}{2} \frac{\tilde{\Lambda}}{2} |2\tau| \right) \theta_3 \left( \frac{1}{2} |2\tau| \right).$$

Introduce the functions

$$\dot{\alpha}(k) := \frac{\alpha^A(k)}{\alpha^0(k)} = \frac{\theta_3 \left( 2A(k) - \frac{1}{2} \frac{\tilde{\Lambda}}{2} |2\tau| \right)}{\theta_3 \left( 2A(k) - \frac{1}{2} |2\tau| \right)}$$

(4.3)

$$\dot{\beta}(k) := \frac{\beta^A(k)}{\beta^0(k)} = \frac{\theta_3 \left( -2A(k) + \frac{1}{2} \frac{\tilde{\Lambda}}{2} |2\tau| \right)}{\theta_3 \left( -2A(k) + \frac{1}{2} |2\tau| \right)}.$$ 

(4.4)

Evidently, both functions $\dot{\alpha}(k)$ and $\dot{\beta}(k)$ have square root singularities at the points $\pm ia$ if $\tilde{\Lambda} \notin \mathbb{Z}$. Moreover,

$$\lim_{k \to \infty} \dot{\alpha}(k) = \lim_{k \to \infty} \dot{\beta}(k) = \frac{\theta_3 \left( \frac{\Lambda}{2} |2\tau| \right)}{\theta_3 \left( 0 |2\tau| \right)}.$$ 

Due to the first three properties of the Abel map we get

$$\dot{\alpha}_+(k) = \dot{\beta}_-(k) \quad \text{and} \quad \dot{\beta}_+(k) = \dot{\alpha}_-(k) \quad \text{for} \quad k \in [ic, ia] \cup [-ia, -ic],$$

$$\dot{\alpha}_+(k) = e^{-i\Lambda} \dot{\alpha}_-(k) \quad \text{and} \quad \dot{\beta}_+(k) = e^{i\Lambda} \dot{\beta}_-(k) \quad \text{for} \quad k \in [ia, -ia],$$

$$\alpha(-k) = \beta(k) \quad \text{for} \quad k \in C \setminus [ic, -ic].$$

Now introduce the function

$$\tilde{\gamma}(k) = \sqrt[k^2 + a^2]{k^2 + c^2},$$ 

(4.5)

defined uniquely on the set $C \setminus ([ic, ia] \cup [-ia, -ic])$ by the condition $\arg \tilde{\gamma}(0) = 0$. This function satisfies the jump conditions

$$\dot{\gamma}_+(k) = i \dot{\gamma}_-(k), \quad k \in [ic, ia],$$

$$\dot{\gamma}_+(k) = -i \dot{\gamma}_-(k), \quad k \in [ia, -ic].$$
Then the vector function

\[(4.6) \quad m^{\text{mod}}(k) = \left( \tilde{\gamma}(k) \frac{\dot{\alpha}(k)}{\alpha(\infty)}, \tilde{\gamma}(k) \frac{\dot{\beta}(k)}{\beta(\infty)} \right) \]

solves problem \((3.1)-(3.4)\).

Note that both components of the vector-valued function \(m^{\text{mod}}(k)\) are bounded everywhere except for small vicinities of the points of the set \(G\), where they have singularities of the type \((k - \kappa)^{-1/4}\), \(\kappa \in G\).

**Remark 4.1.** We observe that

\[\hat{\alpha}_\pm(0) = \frac{\theta_3\left(\mp\tau + 1 - \hat{\Lambda} \mid 2\tau \right)}{\theta_3\left(\mp\tau + 1 \mid 2\tau \right)}, \quad \hat{\beta}_\pm(0) := \frac{\theta_3\left(\mp\tau + 1 - \hat{\Lambda} \mid 2\tau \right)}{\theta_3\left(\mp\tau + 1 \mid 2\tau \right)}.\]

This means that for \(\hat{\Lambda} = \frac{1}{2} \pmod{n}\) we have \(m_\pm^{\text{mod}}(0) = (0, 0)\). From Theorem 1.1 it follows then that for \(\Lambda = 2\pi\hat{\Lambda} = \pi(2n + 1)\), \(n \in \mathbb{Z}\) the matrix model R-H problem associated with the jump \((3.2)\) does not have an invertible solution.

**Remark 4.2.** For \(\hat{\Lambda} \in \mathbb{Z}\) we have \(\hat{\alpha}(\pm ia) = \hat{\beta}(\pm ia) = 1\). By \((4.3), (4.6)\), therefore:

\[m^{\text{mod}}(\pm ia) = (0, 0), \quad \text{as} \quad \Lambda = 2\pi n.\]

Thus the points \((x, t)\) for which \(\hat{\Lambda} \in \mathbb{Z}\) are those points where the vector model solution does not have singularities at the points \(\pm ia\). However, the matrix model solution will have fourth order singularities at \(\pm ia\) for these pairs \((x, t)\).

**Lemma 4.3.** For the product \(m_1^{\text{mod}}(k)m_2^{\text{mod}}(k) := p(k)\) the following formula is valid:

\[p(k) = \tilde{\gamma}^2(k) \frac{\theta_3(2A(k) - \frac{1}{2} + \hat{\Lambda}) \theta_3(-2A(k) + \frac{1}{2} - \hat{\Lambda}) \theta_3(0)^2}{\theta_3(2A(k) - \frac{1}{2})^2(\theta_3(\Lambda))^2}.\]

Moreover, for \(k \to \infty\)

\[(4.7) \quad p(k) = 1 + \frac{1}{2k^2} \left( 8\Gamma^{2} \left. \frac{d^2}{dv^2} \log \theta_3(v) \right|_{v=\hat{\Lambda}} + Q \right) + O\left(\frac{1}{k^4}\right),\]

\[Q = -a^2 + c^2 \left(1 - 2 E(\frac{\pi}{2}) K(\frac{\pi}{2})\right).\]

Here \(\theta_3(\cdot) = \theta_3(\cdot \mid 2\tau)\), \(E(s)\) and \(K(s)\) are the standard complete elliptic integrals and \(\Gamma\) is given by formula \((4.1)\).

