RELATIVE OSCILLATION THEORY FOR DIRAC OPERATORS

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Abstract. We develop relative oscillation theory for one-dimensional Dirac operators which, rather than measuring the spectrum of one single operator, measures the difference between the spectra of two different operators. This is done by replacing zeros of solutions of one operator by weighted zeros of Wronskians of solutions of two different operators. In particular, we show that a Sturm-type comparison theorem still holds in this situation and demonstrate how this can be used to investigate the number of eigenvalues in essential spectral gaps. Furthermore, the connection with Krein's spectral shift function is established. As an application we extend a result by K.M. Schmidt on the finiteness/infiniteness of the number of eigenvalues in essential spectral gaps of perturbed periodic Dirac operators.

1. Introduction

To set the stage, let $I = (a, b) \subseteq \mathbb{R}$ (with $-\infty \le a < b \le \infty$) be an arbitrary interval and consider the Dirac differential expression

$$\tau = \frac{1}{\mathrm{i}}\sigma_2 \frac{d}{dx} + \phi(x). \tag{1.1}$$

Here

$$\phi(x) = \phi_{\mathrm{el}}(x)\mathbb{1} + \phi_{\mathrm{am}}(x)\sigma_1 + (m + \phi_{\mathrm{sc}}(x))\sigma_3, \tag{1.2}$$

σ_1, σ_2, σ_3 denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

and m, ϕ_{sc}, ϕ_{el}, and ϕ_{am} are interpreted as mass, scalar potential, electrostatic potential, and anomalous magnetic moment, respectively (see [19], Chapter 4). As usual we require $m \in [0, \infty)$ and $\phi_{\mathrm{sc}}, \phi_{\mathrm{el}}, \phi_{\mathrm{am}} \in L^1_{loc}(I)$ real-valued. We don't include a magnetic moment $\hat{\tau} = \tau + \sigma_2\phi_{\mathrm{mg}}(x)$ since it can be easily eliminated by a simple gauge transformation $\tau = U\hat{\tau}U^{-1}$, $U = \exp(\mathrm{i}\int^x \phi_{\mathrm{mg}}(r)dr)$ (there is also a gauge transformation which gets rid of ϕ_{am} or ϕ_{el} (see [7], Section 7.1.1)).

If τ is limit point at both a and b, then τ gives rise to a unique self-adjoint operator H when defined maximally (cf., e.g., [7], [21], [20]). Otherwise, we need to fix a boundary condition at each endpoint where τ is limit circle.

Explicitly, H is given by

$$H: \begin{array}{ccc} \mathfrak{D}(H) & \to & L^2(I, \mathbb{C}^2) \\ f & \mapsto & \tau f \end{array}, \tag{1.4}$$

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where

$$\mathfrak{D}(H) = \{f \in L^2(I,\mathbb{C}^2) |\ f \in AC_{loc}(I,\mathbb{C}^2),\ \tau f \in L^2(I,\mathbb{C}^2),\ W_a(u_-,f) = W_b(u_+,f) = 0\} \tag{1.5}$$

with

$$W_x(f,g) = \mathrm{i}\langle f^*(x), \sigma_2 g(x)\rangle = f_1(x)g_2(x) - f_2(x)g_1(x) \tag{1.6}$$

the usual Wronskian (we remark that the limit $W_{a,b}(.,..) = \lim_{x\to a,b} W_x(.,..)$ exists for functions as in (1.5)). Here the function u_- (resp. u_+) used to generate the boundary condition at a (resp. b) can be chosen to be a nontrivial solution of $\tau u = 0$ if τ is limit circle at a (resp. b) and zero else.

We refer to the monographs [7], [21], [22] for background and also [19] for further information about Dirac operators and their applications.

However, even though the Dirac operator is as important to relativistic quantum mechanics as the Schrödinger operator to nonrelativistic quantum mechanics, much less is known about its discrete spectrum. The main reason of course being that in contradistinction to typical Schrödinger operators, Dirac operators are not bounded from below and thus approaches relying on semi-boundedness are not applicable.

