

BOUND STATES OF DISCRETE SCHRÖDINGER OPERATORS WITH SUPER-CRITICAL INVERSE SQUARE POTENTIALS

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ABSTRACT. We consider discrete one-dimensional Schrödinger operators whose potentials decay asymptotically like an inverse square. In the super-critical case, where there are infinitely many discrete eigenvalues, we compute precise asymptotics of the number of eigenvalues below a given energy E as this energy tends to the bottom of the essential spectrum.

1. INTRODUCTION

This paper is concerned with discrete one-dimensional Schrödinger operators in $\ell^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. That is,

$$Hu(n) = -\Delta u(n) + V(n)u(n), \quad \Delta u(n) = u(n+1) - 2u(n) + u(n-1),$$

where we impose a Dirichlet boundary condition, $u(0) = 0$.

In the free case, $V \equiv 0$, the spectrum of $H = -\Delta$ is the interval $[0, 4]$. If $V(n) \rightarrow 0$ as $n \rightarrow \infty$, then V is compact and hence zero is the bottom of the essential spectrum of H . We are interested in the discrete spectrum of H below zero. It consists of simple eigenvalues, so-called bound states, that can only accumulate at zero. Thus, for $E \geq 0$, we define

$$N_E(H) = \dim \text{Ran} P_H((-\infty, -E]),$$

where P_H is the family of spectral projections associated with H by the spectral theorem.

It is well known that $V(n) \sim -n^{-2+\varepsilon}$ produces finitely many discrete eigenvalues if $\varepsilon < 0$ and infinitely many if $\varepsilon > 0$ and so inverse square decay is critical for the existence of infinitely many discrete eigenvalues below the essential spectrum. Furthermore, by the discrete analogue of Kneser's theorem, the discrete spectrum below zero of the operator H with potential $V(n) = -cn^{-2}$ is finite when $c \leq \frac{1}{4}$ and infinite when $c > \frac{1}{4}$. Surprisingly, this discrete analog turns out much harder to prove than the continuum result and it is not so well known. It was first shown by Naïman [9] and later rediscovered in [2] (see also [8] for a more general result).

Our goal is to study the behavior of $N_E(H)$ as $E \downarrow 0$ in the super-critical case $c > \frac{1}{4}$. This question is natural since this case describes the borderline behavior at

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the transition between finitely and infinitely many bound states as pointed out in the previous paragraph. In addition, we were also motivated by recent results on a connection between singular spectrum embedded in the essential spectrum and the discrete spectrum of a given Schrödinger operator.

Let us sketch this connection briefly. For operators of the form $H = -\Delta + V$ on the whole line, that is, acting in $\ell^2(\mathbb{Z})$, it was shown by Killip and Simon that the absence of bound states implies that the potential vanishes identically [6]. In other words, $\sigma(H) \subseteq [0, 4]$ implies $V \equiv 0$. Damanik et al. found a simple variational proof of this statement that also allowed them to extend the result to operators in $\ell^2(\mathbb{Z}^2)$ [1]. It does not hold in dimensions three or higher, nor on the half-line, as shown in [1]. The counterexamples in $\ell^2(\mathbb{Z}^d)$, $d \geq 3$, are given by sparse non-decaying potentials, whereas a possible counterexample in $\ell^2(\mathbb{Z}_+)$ is given by $V(n) = (-1)^n/n$. Damanik and Killip then showed that, on the half-line, the absence of bound states implies the absence of embedded singular spectrum. In fact, absence of embedded singular spectrum follows already when there are finitely many bound states [3]. This raised the natural question of what can be said when there are infinitely many bound states. Is it true that sufficiently fast decay, for example finiteness of the moments

$$M_\gamma = \sum_{n \geq 1} (\text{dist}(E_n, [0, 4]))^\gamma,$$

implies the absence of embedded singular spectrum? Here, $\{E_n\}_{n \geq 1}$ denote the discrete eigenvalues of H outside the essential spectrum $[0, 4]$. Note that the Damanik-Killip result can be phrased as follows: $M_0 < \infty$ implies that $\sigma_{\text{sing}}(H) \cap [0, 4] = \emptyset$. Damanik et al. then found an example with 2 as an embedded eigenvalue (and hence $\sigma_{\text{sing}}(H) \cap [0, 4] \neq \emptyset$) and $M_\gamma < \infty$ for every $\gamma > 0$ [4]¹. This example has the potential $V(n) = (1 + \varepsilon)(-1)^n/n$ for some small $\varepsilon > 0$. The core of the proof consists of using methods developed in [1] and [2] to compare this sign-indefinite potential with two sign-definite ones, both of the form $-c_j n^{-2}$, for different constants $c_1, c_2 > 1/4$. This clearly motivates a study of the bound state structure of super-critical inverse square potentials. The estimates for their bound states derived in [4] were sufficient to prove the result mentioned above, but it is desirable to have a more complete understanding; in particular since in the continuous case the bound state structure has been studied in detail earlier. Namely, Kirsch and Simon considered operators of the form $-\frac{d^2}{dx^2} + V$ and carried out an investigation of $N_E(H)$ for super-critical inverse square potentials [7] (see also Schmidt [10] for extensions). They proved that the bound states approach the essential spectrum geometrically. More precisely, they showed that