**Proof.** Function \(p(k)\) is holomorphic at infinity and due to \((3.4)\) \(p(\infty) = 1\). By property \((3.3)\) it is an even function, therefore, it has an expansion of the form \(p(k) = 1 + \frac{k^2 + \epsilon^2}{2k^2} + O(k^{-4})\), and \(\log p(k) = \frac{k^2 + \epsilon^2}{2k^2} + O(k^{-4})\). Hence, \(p(k) = 1 + \log p(k) + O(k^{-4})\) as \(k \to \infty\). Note that

\[\log p(k) = \frac{1}{2} \log \left( \frac{k^2 + \alpha^2}{k^2 + c^2} \right) + \log \left( \frac{\theta_3(2A(k) - \frac{1}{2} - \hat{\Lambda})}{\theta_3(\Lambda)} \right)\]

\[+ \log \left( \frac{\theta_3(-2A(k) + \frac{1}{2} - \hat{\Lambda})}{\theta_3(\Lambda)} \right) - 2 \log \left( \frac{\theta_3(2A(k) - \frac{1}{2})}{\theta_3(0)} \right)\]
Taking into account the properties of the Abel integral, expanding each logarithmic term and using the general identity

$$\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 = (\log f)''$$

for twice differentiable functions, we see that

$$(4.8) \quad L = 8\Gamma^2 \left( \frac{d^2}{dv^2} \log \theta_3(v) \big|_{v=\lambda} - \frac{d^2}{dv^2} \log \theta_3(v) \big|_{v=0} \right) + a^2 - c^2.$$

Let us take a closer look at the term $\frac{d^2}{dv^2} \log \theta_3(v) \big|_{v=0}$. The first logarithmic derivative of the theta-function is related to the Jacobi zeta function $Z(u)$ via \([2] [5]\)

$$Z(u) = \frac{d}{dv} \log \Theta(v) \big|_{v=u}$$

where

$$\Theta(v) := \theta_3 \left( \frac{v}{2K(s)} + \frac{1}{2} \right), \quad Z(u) := \int_0^u \left( \text{dn}^2(y, s) - \frac{E(s)}{K(s)} \right) dy.$$

Here $K(s)$ and $E(s)$ are the complete elliptic integrals of the first and second kind respectively, $\text{dn}(y, s)$ is the delta amplitude, and $s = a/c$ the modulus. From the above equalities we obtain

$$(4.9) \quad \frac{d^2}{dv^2} \log \theta_3(v) = 4K^2(s) \text{dn}^2(2K(s)v - K(s), s) - 4K(s)E(s).$$

Evaluating at $v = 0$ and using $\text{dn}(K(s), s) = \sqrt{1 - s^2}$ results in

$$\frac{d^2}{dv^2} \log \theta_3(v) \big|_{v=0} = 4K^2(s)\frac{c^2 - a^2}{c^2} - 4K(s)E(s).$$

Substituting this in \((4.8)\) we conclude that

$$(4.10) \quad Q = -32\Gamma^2K^2 \left( \frac{a}{c} \right) \frac{c^2 - a^2}{c^2} + 32\Gamma^2K \left( \frac{a}{c} \right) E \left( \frac{a}{c} \right) + a^2 - c^2.$$

Moreover, from Lemma 2.3 of \([13]\) it follows that

$$16\Gamma^2K^2 \left( \frac{a}{c} \right) = -c^2.$$

Applying this formula to \((4.10)\) we obtain $Q$ as in \((4.7)\). \qed

**Lemma 4.4.** The continuous function

$$8\Gamma^2 \frac{d^2}{dv^2} \log \theta_3(v), \quad v \in \mathbb{R},$$

of period 1 attains its maximum at $v = 0$ and its minimum at $v = \frac{1}{2}$.

**Proof.** This follows immediately from \((4.9)\) and the positivity of the delta amplitude $\text{dn}(y, s)$, as well as the fact that it attains its maximum (minimum) at $y = 0 + 2nK(s)$ ($y = K(s) + 2nK(s)$) for $n \in \mathbb{Z}$ \((5)\). \qed

Recall now that we investigated the asymptotics for large $k$ of the solution for the jump problem \((3.2)\) with $\tilde{\Lambda} = \frac{\Lambda}{2\pi}$ and $\Lambda$ given by formula \((2.31)\). Lemma 4.3 implies

$$(4.11) \quad m_1^{\text{mod}}(k)m_2^{\text{mod}}(k) = 1 + \frac{m_{\text{mod}}(x, t)}{2k^2} + O(k^{-4}),$$
where

\[ q^{\text{mod}}(x, t) = 8 \Gamma^2 \frac{d^2}{dv^2} \log \theta_3(v + \frac{tB + \Delta}{2\pi}) \bigg|_{v=0} + Q, \]

and \( Q \) is defined by (4.7). Let us show that in fact for any fixed \( \xi \) formula (4.12) represents the classical one-gap solution for the KdV equation associated with the spectrum \([-c^2, -a^2] \cup \mathbb{R}_{+}\) and with the initial Dirichlet divisor \( p_0 \) defined uniquely by the Jacobi inversion

\[ \int_{-a^2}^{p_0} d\hat{\omega} = -i\Delta, \quad p_0 = (\lambda(0, 0, \xi), \pm). \]

Here \( d\hat{\omega} \) is the normalized holomorphic Abel differential of the first kind on the elliptic Riemann surface \( \hat{M} = M(\xi) \) associated with the function

\[ R(\lambda, \xi) = \sqrt{\lambda + c^2}(\lambda + a(\xi)^2), \]

with cuts along the spectrum. Let \( \hat{b}, \hat{a} \) be the canonical basis on \( \hat{M} \), where the cycle \( \hat{b} \) surrounds the interval \([-c^2, -a^2]\) counterclockwise on the upper sheet and the cycle \( \hat{a} \) supplements \( \hat{b} \) by passing along the gap \([-a^2, 0]\) in the positive direction on the lower sheet and then changing the sheet. The normalization for \( d\hat{\omega} \) is given by formula \( \int_{\hat{b}} d\hat{\omega} = 2\pi i \).

Denote \( \int_{\hat{b}} d\hat{\omega} = \hat{\tau} \). It is straightforward to check that \( \hat{\tau} = 4\pi i \tau \) (cf. (4.2)).

Furthermore, let \( \hat{A}(p) := \int_{\hat{b}} d\hat{\omega} \) be the associated Abel map and

\[ K := -\hat{A}(-a^2) = -\frac{\tau}{2} + \pi i \]

be the Riemann constant. Introduce the wave and frequency numbers

\[ q \quad \text{where} \quad Q \quad \text{and} \quad A \]

of the second kind

\[ V \quad \text{(4.14)} \]

and the normalization conditions \( \int_{\hat{b}} d\hat{\Omega}_3 = 0 \). Thus,

\[ iV := \int_{\hat{b}} d\hat{\Omega}_1, \quad iW := \int_{\hat{b}} d\hat{\Omega}_3. \]

The following result is obtained in [13].