Our aim in the present paper is to develop what we will call relative oscillation theory for a pair of Dirac operators H_1 and H_0 associated with two potentials ϕ_1 and ϕ_0 as above. As we will show, it turns out to be an effective tool for both counting eigenvalues in essential spectral gaps as well as for investigation the accumulation of eigenvalues at the boundary of an essential spectral gap.

Let $\langle f, g\rangle = f_1^* g_1 + f_2^* g_2$ and $|f| = \sqrt{|f_1|^2 + |f_2|^2}$ denote the scalar product and norm in $\mathbb{C}^2$. Our key ingredient will be the Wronskian of two (nontrivial) real-valued solutions u_0 and u_1 satisfying $\tau_0 u_0 = \lambda_0 u_0$ and $\tau_1 u_1 = \lambda_1 u_1$. Then we define a Prüfer angle for the Wronskian $W(u_0, u_1)$ via

$$\begin{pmatrix} W_x(u_1,u_0) \\ W_x(u_1, -\mathrm{i}\sigma_2 u_0)\end{pmatrix} = R(x)\begin{pmatrix}\sin(\psi(x)) \\ \cos(\psi(x))\end{pmatrix}. \tag{1.7}$$

Note that $\psi(x)$ is uniquely determined up to a multiple of 2π by the requirement that $\psi(x)$ should be continuous since the two Wronskians cannot vanish simultaneously.

The total difference

$$\#_{(c,d)}(u_0,u_1) = \lceil \Delta_{1,0}(d)/\pi\rceil - \lfloor \Delta_{1,0}(c)/\pi\rfloor - 1 \tag{1.8}$$

will then be called the weighted number of sign flips of the Wronskian $W(u_0, u_1)$ in the interval $(c, d) \subset I$ (with $a < c < d < b$). Here $\lfloor x\rfloor = \max\{n \in \mathbb{Z} \,|\, n \le x\}$ and $\lceil x\rceil = \min\{n \in \mathbb{Z} \,|\, n \ge x\}$ are the usual floor and ceiling functions.

In fact, $\#_{(c,d)}(u_0, u_1)$ counts the number of sign flips of $W(u_0, u_1)$ where a sign flip is counted as $+1$ if ψ increases along the sign flip and as -1 if ψ decreases. Moreover, one can show that a zero x_0 is counted as $+1$ if $\langle u_0(x_0), \Delta\phi(x_0)u_0(x_0)\rangle > 0$ and as -1 if $\langle u_0(x_0), \Delta\phi(x_0)u_0(x_0)\rangle < 0$, where

$$\Delta\phi = \phi_1 - \phi_0. \tag{1.9}$$

We will also set

$$\#(u_0,u_1) = \lim_{c\downarrow a, d\uparrow b} \#_{(c,d)}(u_0,u_1) \tag{1.10}$$

provided this limit exists. This will for example be the case if the perturbation is of a definite sign, $\Delta\phi(x) \ge 0$ or $\Delta\phi(x) \le 0$, at least for x near a and b. We will call

$\tau_1 - \lambda_1$ relatively nonoscillatory with respect to $\tau_0 - \lambda_0$ if $\#(u_0, u_1)$ is finite and relatively oscillatory otherwise.

Our first result implies that if we choose u_0 and u_1 to be Weyl solutions, then the weighted number of sign flips counts precisely the eigenvalue difference. Recall that a solution $u_-(z,.)$ of $\tau u = zu$ is called Weyl solution at a if it is square integrable near a and fulfills the boundary condition of H at a (if there is any, i.e., if τ is limit circle at a). Such a solution is unique up to a constant if it exists (e.g. if $z \notin \sigma_{ess}(H)$) and it can be chosen to be real for $z \in \mathbb{R}$. Similarly a Weyl solution $u_+(z,.)$ at b is defined.

Finally, denote by $P_\Omega(H)$, $\Omega \subseteq \mathbb{R}$, the family of spectral projections associated with the self-adjoint operator H (see e.g. [18]).