$$(1.1) \quad \lim_{E \downarrow 0} \frac{N_E(-\Delta + V)}{-\ln(E)} = \frac{1}{2\pi} \sqrt{c - \frac{1}{4}},$$

where $V(x) = -\frac{c}{1+x^2} + W(x)$ and W is such that $N_0(-\Delta + \gamma W) < \infty$ for all $\gamma \in \mathbb{R}$.² Let us also note that in the continuous case, a study of inverse square potentials

¹It was later realized by Damanik and Remling that $M_\gamma < \infty$ does indeed put some restrictions on the embedded singular spectrum; see [5]. For example, if $M_\gamma < \infty$ for every $\gamma > 0$, then the embedded singular spectrum is supported on a set of zero Hausdorff dimension.

²This is the half-line version of the Kirsch-Simon result. On the whole line, one has to multiply the right-hand side of (1.1) by two; compare [6, Theorem 1].

is further motivated by the fact that they arise in the partial wave analysis of rotationally symmetric partial differential operators.

We will prove the discrete analogue of the Kirsch-Simon result in this paper, that is, geometric approach of the bound states to the essential spectrum.

On the one hand, as explained above, this result is relevant to the investigations and results described in the previous paragraph that led the authors of [4] to ask if the Kirsch-Simon result has a discrete analogue; compare [4, Sect. 2]. On the other hand, the proof of Kirsch and Simon uses some arguments that do not carry over directly to the discrete case: They scale the spatial variable and use exact solvability of the Euler differential equation. Spatial scaling is not possible in the discrete case and, while there exists a discrete Euler equation, it is not symmetric.

Theorem 1. *Suppose*

$$V(n) = -\frac{c}{n^2} + W(n), \quad c > \frac{1}{4},$$

where W is a decaying sequence such that $N_0(-\Delta + \gamma W) < \infty$ for all $\gamma \in \mathbb{R}$. Then

$$(1.2) \quad \lim_{E \downarrow 0} \frac{N_E(-\Delta + V)}{-\ln(E)} = \frac{1}{2\pi} \sqrt{c - \frac{1}{4}}.$$

- Remarks.* (i) We say that a sequence W is decaying if $W(n) \rightarrow 0$ as $n \rightarrow \infty$.
 (ii) The hypothesis on W is satisfied, for example, if $\sum_{n>0} n|W(n)| < \infty$. See [12, Thm. 5.10].
 (iii) An analogous result holds on the whole line. This follows quickly from the half-line result since the whole-line case can be reduced to the half-line case by Dirichlet decoupling.
 (iv) For perturbations of the form $V(n) = \frac{c}{n^2} + W(n)$, an analogous result holds near the top of the essential spectrum.

2. PROOF OF THEOREM 1

As a preparation we state the discrete analog of Proposition 5 from [7]. The proof is analogous.

Lemma 2. *Let V, W be decaying sequences. Then for every $E > 0$ and $0 < \varepsilon < 1$ we have*

$$\begin{aligned} N_E(-\Delta + V + W) &\leq N_E\left(-\Delta + \frac{1}{1-\varepsilon}V\right) + N_E\left(-\Delta + \frac{1}{\varepsilon}W\right), \\ N_E(-\Delta + V + W) &\geq N_E(-\Delta + (1-\varepsilon)V) - N_E\left(-\Delta - \frac{1-\varepsilon}{\varepsilon}W\right). \end{aligned}$$

Now we come to the proof of our main theorem. We start with

$$V_c(n) = -\frac{c}{n^2}$$

and replace it by $V_{E,c}$ which is just $V_c - E$ on $\{n : V_c(n) \leq -E\}$ and equal to V otherwise. To investigate the asymptotics of $N_E(-\Delta + V_{E,c})$ we split our domain into two parts by cutting at $\sqrt{\frac{c}{E}}$. For the first part, we will compute the asymptotics of N_E directly. The remaining part does not contribute to N_E . Then we use Lemma 2 to show that N_E has the same asymptotics for $V_{E,c}$ and $V = V_c + W$.

Lemma 3. *We have*

$$\lim_{E \downarrow 0} \frac{N_E(-\Delta + V_{E,c})}{-\ln(E)} = \frac{1}{2\pi} \sqrt{c - \frac{1}{4}}.$$

Proof. We first decompose $-\Delta + V_{E,c}$ into two parts by imposing an additional Dirichlet boundary condition at $\lfloor \sqrt{\frac{c}{E}} \rfloor$. Since this constitutes a rank-one resolvent perturbation it will not affect the limit. By the choice of our cut point, the part with $n > \lfloor \sqrt{\frac{c}{E}} \rfloor$ does not contribute and by oscillation theory (see e.g. [11] or [12, Ch. 4]) it suffices to count the number of sign flips of some solution of $(-\Delta + V_{E,c})u = -Eu$ on $(1, \sqrt{\frac{c}{E}})$, that is, the number of sign flips of some solution of $(-\Delta + V_c)u = 0$ on $(1, \sqrt{\frac{c}{E}})$.