**Lemma 4.5.** Let \( B = B(\xi) \) be as in Lemma 2.3 (c) and \( \Gamma = \Gamma(\xi) \) be given by (4.1). Then the following identities hold

\[ tB = Vx - 4Wt, \quad 4\pi i\Gamma = -V. \]

Recall now that the one-gap solution corresponding to the spectrum \([-c^2, -a^2] \cup \mathbb{R}_{+}\) and to the initial divisor (4.13), can be expressed by the Its-Matveev formula (22):

\[ q^{IM}(x, t) = -2 \frac{d^2}{dx^2} \log \theta(iVx - 4iWt - \hat{A}(p_0) - K | \hat{\tau}) + Q, \]

where \( Q \) can be represented by (4.7) (cf. [13]). Here \( \theta(2\pi iv | \tau) = \theta_3(v | 2\tau) \). Since \( \hat{A}(p_0) + K = -i\Delta \), it follows from Lemma 4.5 that that

\[ q^{\text{mod}}(x, t) = q^{IM}(x, t). \]
Recall now that the trace formula for \( q^{\text{mod}}(x,t) \) reads:

\[
q^{\text{mod}}(x,t) = -c^2 - a^2 - 2\lambda(x,t),
\]

where \( p(x,t) = (\lambda(x,t), \pm) \in \mathbb{M} \) with \( \lambda(x,t) \in [-a^2, 0] \), is the unique solution of the Jacobi inversion problem

\[
\int_{p_0}^{p(x,t)} d\hat{\omega} = i(V x - 4W t) \pmod{2\pi i}.
\]

Evidently \( \lambda(x,t) = 0 \) corresponds to the local minimum of \( q^{\text{mod}}(x,t) \). According to Lemma 4.4

\[
\lambda(x,t) = 0 \text{ iff } \tilde{\Lambda} = \frac{\mathcal{B} + \Delta}{2\pi} = \frac{1}{2} \pmod{\mathbb{Z}}.
\]

**Lemma 4.6.** Let \( \gamma \) be a fixed small number and let \( t \) be sufficiently large, such that \( a^2(\xi) > 2t\gamma \) uniformly for \( \xi \in \mathcal{I}_\epsilon \). If

\[
0 \geq \lambda(x,t) > -\frac{1}{t\gamma},
\]

then there exist \( n \in \mathbb{Z} \) such that

\[
|\tilde{\Lambda} - \frac{1}{2} - n| < \frac{C}{a(-c^2/2 + \varepsilon)t^{3/2}}, \quad \gamma \in (0, \frac{1}{8}).
\]

Here \( C \) depends on \( c \) only.

**Proof.** Since \( \lambda(x,t) = 0 \) corresponds to \( \tilde{\Lambda} = \frac{1}{2} \pmod{n} \), the Jacobi inversion problem (4.17) can be represented as

\[
\int_{p_0}^{p(x,t)} d\hat{\omega} = i(\Lambda - \pi) \pmod{2\pi i}.
\]

Taking into account that \( |\hat{\omega}'| \leq \frac{C}{|a|\sqrt{|\lambda|}} \), we get (4.18). \( \square \)

At the end of this section we discuss one more interesting property of the function \( m_1^{\text{mod}}(k)m_2^{\text{mod}}(k) = p(k) \).

**Lemma 4.7.** The function \( p(k), k \in \mathbb{C} \), admits the following representation:

\[
p(k) = \frac{k^2 - \lambda(x,t)}{\sqrt{(k^2 + a^2)(k^2 + c^2)}}.
\]

**Proof.** Given (4.3) and (4.4), consider the function

\[
\tilde{p}(k) = p(k)k^{-2}(k) = \frac{\hat{\alpha}(k)\hat{\beta}(k)}{\hat{\alpha}(\infty)\hat{\beta}(\infty)}.
\]

By the symmetry property we have \( \tilde{p}(-k) = \tilde{p}(k) \). Moreover, this function does not have jumps for \( k \in [-ic, ic] \), and \( \tilde{p}(k) \to 1 \) as \( k \to \infty \). Thus, it must be a meromorphic (in fact, rational) function of \( \lambda = k^2 \) in the whole complex plane.

Due to (4.13) and (4.17) the function \( \hat{\alpha}(k)\hat{\beta}(k) \) has the only zero, simple with respect to \( \lambda \), at the point \( \lambda = \lambda(x,t) \), and the only simple pole (again with respect to \( \lambda \)) at \( \lambda = -a^2 \). We conclude that

\[
\tilde{p}(k) = \frac{\hat{\alpha}(k)\hat{\beta}(k)}{\hat{\alpha}(\infty)\hat{\beta}(\infty)} = \frac{k^2 - \lambda(x,t)}{k^2 + a^2},
\]

which together with (4.5) implies (4.19). \( \square \)
In turn, decomposing (4.19) with respect to \( \frac{1}{\sqrt{2}} \), we get the trace formula (4.16). It proves once more a well-known result that (4.16) is the same as (4.12). Moreover, the property of the combination of theta functions involved in \( m_{1 \text{mod}}^n(k) m_{2 \text{mod}}^n(k) \) to be a rational function of the spectral parameter \( \lambda \) is tightly connected with the analogous property of the product of two branches of the Baker–Akhiezer function. It allows us to expect that this approach may considerably simplify the evaluation of asymptotics in the case of finite gap backgrounds.

5. The matrix model R-H problem solution and its properties

In this section we construct a symmetric matrix solution of the R-H problem associated with the jump problem (3.2) which is holomorphic in \( C \setminus [ic, -ic] \), has singularities of order \( O(k-\kappa)^{-1/4} \) for \( \kappa \in G \) and is invertible for all \( \Delta(\xi) \neq \pi(2n+1), n \in \mathbb{Z} \).