Theorem 1.1. *Let H_0, H_1 be self-adjoint operators associated with τ_0, τ_1, respectively, and separated boundary conditions. Suppose*

(i) $\Delta\phi \le 0$*, near singular endpoints,*
(ii) $\lim_{x\to a} \Delta\phi(x) = 0$ *if a is singular and* $\lim_{x\to b}\Delta\phi(x) = 0$ *if b is singular,*
(iii) *H_0 and H_1 are associated with the same boundary conditions near a and b, that is, $u_{0,-}(\lambda)$ satisfies the boundary condition of H_1 at a (if any) and $u_{1,+}(\lambda)$ satisfies the boundary condition of H_0 at b (if any).*

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then

$$\dim \operatorname{Ran} P_{(-\infty,\lambda_0)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#(u_{1,\mp}(\lambda_0), u_{0,\pm}(\lambda_0)). \tag{1.11}$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then $\tau_1 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_0$ and

$$\begin{aligned} &\dim \operatorname{Ran} P_{[\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1]}(H_0) \\ &\qquad = \#(u_{1,\mp}(\lambda_1), u_{0,\pm}(\lambda_1)) - \#(u_{1,\mp}(\lambda_0), u_{0,\pm}(\lambda_0)). \end{aligned} \tag{1.12}$$

The proof will be given at the end of Section 2.

Remark 1.2. *Note that condition (ii) implies $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ (cf. Lemma 2.7 below). In addition, (ii) implies that any function which is in $\mathfrak{D}(\tau_0)$ near a (or b) is also in $\mathfrak{D}(\tau_1)$ near a (or b), and vice versa. Hence condition (iii) is well-posed.*

In the case where the resolvent difference of H_1 and H_0 is trace class, the difference in (1.12) as opposed to (1.11) can be avoided if we replace the left-hand side by Krein's spectral shift function $\xi(\lambda, H_1, H_0)$ (see [23] for more information on Krein's spectral shift function). In order to fix the unknown constant in the spectral shift function, we will require that H_0 and H_1 are connected via a path within the set of operators whose resolvent difference with H_0 are trace class. Hence we will require

Hypothesis 1.3. *Suppose H_0 and H_1 are self-adjoint operators associated with τ_0 and τ_1 and separated boundary conditions. Assume that*

- *$\Delta\phi$ is relatively bounded with respect to H_0 with H_0-bound less than one, and*
- *$\sqrt{|\Delta\phi|}(H_0 - z)^{-1}$ is Hilbert–Schmidt for one (and hence for all) $z \in \rho(H_0)$.*

It was shown in [6, Sect. 8] that these conditions ensure that we can interpolate between H_0 and H_1 using operators H_ε, $\varepsilon \in [0,1]$, such that the resolvent difference of H_0 and H_ε is continuous in ε with respect to the trace norm. Hence we can fix $\xi(\lambda, H_1, H_0)$ by requiring $\varepsilon \mapsto \xi(\lambda, H_\varepsilon, H_0)$ to be continuous in $L^1(\mathbb{R}, (\lambda^2+1)^{-1}d\lambda)$, where we of course set $\xi(\lambda, H_0, H_0) = 0$. While ξ is only defined a.e., it is constant on the intersection of the resolvent sets $\mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$, and we will require it to be continuous there. In particular, note that by Weyl's theorem the essential spectra of H_0 and H_1 are equal, $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$. Then we have the following result:

Theorem 1.4. *Let H_0, H_1 satisfy Hypothesis 1.3. Then for every $\lambda \in \mathbb{R} \cap \rho(H_0) \cap \rho(H_1)$ we have*

$$\xi(\lambda, H_1, H_0) = \#(\psi_{0,\pm}(\lambda), \psi_{1,\mp}(\lambda)). \tag{1.13}$$

Again, the proof will be given at the end of Section 2.

In particular, this result implies that under these assumptions $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for every λ in an essential spectral gap.

Concerning the history of these results we mention that the analogs of Theorem 1.1 and Theorem 1.4 were first given in the case of Sturm–Liouville operators by Krüger and Teschl [6], [4] extending earlier work of Gesztesy, Simon, and Teschl [3] which corresponded to the case $H_1 = H_0$. In the case of Dirac operators the case $H_1 = H_0$ was first given in Teschl [17].

Finally, we will show how $\#(u_0, u_1)$ can be used to settle the question whether the eigenvalues introduced by a given perturbation will accumulate at a boundary point of the essential spectrum and apply this to the case of perturbed periodic Dirac operators.