Unfortunately, $(-\Delta + V_c)u = 0$ is not explicitly solvable, but

$$\tilde{u}_c(n) = \sqrt{n} \exp\left(i\sqrt{c - \frac{1}{4}} \ln(n)\right)$$

solves $(-\Delta + \tilde{V}_c)\tilde{u} = 0$ with the complex-valued potential

$$\tilde{V}_c(n) = \frac{\Delta \tilde{u}_c(n)}{\tilde{u}_c(n)} = -\frac{c}{n^2} + O\left(\frac{1}{n^3}\right).$$

Moreover, it is straightforward to check (cf. [12, Lemma 7.10], resp. [8]) that $-\Delta u + V_c u = 0$ has a solution u_c which asymptotically looks like $\tilde{u}_c(n)$. Taking the real part of u_c , we see that the number of sign flips behaves to leading order like $-\frac{1}{2\pi} \sqrt{c - \frac{1}{4}} \ln(E)$. □

Let us prove the upper bound in (1.2). By Lemma 2,

$$\begin{aligned} N_E(-\Delta + V_c + W) &= N_E(-\Delta + (V_c - \chi_{(1-\varepsilon)E,c}) + N_E(\chi_{(1-\varepsilon)E,c} + W)) \\ &\leq N_E(-\Delta + V_{E,c/(1-\varepsilon)}) + N_E(-\Delta + \frac{1}{\varepsilon}(\chi_{(1-\varepsilon)E,c} + W)), \end{aligned}$$

where $\chi_{E,c} = E\chi_{(0, \sqrt{c/E})}$ and χ_Ω is the characteristic function of the set Ω . Using

$$N_E(-\Delta + \frac{1}{\varepsilon}(\chi_{(1-\varepsilon)E,c} + W)) \leq N_0(-\Delta + \frac{1}{\varepsilon}W),$$

the assumption on W , and Lemma 3, we see that

$$\limsup_{E \downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \leq \frac{1}{2\pi} \sqrt{\frac{c}{1-\varepsilon} - \frac{1}{4}}$$

for every $0 < \varepsilon < 1$, that is,

$$(2.1) \quad \limsup_{E \downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \leq \frac{1}{2\pi} \sqrt{c - \frac{1}{4}}.$$

It remains to show the lower bound in (1.2). By Lemma 2,

$$\begin{aligned} N_E(-\Delta + V_c + W) &= N_E(-\Delta + (V_c - \chi_{E/(1-\varepsilon),c}) + N_E(\chi_{E/(1-\varepsilon),c} + W)) \\ &\geq N_E(-\Delta + V_{E,(1-\varepsilon)c}) - N_E(-\Delta - \frac{1-\varepsilon}{\varepsilon}(\chi_{E/(1-\varepsilon),c} + W)). \end{aligned}$$

Observe that it suffices to show that the second summand does not contribute to the limit. Invoking Lemma 2 a second time we have

$$N_E(-\Delta - \frac{1-\varepsilon}{\varepsilon}(\chi_{E/(1-\varepsilon),c} + W)) \leq N_E(-\Delta - \frac{1}{\varepsilon}\chi_{E/(1-\varepsilon),c}) + N_E(-\Delta - \frac{1-\varepsilon}{\varepsilon^2}W).$$

The second term is bounded for fixed ε as $E \downarrow 0$ by assumption and it remains to investigate the first one. As before we impose a Dirichlet boundary condition at

$\lfloor \sqrt{\frac{c(1-\varepsilon)}{E}} \rfloor$ and we need to count the sign flips of the solution of $-\Delta u - \frac{E}{\varepsilon(1-\varepsilon)}u = -Eu$ on $(0, \sqrt{\frac{c(1-\varepsilon)}{E}})$. Since this equation is explicitly solvable we obtain

$$N_E(-\Delta - \frac{1}{\varepsilon}\chi_{E/(1-\varepsilon),c}) = \sqrt{c(1-\varepsilon-\frac{1}{\varepsilon})} + O(E).$$

Hence

$$\liminf_{E \downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \geq \frac{1}{2\pi} \sqrt{(1-\varepsilon)c - \frac{1}{4}}$$

for every $0 < \varepsilon < 1$ and thus,

$$(2.2) \quad \liminf_{E \downarrow 0} \frac{N_E(-\Delta + V_c + W)}{-\ln(E)} \geq \frac{1}{2\pi} \sqrt{c - \frac{1}{4}}.$$

Combining (2.1) and (2.2), we obtain the assertion of the theorem. \square

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