Introduce the function

\[
\gamma(k) = \tilde{\gamma}(k) \sqrt{\frac{c}{a}} = i \sqrt{\frac{k^2 + a^2}{k^2 + c^2}} \sqrt{\frac{c}{a}}
\]

Then

\[
\gamma(-k) = \gamma(k), \quad k \in C \setminus [ic, ia] \cup [-ia, -ic]; \quad \gamma(0) = 1,
\]

and

\[
[\gamma(k) \pm e^{-2\Gamma(k)}]_+ = i [\gamma(k) \mp e^{-2\Gamma(k)}]_-, \quad k \in [ic, ia];
\]

\[
[\gamma(k) \pm e^{-2\Gamma(k)}]_+ = -i [\gamma(k) \mp e^{-2\Gamma(k)}]_-, \quad k \in [-ia, -ic].
\]

Moreover, the function \( \gamma(k) - \gamma^{-1}(k) \) has a second-order zero at the point \( k = 0 \).

Introduce now the functions

\[
\mu^\Lambda(k) = \theta_3 \left( A(k) - \frac{1}{4} - \tilde{\Lambda} \mid \tau \right); \quad \nu^\Lambda(k) = \theta_3 \left( -A(k) - \frac{1}{4} - \tilde{\Lambda} \mid \tau \right).
\]

One can see that the function \( \mu^0(k) = \theta_3 \left( A(k) - \frac{1}{4} \mid \tau \right) \) does not have zeros in the domain \( C \setminus ([ic, ia] \cup [-ic, -ic]) \) which we identified with the upper sheet of the Riemann surface (this function has a zero at the projection of the point \( k = 0 \) on the lower sheet). However,

\[
\nu^0(k) = C_{\pm} k (1 + o(1)), \quad \text{as } k \to \pm 0; \quad C_+ C_- \neq 0.
\]

Set

\[
(5.1) \quad \mu(k) := \frac{\mu^\Lambda(k)}{\mu^0(k)}, \quad \nu(k) := \frac{\nu^\Lambda(k)}{\nu^0(k)}, \quad \tilde{\mu}(k) := \mu(-k), \quad \tilde{\nu}(k) := \nu(-k).
\]

Then \( \mu(k), \nu(k), \tilde{\mu}(k) \) and \( \tilde{\nu}(k) \) satisfy the jump conditions

\[
(5.2) \quad \mu_+(k) = e^{-i\Lambda} \mu_-(k), \quad \nu_+(k) = e^{i\Lambda} \nu_-(k),
\]

\[
\tilde{\mu}_+(k) = e^{i\tilde{\Lambda}} \tilde{\mu}_-(k), \quad \tilde{\nu}_+(k) = e^{-i\tilde{\Lambda}} \tilde{\nu}_-(k)
\]

for \( k \in [ia, -ia] \), as well as

\[
(5.3) \quad \mu_+(k) = \nu_-(k), \quad \nu_+(k) = \mu_-(k),
\]

\[
\tilde{\mu}_+(k) = \tilde{\nu}_-(k), \quad \tilde{\nu}_+(k) = \tilde{\mu}_-(k),
\]

for \( k \in [ic, ia] \cup [-ic, -ic] \).
Lemma 5.1. Let

\[ M^\text{mod}(k) := \begin{pmatrix} \gamma(k) + \gamma^{-1}(k) & \mu(k) \\ \gamma(k) - \gamma^{-1}(k) & \nu(k) \end{pmatrix}, \quad k \in \mathbb{C} \setminus \{ic, -ic\}. \]

Then the matrix \( M^\text{mod}(k) \) is holomorphic in \( \mathbb{C} \setminus \{ic, -ic\} \) and possesses the following properties:

- It solves the jump problem \( M_+^\text{mod}(k) = M_-^\text{mod}(k)v^\text{mod}(k) \) with the jump matrix given by formula \( (3.2) \).
- It satisfies the symmetry condition

\[ M^\text{mod}(k) = \sigma_1 M^\text{mod}(-k) \sigma_1. \]

- It has singularities only at points of the set \( \mathcal{G} \) of the following type:

\[ M^\text{mod}(k) = O((k - \kappa)^{-1/4}) \quad \text{as} \quad k \to \kappa \in \mathcal{G}. \]

- Its determinant does not depend on \( k \) and is given by

\[ \det M^\text{mod}(k) = 4 \frac{\theta_3 \left( \frac{\pi}{2} - \Lambda | \tau \right) \theta_3 \left( -\frac{\pi}{2} - \Lambda | \tau \right)}{\theta_3 \left( \frac{\pi}{2} | \tau \right)^2}. \]

- \( \det M^\text{mod}(k) = 0 \) iff \( \Lambda = \pi(2n + 1) \).
- \( \mu(\infty) > 0, \nu(\infty) > 0 \) uniformly with respect to \( \Lambda \in \mathbb{R} \) and \( \xi \in \mathcal{I}_\varepsilon \).
- The following formula is valid:

\[ m^\text{mod}(k) = \frac{\sqrt{ac}}{(c + a)\mu(\infty) + (c - a)\nu(\infty)} (1, 1) M^\text{mod}(k). \]

Proof. That \( M^\text{mod}(k) \) satisfies the jump condition \( (3.2) \) follows immediately from \( (5.2) \) and \( (5.3) \), while the symmetry condition follows from \( (5.1) \).

As already mentioned above, \( \mu^0(k) \) (\( \mu^0(k) \)) has no zeros on the upper sheet of the Riemann surface, while \( \nu^0(k) \) (\( \nu^0(k) \)) has from both sides a simple zero at \( k = 0 \). However, because of the second-order zero of \( \gamma(k) - \gamma^{-1}(k) \), the resulting singularity is absorbed. In fact, \( M^\text{mod}(k) \) vanishes off-diagonal entries at the origin. So we conclude that the only singularities that can arise, come from \( \gamma(k) \) and \( \gamma^{-1}(k) \) for \( k \to \kappa \in \mathcal{G} \), which are of order \( O((k - \kappa)^{-1/4}) \).

The determinant \( \det M^\text{mod}(k) \) is holomorphic away from the contour, with at most square root singularities at \( \kappa \in \mathcal{G} \). As \( \det v(k) = 1 \), we can conclude, as in Theorem \( (1.1) \) that \( \det M^\text{mod}(k) \) must be constant. Hence, we need to evaluate the determinant just at one point, which we choose to be \( k = 0 \). As the off-diagonal entries vanish at the origin because of the second-order zero of \( \gamma(k) - \gamma^{-1}(k) \), we just need to multiply the diagonal entries, which evaluates to \( (5.4) \). As the zeros of \( \theta_3(z) \) lie at \( z = -\frac{1}{2} + \frac{\pi}{2} (\text{mod} 1, \tau) \), we immediately conclude that the determinant of \( M^\text{mod}(k) \) vanishes, if and only if \( \Lambda = 2\pi \tilde{\Lambda} = \pi(2n + 1) \) for \( n \in \mathbb{Z} \).