We first recall some basic facts from the theory of periodic Dirac operators (cf., e.g., [21], Chapter 12, [22], Chapter 16). Let H_0 be a Dirac operator associated with periodic potential ϕ_0 of period $\alpha > 0$, that is, $\phi_0(x + \alpha) = \phi_0(x)$, $x \in I = (a, \infty)$. The essential spectrum of H_0 is purely absolutely continuous and consists of a countable number of bands, that is,

$$\sigma_{ess}(H_0) = \bigcup_{j \in \mathbb{Z}} [E_{2j}, E_{2j+1}] \tag{1.14}$$

with $\cdots E_{2j} < E_{2j+1} \le E_{2j+2} < E_{2j+3} \cdots$. In addition, in every essential spectral gap there can be at most one eigenvalue.

Moreover, Floquet theory implies the existence of an (anti-)periodic solution $u_0(E_j, x)$ at each boundary point of the essential spectrum.

To phrase our result, we recall the iterated logarithm $\log_n(x)$ which is defined recursively via

$$\log_0(x) = x, \qquad \log_n(x) = \log(\log_{n-1}(x)).$$

Here we use the convention $\log(x) = \log|x|$ for negative values of x. Then $\log_n(x)$ will be continuous for $x > \mathrm{e}_{n-1}$ and positive for $x > \mathrm{e}_n$, where $\mathrm{e}_{-1} = -\infty$ and $\mathrm{e}_n = \mathrm{e}^{\mathrm{e}_{n-1}}$. Abbreviate further

$$L_n(x) = \frac{1}{\log_{n+1}'(x)} = \prod_{j=0}^{n} \log_j(x).$$

Explicitly we have

$$L_0(x) = x, \quad L_1(x) = x\log(x), \quad L_2(x) = x\log(x)\log(\log(x)), \quad \ldots$$

With this notation we have the following result:

Theorem 1.5. *Let E_j be a boundary point of the essential spectrum of the periodic operator H_0 and let $u_0(x)$ be a corresponding (anti-)periodic solution of $\tau_0 u_0 = E_j u_0$.*

Suppose

$$\phi_1(x) = \phi_0(x) - \frac{1}{4}\sum_{k=0}^{n} \frac{1}{L_k(x)^2}\phi_{1,k} + o(L_n(x)^{-2}) \tag{1.15}$$

for some constant matrices $\phi_{1,k}$, $0 \le k \le n$, and define

$$A = \frac{2}{\alpha}\int_0^\alpha \frac{\langle u_0(x), ((m+\phi_{0,\mathrm{sc}}(x))\sigma_3 + \phi_{0,\mathrm{am}}(x)\sigma_1)u_0(x)\rangle}{|u(x)|^4}dx,$$

$$B_k = -\frac{1}{\alpha}\int_0^\alpha \langle u_0(x), \phi_{1,k}u_0(x)\rangle dx, \quad 0 \le k \le n. \tag{1.16}$$

Then the eigenvalues of H_1 accumulate at E_j if

$$AB_0 = \cdots = AB_{n-1} = 1 \quad \text{and} \quad AB_n > 1 \tag{1.17}$$

and the do not accumulate at E_j if

$$AB_0 = \cdots = AB_{n-1} = 1 \quad \text{and} \quad AB_n < 1. \tag{1.18}$$

The proof will be given at the end of Section 4.

In the case of Sturm–Liouville operators this result originates in the work of Rofe-Beketov [8]–[11] (see also the recent monograph [13]) who proved the case $n = 0$. His work was recently improved by Schmidt [15] who gave a new proof and obtained the cases $n = 0, 1$. Extending the approach by Schmidt the general case was obtained in Krüger and Teschl [5]. Schmidt also established the case $n = 0, 1$ for Dirac operators in [16]. In his paper [16] he also gives an equivalent formulation for the criterion in terms of the gradient of the Floquet discriminant and shows how the above criterion can be applied to radial Dirac operators via a transformation from [14]. In fact, if