Next let us consider \( \mu(\infty) \) as a function of \( \tilde{\Lambda} \in \mathbb{R} \),

\[ \mu(\infty, \tilde{\Lambda}) = \frac{\theta_3(\tilde{\Lambda})}{\theta_3(0)}. \]

As \( \theta_3(z) \) does not have real roots, is periodic on the real axis and \( \mu(\infty, 0) = 1 \), we can conclude that \( \mu(\infty, \tilde{\Lambda}) > C \) for all \( \tilde{\Lambda} \in \mathbb{R} \) and some positive constant \( C \). The same reasoning applies to \( \nu(\infty) \).

Finally, applying \( (1.15) \) in our case gives us \( (5.5) \), which exists for all values of \( \tilde{\Lambda} \) by the positivity of \( \mu(\infty) \) and \( \nu(\infty) \). \( \Box \)
Corollary 5.2. For all pairs \((x, t) \in \mathcal{D}_z\) (cf. (1.5)) the estimate
\[
(5.6) \quad |\det M_{\text{mod}}(k)| \geq \frac{C}{a^2(-c^2/2 + \varepsilon)t^r},
\]
is valid uniformly with respect to \(\xi \in \mathcal{I}_z\).

Proof. Estimate (5.6) follows immediately from Lemmas 4.6, 5.1 and the fact that \(\theta_3^*(\xi + \frac{1}{2}) \neq 0\).

6. The Matrix Solution of the Parametrix Problem

In this section we study the matrix solutions of the local R-H problems in vicinities of the points \(\pm ia\). Consider first the point \(ia\). Let \(\mathcal{B}^U = \mathcal{B}^U(\varepsilon)\) be a neighbourhood of this point such that \(\partial \mathcal{B}^U \cap [ia, 0] = \{i(a - \rho)\} = \{ib\}\). To describe the boundary of \(\mathcal{B}^U\) in more details, introduce in it a local change of variables
\[
(6.1) \quad w^{3/2}(k) = -\frac{3it}{2}(g(k) - g_{\pm}(ia)), \quad k \in \mathcal{B}^U,
\]
with the cut along the interval \(J := [ic, ia] \cap \overline{\mathcal{B}^U}\). We observe that
\[
(6.2) \quad w^{3/2}(k) = P(a)e^{\frac{3i}{2}t}(k - ia)^{3/2}(1 + O(k - ia)), \quad P(a) > 0.
\]
Indeed, from (2.22) and Lemma 2.3 it follows that for \(is \rightarrow ia \pm 0\)
\[
\text{Re}(-ig(is)) = 12 \int_{a \pm 0}^{s} \left(\frac{c^2 - a^2}{2} + \xi - s^2\right) \sqrt{\frac{a + s}{c^2 - s^2} - \sqrt{s - a}ds}
\]
\[
= -8 \left(\frac{c^2 - 3a^2}{2} + \xi\right) \sqrt{\frac{2a}{c^2 - a^2}(a - s)^{3/2}(1 + O(a - s))}.
\]
Since \(a(\xi)\) is a monotonous function with \(a\left(\frac{c^2}{4}\right) = c\) and \(a\left(-\frac{c^2}{2}\right) = 0\), this implies (6.2) with \(P(a) > 0\). Thus, \(w(k)\) is a holomorphic function in \(\mathcal{B}^U\) with \(w(ia) = 0\), \(w'(k) \neq 0\).

Till now we did not specify a particular shape of the boundary of \(\mathcal{B}^U\) and the shape of the contour \(\Sigma^U_i\) inside \(\mathcal{B}^U\). Treating \(w(k)\) as a conformal map, let us think of \(\mathcal{B}^U\) as a preimage of a disc \(\mathcal{O}\) of radius \(P^{2/3}(a)\rho t^{2/3}\) centered at the origin. Since \(w(k) = P_1(a)\rho^{2/3}(ik + a)(1 + o(1))\), the function \(w(k)\) maps the interval \([ia, ic] \cap \mathcal{B}^U\) into the negative half axis. We can always choose the contours \(\Sigma^U_i \cap \mathcal{B}^U\) to be contained in the preimage of the rays \((a, 0) = \pm \frac{2\pi}{3}\).

Next, in \(\mathcal{B}^U\) introduce the function
\[
\rho(k) := \frac{\sqrt{\chi(k)}}{\rho(k)} e^{\frac{\pi}{4}i} e^{\frac{\pi\rho}{2}}, \quad k \in \mathcal{B}^U \cap \{k: \pm \text{Re} \, k > 0\},
\]
where \(\chi\) and \(F\) are defined by (2.7) and (2.25) respectively, and \(B = -2g_+(ia)\). By (2.8), (2.13) and Lemma 2.4 we conclude that
\[
\rho_+(k) = \frac{\sqrt{\chi(k)}}{F_+(k)} e^{\frac{\pi\rho}{2}}, \quad \rho_-(k) = \frac{\sqrt{\chi(k)}}{F_-(k)} e^{\frac{\pi\rho}{2}}, \quad k \in [ic, 0] \cap \mathcal{B}^U.
\]
Therefore,
\[
(6.3) \quad \rho_+(k)\rho_-(k) = 1, \quad k \in J; \quad \rho_+(k) = \rho_-(k)e^{-i\Delta - uB}, \quad k \in J',
\]
where we defined
\[
J := [ic, ia] \cap \overline{\mathcal{B}^U}, \quad J' := [ia, ib] = [ia, 0] \cap \overline{\mathcal{B}^U}.\]
Denote also
\[ L_1 = \Sigma^U \cap B_U \cap \{ \Re k \geq 0 \}; \quad L_2 = \Sigma^U \cap B_U \cap \{ \Re k \leq 0 \}. \]

Recall that the vector function \( m^{(3)}(k) \) satisfies the jump condition \( m_+^{(3)}(k) = m_-^{(3)}(k) w^{(3)}(k) \), with the jump matrix (2.30). Redefine now \( m^{(3)}(k) \) inside the domains \( B_U \) and \( B_L := \{ k : -k \in B_U \} \) by formula
\[
(6.4) \quad m^{(4)}(k) = \begin{cases} 
  m^{(3)}(k)[r(k)]^{-\sigma_3}, & k \in B_U, \\
  m^{(4)}(-k)\sigma_1, & k \in B_L, \\
  m^{(3)}(k), & k \in \mathbb{C} \setminus (B_U \cup B_L).
\end{cases}
\]