$$\tau_k = \frac{1}{\mathrm{i}}\sigma_2\frac{d}{dr} + \frac{k}{r}\sigma_1 + m\sigma_3 + \phi_{\mathrm{el}}(x)\mathbb{1}, \qquad r \in (0,\infty), \tag{1.19}$$

is a radial Dirac operator (i.e. one which arises by separation of variables in spherical coordinates [19, Sect. 4.6.6]), then the unitary transformation ([14, Lem. 3])

$$Uf(r) = \begin{pmatrix} \cos(\vartheta(r)) & -\sin(\vartheta(r)) \\ \sin(\vartheta(r)) & \cos(\vartheta(r)) \end{pmatrix}\begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}, \quad \vartheta(r) = \frac{1}{2}\arctan\Big(\frac{k}{m\,r}\Big), \tag{1.20}$$

transforms τ to

$$U^*\tau U = \frac{1}{\mathrm{i}}\sigma_2\frac{d}{dr} + \sqrt{m^2 + \frac{k^2}{r^2}}\sigma_3 + \frac{km}{2(m^2r^2+k^2)}\mathbb{1} + \phi_{\mathrm{el}}(r)\mathbb{1}. \tag{1.21}$$

Since

$$\sqrt{m^2 + \frac{k^2}{r^2}}\sigma_3 + \frac{km}{2(m^2r^2+k^2)}\mathbb{1} = m\sigma_3 + \frac{k}{2m}(k\sigma_3 + \mathbb{1})\frac{1}{r^2} + O(r^{-4}) \tag{1.22}$$

our result is directly applicable to this situation.

We also refer to [16] and the recent work by Cojuhari [2] for more on the history of this problem and references to related results. Analogous results for the discrete case, Jacobi matrices, can be found in [1].

2. Relative Oscillation Theory

After these preparations we are now ready to develop relative oscillation theory. Our presentation closely follows [6].

Definition 2.1. *For τ_0, τ_1 possibly singular Dirac operators as in (1.1) on (a,b), we define*

$$\underline{\#}(u_0,u_1) = \liminf_{d\uparrow b,\, c\downarrow a} \#_{(c,d)}(u_0,u_1) \quad \text{and} \quad \overline{\#}(u_0,u_1) = \limsup_{d\uparrow b,\, c\downarrow a} \#_{(c,d)}(u_0,u_1), \tag{2.1}$$

where $\tau_j u_j = \lambda_j u_j$, $j = 0,1$.

We say that $\#(u_0,u_1)$ exists, if $\overline{\#}(u_0,u_1) = \underline{\#}(u_0,u_1)$, and write

$$\#(u_0,u_1) = \overline{\#}(u_0,u_1) = \underline{\#}(u_0,u_1) \tag{2.2}$$

in this case.

By Lemma 3.1 below one infers that $\#(u_0,u_1)$ exists if $\phi_0 - \lambda_0 - \phi_1 + \lambda_1$ has the same definite sign near the endpoints a and b. On the other hand, note that $\#(u_0,u_1)$ might not exist even if both a and b are regular, since the difference of Prüfer angles might oscillate around a multiple of π near an endpoint. Furthermore, even if it exists, one has $\#(u_0,u_1) = \#_{(a,b)}(u_0,u_1)$ only if there are no zeros at the endpoints (or if $\phi_0 - \lambda_0 - \phi_1 + \lambda_1 \geq 0$ at least near the endpoints).

We begin with our analog of Sturm's comparison theorem for zeros of Wronskians. We will also establish a triangle-type inequality which will help us to provide streamlined proofs below. Both results follow as in [6].

Theorem 2.2 (Comparison theorem for Wronskians)**.** *Suppose u_j satisfies $\tau_j u_j = \lambda_j u_j$, $j = 0,1,2$, where $\lambda_0 - \phi_0 \leq \lambda_1 - \phi_1 \leq \lambda_2 - \phi_2$.*

If $c < d$ are two zeros of $W_x(u_0,u_1)$ such that $W_x(u_0,u_1)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0,u_2)$ in (c,d). Similarly, if $c < d$ are two zeros of $W_x(u_1,u_2)$ such that $W_x(u_1,u_2)$ does not vanish identically, then there is at least one sign flip of $W_x(u_0,u_2)$ in (c,d).