By use of (6.3) we get \( m_+^{(4)}(k) = m_-^{(4)}(k) v^{(4)}(k) \) with
\[
(6.5) \quad v^{(4)}(k) = \begin{cases} 
  (i e^{-4/3 w(k) 3/2} 0) \begin{pmatrix} 1 \\ i \sigma_1 \end{pmatrix}, & k \in J', \\
  (1 e^{4/3 w(k) 3/2} 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & k \in J, \\
  (1 -e^{4/3 w(k) 3/2} 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & k \in L_1, \\
  r(k)^{-\sigma_3}, & k \in \partial B^U, \\
  \sigma_1 [v^{(4)}(-k)]^{-1} \sigma_1, & k \in \partial B_L \cup G^L, \\
  v^{(3)}(k), & k \in \Sigma^{(4)} \setminus (G^U \cup G^L),
\end{cases}
\]

where
\[
(6.6) \quad \Sigma^{(4)} = \{ i c, -ic \} \cup \Sigma^U \cup \Sigma^L \cup \Sigma^U_1 \cup \Sigma^L_1, \\
G^U = J \cup J' \cup L_1 \cup L_2, \quad G^L = \{ k : -k \in G^U \},
\]
and the orientation preserving symmetries hold. In particular, \( \partial B^L \) should be oriented clockwise.

We observe that transformation (6.4) applied in \( B^U \) to the matrix model problem solution,
\[
(6.7) \quad M(k) := M^{(\text{mod})}(k)[r(k)]^{-\sigma_3}, \quad k \in B^U,
\]
leads to wiping out of the jump along \( J' \), i.e. in \( B^U \) the matrix \( M \) satisfies the jump condition \( M_+(k) = i M_-(k)\sigma_1, k \in J \). Next by (6.1), the function \( w^{1/4}(k) \) has the
following jump along the interval $J$:

$$w_+^{1/4}(k) = w_-^{1/4}(k), \quad k \in J.$$ 

Put $O = w(B)$. It is now straightforward to check that the matrix

$$N(w) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} \\
-w^{-1/4} & w^{-1/4}
\end{pmatrix}, \quad w \in O,$$

solves the jump problem

$$N_+(w(k)) = iN_-(w(k))\sigma_1, \quad k \in J.$$ 

Therefore, in $B^U$ we have $M(k) = H(k)N(w(k))$, where $H(k)$ is a holomorphic matrix function in $B^U$. Moreover, since $\det N(w) = \det[\sigma_3] = 1$, we have

(6.8) \hspace{1cm} \det H(k) = \det M^{mod}(k) = \det M(k).$$

According to (2.3) we get then

(6.9) \hspace{1cm} M^{mod}(k) = H(k)N(w(k))r(k)^{\sigma_3}, \quad k \in \partial B^U.$$

Next, by property (b) of Lemma 2.3 $w_+(k)^{3/2} = -w_-(k)^{3/2}, \quad k \in J$, that is

$$u^{(4)}(k) = d_-(k)^{\sigma_3}S d_+(k)^{-\sigma_3}, \quad k \in B^U,$$

where

$$d(k) := \tilde{d}(w(k)), \quad \tilde{d}(w) = e^{2/3w^{3/2}},$$

and

$$S = \begin{cases}
\begin{cases}
i \sigma_1, & k \in J, \\
1 & 0, \quad k \in J', \\
1 & i, \quad k \in L_1, \\
0 & 1, \quad k \in L_2
\end{cases}, & 
\begin{cases}
i \sigma_1, & k \in J, \\
1 & 0, \quad k \in J', \\
1 & i, \quad k \in L_1, \\
0 & 1, \quad k \in L_2
\end{cases}
\end{cases}$$

Let us consider the constant matrix $S$ as the jump matrix on the contour $\Gamma := w(G^U)$ (see (6.6)). Let $A(w)$ be the matrix solution of the jump problem

$$A_+(w) = A_-(w)S, \quad w \in \Gamma,$$

satisfying the boundary condition

$$A(w) = N(w)\Psi(w)\tilde{d}(w)^{\sigma_3}, \quad w \in \partial O, \quad t \rightarrow \infty,$$

where

$$\Psi(w) = I + \frac{C}{w^{3/2}}(1 + O(w^{-3/2})), \quad w \rightarrow \infty,$$

is an invertible matrix and $C$ is a constant matrix with respect to $w$, $t$ and $\xi$. The solution $A(w)$ can be expressed via the Airy functions and their derivatives in a standard way (see, for example, [8], [9] Chapter 3, [10] or [11]). In particular,

(6.10) \hspace{1cm} C = \frac{1}{48} \begin{pmatrix}
-1 & 6 \\
-6 & 1
\end{pmatrix},$$

and in the domain between the contours $w(J')$ and $w(L_1)$ we have

$$A(w) = \sqrt{2\pi} \begin{pmatrix}
-y_1'(w) & iy_2'(w) \\
y_1(w) & iy_2(w)
\end{pmatrix},$$
where \( y_1(w) = \Lambda i(w) \) and \( y_2(w) = e^{-2\pi i} \Lambda i(e^{-2\pi i} w) \). The precise formula for \( A(w) \) in the other domains can be obtained by simple multiplication on the jump matrix \( S \), but it is not important for us.

Define the matrix

\[
M^{par}(k) := H(k)A(w(k))d(k)^{-\sigma_3}, \quad k \in \mathcal{B}^U.
\]

This matrix then solves in \( \mathcal{B}^U \) the jump problem

\[
M^{par}_+(k) = M^{par}_-(w^{(4)}(k)), \quad k \in \mathcal{G}^U = J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2,
\]

and satisfies for sufficiently large \( t \) the boundary condition

\[
M^{par}_+(k) = H(k)N(w(k))\Psi(w(k)) = M(k)\Psi(w(k)), \quad k \in \partial \mathcal{B}^U.
\]

In \( \mathcal{B}^L \) we define \( M^{par}(k) \) by symmetry

\[
M^{par}(k) = \sigma_1 M^{par}(-k)\sigma_1.
\]

7. Completion of asymptotical analysis

The aim of this section is to establish that the solution \( m^{(4)}(k) \) is well approximated by \( (1 \ 1) M^{par}(k) \) inside the domain \( \mathcal{B} := \mathcal{B}^U \cup \mathcal{B}^L \) and by \( (1 \ 1) M^{mod}(k) \) in \( \mathbb{C} \setminus \mathcal{B} \). We follow the well-known approach via singular integral equations (see e.g., [9], [18], [21] Chapter 4, 23). To simplify notation we introduce (see formula (6.6))

\[
\hat{\Sigma} = \Sigma^{(4)} \cup \partial \mathcal{B}^U \cup \partial \mathcal{B}^L.
\]

Set

\[
\hat{m}(k) = m^{(4)}(k)(M^{as}(k))^{-1}, \quad M^{as}(k) := \begin{cases} M^{par}(k), & k \in \mathcal{B}, \\ M^{mod}(k), & k \in \mathbb{C} \setminus \mathcal{B}. \end{cases}
\]

Formula (6.11) implies that \( \hat{m} \) does not have jumps inside \( \mathcal{B} \). Let us compute the jump of this vector on \( \partial \mathcal{B}^U \) by use of (6.5), (6.9), (6.7) and (6.12):

\[
\hat{m}_+ = m^{(4)}(M^{par}_+)^{-1} = m^{(4)}_-(r^{-\sigma_3})\Psi^{-1}M^{par}_+^{-1} = m^{(4)}_-(M^{mod}_-)^{-1}M^{mod}_-r^{-\sigma_3}\Psi^{-1}M^{par}_+^{-1} = \hat{m}_-M^{mod}_+\Psi^{-1}M^{par}_+^{-1}.
\]

Here we took into account (6.7) and that \( M^{mod} \) does not have a jump on \( \partial \mathcal{B}^U \). Note also that both matrices \( M^-(k) \) and \( M^{mod}(k) \) are bounded with respect to \( t \) uniformly on \( \partial \mathcal{B}^U \).

Next, the structure of the matrix \( \Psi(w(k)) \) implies that

\[
\Psi^{-1}(w(k)) = 1 + \frac{\mathcal{F}(k, t)}{t(g(k) - g_{\pm}(ia))}, \quad \|\mathcal{F}(k, t)\| \leq O(1), \quad t \to \infty,
\]

where the matrix norm estimate \( O(1) \) is uniform with respect to \( k \) on the compact \( \partial \mathcal{B}^U(\xi) \cup \partial \mathcal{B}^L(\xi) \), and uniform with respect to \( \xi \in \mathcal{I}_\varepsilon = (-\varepsilon/2, \varepsilon/2) \). Hence \( \hat{m}(k) \) solves the jump problem

\[
\hat{m}_+(k) = \hat{m}_-(k)\hat{v}(k),
\]

where

\[
\hat{v}(k) = \begin{cases} \mathbb{I} + \frac{\mathcal{F}(k, t)}{t(g(k) - g_{\pm}(w))} M(k)^{-1}, & k \in \partial \mathcal{B}^U, \\ \sigma_1\hat{v}(-k)^{-1}\sigma_1, & k \in \partial \mathcal{B}^L, \\ M^{mod}(k)w^{(4)}(k)(M^{par}(k))^{-1}, & k \in \hat{\Sigma} \setminus \mathcal{B}, \end{cases}
\]
and satisfies the symmetry and normalization conditions:
\[
\hat{m}(k) = \hat{m}(-k)\sigma_1, \quad \hat{m} \to (\alpha, \alpha), \quad k \to \infty.
\]

where
\[
\alpha = \frac{\sqrt{ac}}{(c + a)\mu(\infty) + (c - a)\nu(\infty)},
\]
is taken from [5, 5]. Note that we do not need to establish uniqueness for this R-H problem, that is why for convenience we change the orientation on \(\partial B^L\) to counterclockwise. Note also that the constant matrix (6.10) plays a significant role in evaluating the second term of expansion of \(q(x, t)\) with respect to \(t\). However, here we justify the main term of asymptotics, and for this purpose it is sufficient to use a rough estimate (7.2). Abbreviate \(W(k) = \hat{v}(k) - I\). Recall that in fact \(v^{(4)} = v^{\text{mod}}\) on \([ib, -ib]\). Hence

\[
W(k) = \begin{cases}
\frac{1}{t(g(k) - g_2(\xi))} M_+(k) F(k, t) M_+^{-1}(k), & k \in \partial B^U, \\
\sigma_1 W(-k) \sigma_1, & k \in \partial B^L, \\
0, & k \in [ib, -ib], \\
M_-^{\text{mod}}(k)(v^{(4)}(k) - I)(M_+^{\text{mod}}(k))^{-1}, & k \in \bar{\Sigma} \setminus (\bar{B} \cup [-ib, ib]).
\end{cases}
\]

Thus the error vector \(\hat{m}(k)\) has jumps on the contour (cf. (6.6))
\[
\hat{\Sigma} = \text{clos} \left( \Sigma^{(4)} \cup \partial B^U \cup B^L \setminus (g^U \cup g^L \cup [ic, -ic]) \right)
\]
only. We observe that for all \((x, t) \in D_\varepsilon\) the matrix \(W(k)\) is continuous on any smooth part of the contour \(\hat{\Sigma}\) and bounded with respect to \(k\). Moreover, since
\[
\|k^j (v^{(4)}(k) - I)\|_{L_p(\hat{\Sigma} \setminus \partial B)} = O(e^{-C(\varepsilon)t}), \quad p \in [1, \infty], \quad j = 0, 1, 2
\]
the estimates on the higher moments will be used later we get using (7.4), (5.6) and (6.8) the estimate

**Lemma 7.1.** The following estimates hold uniformly with respect to \(\xi \in I_\varepsilon\) and \((x, t) \in D_\varepsilon:\)
\[
\|k^j W(k)\|_{L_p(\hat{\Sigma})} \leq C(\varepsilon)t^{-1 + \gamma}, \quad p \in [1, \infty], \quad j = 0, 1.
\]

Now we are ready to apply the technique of singular integral equations. Since this is well known (see, for example, [9], [18], [23]) we will be brief and only list the necessary notions and estimates.

Let \(C\) denote the Cauchy operator associated with \(\hat{\Sigma}:
\[
(CH)(k) = \frac{1}{2\pi i} \int_{\hat{\Sigma}} h(s) \frac{ds}{s - k}, \quad k \in \mathbb{C} \setminus \hat{\Sigma},
\]
where \(h = (h_1, h_2) \in L^2(\hat{\Sigma})\). Let \(C_+ f\) and \(C_- f\) be its non-tangential limiting values from the left and right sides of \(\hat{\Sigma}\) respectively.