Theorem 2.3 (Triangle inequality for Wronskians)**.** *Suppose u_j, $j = 0,1,2$ are given real-valued non-vanishing vector functions. Then*

$$\underline{\#}(u_0,u_1) + \underline{\#}(u_1,u_2) - 1 \leq \underline{\#}(u_0,u_2) \leq \underline{\#}(u_0,u_1) + \underline{\#}(u_1,u_2) + 1, \tag{2.3}$$

and similarly for $\underline{\#}$ replaced by $\overline{\#}$.

Definition 2.4. *We call τ_1 relatively nonoscillatory with respect to τ_0, if the quantities $\underline{\#}(u_0,u_1)$ and $\overline{\#}(u_0,u_1)$ are finite for all solutions $\tau_j u_j = 0$, $j = 0,1$.*

We call τ_1 relatively oscillatory with respect to τ_0, if one of the quantities $\underline{\#}(u_0,u_1)$ or $\overline{\#}(u_0,u_1)$ is infinite for some solutions $\tau_j u_j = 0$, $j = 0,1$.

Note that this definition is in fact independent of the solutions chosen as a straightforward application of our triangle inequality (cf. Theorem 2.3) shows.

Corollary 2.5. *Let $\tau_j u_j = \tau_j v_j = 0$, $j = 0,1$. Then*

$$|\underline{\#}(u_0,u_1) - \underline{\#}(v_0,v_1)| \leq 4, \quad |\overline{\#}(u_0,u_1) - \overline{\#}(v_0,v_1)| \leq 4. \tag{2.4}$$

The bounds can be improved using our comparison theorem for Wronskians to be ≤ 2 in the case of perturbations of definite sign.

To demonstrate the usefulness of Definition 2.4, we now establish its connection with the spectra of self-adjoint operators associated with τ_j, $j = 0,1$.

Theorem 2.6. *Let H_j be self-adjoint Dirac operators associated with τ_j, $j = 0, 1$. Then*

(i) $\tau_0 - \lambda_0$ *is relatively nonoscillatory with respect to* $\tau_0 - \lambda_1$ *if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$.

(ii) *Suppose* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$ *and* $\tau_1 - \lambda$ *is relatively nonoscillatory with respect to* $\tau_0 - \lambda$ *for one* $\lambda \in [\lambda_0, \lambda_1]$. *Then it is relatively nonoscillatory for all* $\lambda \in [\lambda_0, \lambda_1]$ *if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) < \infty$.

Proof. Item (i) is [17, Thm. 4.5] and item (ii) follows as in [6]. □

For a practical application of this theorem one needs of course criteria when $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for λ inside an essential spectral gap.

Lemma 2.7. *Let* $\lim_{x\to a}(\phi_0(x) - \phi_1(x)) = 0$ *if* a *is singular, and similarly,* $\lim_{x\to b}(\phi_0(x) - \phi_1(x)) = 0$ *if* b *is singular. Then* $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ *and* $\tau_1 - \lambda$ *is relatively nonoscillatory with respect to* $\tau_0 - \lambda$ *for* $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H_0)$.

Proof. Since τ_1 can be written as $\tau_1 = \tau_0 + \tilde{\phi}_0 + \tilde{\phi}_1$, where $\tilde{\phi}_0$ has compact support near singular endpoints and $|\tilde{\phi}_1| < \varepsilon$, for arbitrarily small $\varepsilon > 0$, we infer that $R_{H_1}(z) - R_{H_0}(z)$ is the norm limit of compact operators. Thus $R_{H_1}(z) - R_{H_0}(z)$ is compact and hence $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$.

Let $\delta > 0$ be the distance of λ to the essential spectrum and choose $a < c < d < b$, such that

$$|\phi_1(x) - \phi_0(x)| \le \delta/2, \qquad x \notin (c, d).$$

Clearly $\#_{(c,d)}(u_0, u_1) < \infty$, since both operators are regular on (c, d). Moreover, observe that

$$\phi_0 - \lambda_+ \le \phi_1 - \lambda \le \phi_0 - \lambda_-, \qquad \lambda_\pm = \lambda \pm \delta/2,$$

on $I = (a, c)$ or $I = (d, b)$. Then Theorem 2.6 (i) implies $\#_I(u_0(\lambda_-), u_0(\lambda_+)) < \infty$ and invoking Theorem 2.2 shows $\#_I(u_0(\lambda_\pm), u_1(\lambda)) < \infty$. From Theorem 2.3 and 2.6 (i) we infer

$$\overline{\#}_I(u_0(\lambda), u_1(\lambda)) < \#_I(u_0(\lambda), u_0(\lambda_+)) + \#_I(u_0(\lambda_+), u_1(\lambda)) + 1 < \infty,$$

and similarly for $\underline{\#}_I(u_0(\lambda), u_1(\lambda))$. This shows that $\tau_1 - \lambda$ is relatively nonoscillatory with respect to τ_0. □

Our next task is to reveal the precise relation between the number of weighted sign flips and the spectra of H_1 and H_0. The special case $H_0 = H_1$ is covered by

Theorem 2.8 ([17, Thm. 4.5]). *Let H_0 be a self-adjoint operator associated with τ_0 and suppose $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) = \#(\psi_{0,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_1)). \tag{2.5}$$

Combining this result with our triangle inequality already gives some rough estimates in the spirit of Weidmann [20] who treats the case $H_0 = H_1$.

Lemma 2.9. *For $j = 0, 1$ let H_j be a self-adjoint operator associated with τ_j and separated boundary conditions. Suppose that $(\lambda_0, \lambda_1) \subseteq \mathbb{R}\backslash(\sigma_{ess}(H_0) \cup \sigma_{ess}(H_1))$, then*

$$\begin{aligned} &\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) \\ &\qquad \le \underline{\#}(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \overline{\#}(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)) + 2, \end{aligned} \tag{2.6}$$

respectively,

$$\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) \geq \overline{\#}(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \underline{\#}(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)) - 2. \tag{2.7}$$

Given these preparations the proofs of Theorem 1.1 and Theorem 1.4. can be done as in [6].

Proof of Theorem 1.1. For the proof one can literally follow the arguments in Section 6 of [6]. The only noteworthy difference is that in Lemma 6.4 one has to use the lim sup of the largest eigenvalue and the lim inf of the lowest eigenvalue of $\tilde{\phi}$. □

Proof of Theorem 1.4. For the proof one can literally follow the arguments in Section 7 of [6]. □

3. More on Prüfer angles and the case of regular operators

The purpose of this section is to collect some further facts on Prüfer angles for Wronskians and to prove Theorem 1.1 in the case of regular operators. Even tough the Prüfer angle $\Delta_{1,0}$ introduced below is different from ψ used in the introduction it will be equivalent for our purpose (cf. Definition 4.1 below). We closely follow [6] and we will provide proofs only when there is a significant difference to the Sturm–Liouville case.

We first introduce Prüfer variables for $u \in C(I, \mathbb{R}^2)$ defined by

$$u_1(x) = \rho_u(x)\sin(\theta_u(x)) \qquad u_2(x) = \rho_u(x)\cos(\theta_u(x)). \tag{3.1}$$

If u is never $(0,0)$ and u is continuous, then ρ_u is positive and θ_u is uniquely determined once a value of $\theta_u(x_0)$, $x_0 \in I$ is chosen by the requirement $\theta_u \in C(I, \mathbb{R})$.

The connection with the Wronskian is given by

$$W_x(u,v) = -\rho_u(x)\rho_v(x)\sin(\Delta_{v,u}(x)), \qquad \Delta_{v,u}(x) = \theta_v(x) - \theta_u(x). \tag{3.2}$$

Hence the Wronskian vanishes if and only if the two Prüfer angles differ by a multiple of π. We will call the total difference

$$\#_{(c,d)}(u_0, u_1) = \lceil \Delta_{1,0}(d)/\pi \rceil - \lfloor \Delta_{1,0}(c)/\pi \rfloor - 1 \tag{3.3}$$

the number of weighted sign flips in (c,d), where we have written $\Delta_{1,0}(x) = \Delta_{u_1,u_0}$ for brevity.