As usual, we introduce the operator \(C_W : L^2(\hat{\Sigma}) \cup L^\infty(\hat{\Sigma}) \to L^2(\hat{\Sigma})\) by formula \(C_W f = C_-(fW)\), where \(W\) is our error matrix (7.4). Then,
\[
\|C_W\|_{L^2(\hat{\Sigma})} \leq C\|W\|_{L^\infty(\hat{\Sigma})} \leq O(t^{-1 + \gamma}),
\]
as well as
\[
\|(I - C_W)^{-1}\|_{L^2(\hat{\Sigma})} \leq \frac{1}{1 - O(t^{-1 + \gamma})}
\]
for sufficiently large $t$. Consequently, for $t \gg 1$, we may define a vector function
\[ \mu(k) = (\alpha, \alpha) + (I - C_W)^{-1}C_W((\alpha, \alpha))(k), \]
with $\alpha$ defined by (7.3). Then by (7.5) and (7.6)
\begin{align*}
\|\mu(k) - (\alpha, \alpha)\|_{L^2(\Sigma)} & \leq \|\|I - C_W\|^{-1}\|_{L^2(\Sigma) \to L^2(\Sigma)}\|C_W\|_{L^2(\Sigma) \to L^2(\Sigma)}\|W\|_{L^\infty(\Sigma)} \\
& = O(t^{-1+\gamma}).
\end{align*}
With the help of $\mu$, (7.1) can be represented as
\[ \hat{m}(k) = (\alpha, \alpha) + \frac{1}{2\pi i} \int_{\Sigma} \mu(s)W(s)ds, \]
and in virtue of (7.7) and Lemma 7.1 we obtain as $k \to +i\infty$:
\[ \hat{m}(k) = (\alpha, \alpha) + \frac{1}{2\pi i} \int_{\Sigma} (\alpha, \alpha)W(s)ds + E(k), \]
where
\[ |E(k)| \leq \frac{1}{\text{Im}(k + ic)} \|W\|_{L^2(\Sigma)} \|\mu(k) - (\alpha, \alpha)\|_{L^2(\Sigma)} \leq O(t^{-2+2\gamma}) \frac{\text{Im}(k + ic)}{k}, \]
where $O(t^{-2+2\gamma})$ is uniformly bounded with respect to $\xi \in \mathcal{I}_e$, $(x,t) \in \mathcal{D}_\varepsilon$, and $k \to \infty$. In the regime $\text{Re} \, k = 0$, $\text{Im} \, k \to +\infty$ we have
\[ \frac{1}{2\pi i} \int_{\Sigma} (\alpha, \alpha)W(s)ds = \frac{f_0(\xi, t)}{2i k t^{1-\gamma}}(\alpha, -\alpha) + \frac{f_1(\xi, t)}{2k^2 t^{1-\gamma}}(\alpha, \alpha) \\
+ O(t^{-1+\gamma})O(k^{-3}) + O(t^{-2+\gamma})O(k^{-1}), \]
where $f_{0,1}(\xi, t)$ are uniformly bounded for $t \to \infty$ and $\xi \in \mathcal{I}_e$. Furthermore $O(k^{-s})$ are vector functions depending on $k$ only and $O(t^{-\gamma})$ are as above. Hence, 
\[ m_1^{(4)}(k) = \hat{m}(k)M^{mod}(k) = m^{mod}(k) + \frac{f_0(\xi, t)}{2ik t^{1-\gamma}}(\alpha, -\alpha)M^{mod}(k) \]
\[ + \frac{f_1(\xi, t)}{2k^2 t^{1-\gamma}}m^{mod}(k) + O(t^{-1+\gamma})O(k^{-3}) + O(t^{-2+\gamma})O(k^{-1}). \]
Now we are in a position to apply (1.10), making use of (4.11) and (4.12). Note that since all conjugation steps in the vicinity of $\infty$ involved diagonal matrices with determinant 1, we have
\[ m_1(k)m_2(k) = m_1^{(4)}(k)m_2^{(4)}(k) = m_1^{mod}(k)m_2^{mod}(k) + O(t^{-1+\gamma})O(k^{-2}). \]
Here we used that the entries of $M^{mod}(k)$ are uniformly bounded for $\xi \in \mathcal{I}_e$ and that the $k^{-1}$ term disappears by symmetry (2.17).
Thus, we recomputed in a simpler and more rigorous way formulas (4.11) and (4.12), and rigorously justified asymptotics (1.4), (1.6) of the shock wave in the elliptic zone. We proved the following

**Theorem 7.2.** Let $q(x,t)$ be the unique solution of the initial value problem (1.1), (1.2), (2.1) associated with the initial scattering data (2.15).
Let $\varepsilon$ be a small positive number and let $\gamma \in (0, \frac{1}{2})$ be another given number. For any $\xi \in \mathcal{I}_e$ (cf. (1.3)) let $a(\xi) \in (0, c)$ be defined implicitly by (2.21).
On the Riemann surface $\mathcal{M}(\xi)$ associated with the set $\sigma(\xi) := [-c^2, -a(\xi)^2] \cup [0, \infty]$ let $p_0(\xi)$ be the point given by the Jacobi inversion (4.13) with $\Delta$ defined via (2.15) by formula (2.27). Let $p(x,t) \in \mathcal{M}(\xi)$ be the solution of (4.17), where $W$
and $V$ are defined by (1.14). Let $\lambda(x, t, \xi) \in [-a^2, 0]$ be the projection of $p(x, t)$ on the complex plane, and $D_\varepsilon$ be the domain defined by (1.5).

Let $q^{\text{mod}}(x, t, \xi)$ be given by (1.4), or equivalently, by formula (4.15). For any $\xi \in \mathcal{I}_{\varepsilon}$ fixed, this formula represents the unique finite gap (in fact, periodic) solution of the KdV equation corresponding to the spectrum $\sigma(\xi)$ and to the initial divisor $p_0(\xi)$.

Then for all $x \to \infty$, $t \to \infty$ and $(x, t) \in D_\varepsilon$ the following asymptotics is valid uniformly with respect to $\xi \in \mathcal{I}_{\varepsilon}$:

\begin{equation}
q(x, t) = q^{\text{mod}}(x, t, \xi) + O(t^{-1+\gamma}).
\end{equation}

Acknowledgments. We are grateful to Alexander Minakov for useful discussions. I.E. is indebted to the Department of Mathematics at the University of Vienna for its hospitality and support during the winter of 2019, where this work was done.

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