Next, let us take two real-valued (nontrivial) solutions u_j, $j = 1, 2$, of $\tau_j u_j = \lambda_j u_j$ and associated Prüfer variables ρ_j, θ_j. Since we can replace $\phi \to \phi - \lambda$ it is no restriction to assume $\lambda_0 = \lambda_1 = 0$.

Under these assumptions $W_x(u_0, u_1)$ is absolutely continuous and satisfies

$$W_x'(u_0, u_1) = \langle u_0(x), (\phi_1(x) - \phi_0(x))u_1(x)\rangle. \tag{3.4}$$

Lemma 3.1. *Abbreviate $\Delta_{1,0}(x) = \theta_1(x) - \theta_0(x)$ and suppose $\Delta_{1,0}(x_0) \equiv 0 \bmod \pi$. If $-\langle u_0(x), \Delta\phi(x)u_1(x)\rangle$ is (i) negative, (ii) zero, or (iii) positive for a.e. $x \in (x_0, x_0+\varepsilon)$ respectively for a.e. $x \in (x_0 - \varepsilon, x_0)$ for some $\varepsilon > 0$, then the same is true for $(\Delta_{1,0}(x) - \Delta_{1,0}(x_0))/(x - x_0)$.*

Hence $\#_{(c,d)}(u_0, u_1)$ counts the weighted sign flips of the Wronskian $W_x(u_0, u_1)$, where a sign flip is counted as $+1$ if $-\Delta\phi$ is positive in a neighborhood of the sign flip, it is counted as -1 if $-\Delta\phi$ is negative in a neighborhood of the sign flip. If $\Delta\phi$ changes sign (i.e., it is positive on one side and negative on the other) the Wronskian will not change its sign. In particular, we obtain:

Lemma 3.2. *Let u_0, u_1 solve $\tau_j u_j = 0$, $j = 0, 1$, where $\Delta\phi \le 0$. Then $\#_{(a,b)}(u_0, u_1)$ equals the number sign flips of $W(u_0, u_1)$ inside the interval (a, b).*

In the case $\Delta\phi \ge 0$ we get of course the corresponding negative number except for the fact that zeros at the boundary points are counted as well since $\lfloor -x \rfloor = -\lceil x \rceil$. That is, if $\Delta\phi < 0$, then $\#_{(c,d)}(u_0, u_1)$ equals the number of zeros of the Wronskian in (c, d) while if $\Delta\phi > 0$, it equals minus the number of zeros in $[c, d]$. In the next theorem we will see that this is quite natural. In addition, note that $\#(u, u) = -1$.

Finally, we establish the connection with the spectrum of regular operators. A finite end point is called regular if all entries of ϕ are integrable near this end point. In this case boundary values for all functions exist at this end point. In particular, τ is called regular if both end points a, b are regular. In the regular case the resolvent of H is Hilbert-Schmidt and hence the spectrum is purely discrete (i.e., $\sigma_{ess}(H) = \emptyset$).

Theorem 3.3. *Let H_0, H_1 be regular Sturm–Liouville operators associated with τ_0, τ_1 and the same boundary conditions at a and b. Then*

(3.5)
$$\dim \operatorname{Ran} P_{(-\infty,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#_{(a,b)}(u_{0,\pm}(\lambda_0), u_{1,\mp}(\lambda_1)).$$

The proof will be given below employing interpolation between H_0 and H_1, using $H_\varepsilon = (1-\varepsilon)H_0 + \varepsilon H_1$ together with a careful analysis of Prüfer angles.

It is important to observe that in the special case $H_1 = H_0$, the left-hand side equals $\dim \operatorname{Ran} P_{(\lambda_1,\lambda_0)}(H_0)$ if $\lambda_1 > \lambda_0$ and $-\dim \operatorname{Ran} P_{[\lambda_0,\lambda_1]}(H_0)$ if $\lambda_1 < \lambda_0$. This is of course in accordance with our previous observation that $\#(u_{0,\pm}(\lambda_0), u_{1,\mp}(\lambda_1))$ equals the number of zeros in (a, b) if $\lambda_1 > \lambda_0$ while it equals minus the numbers of zeros in $[a, b]$ if $\lambda_1 < \lambda_0$.

Now let us suppose that $\tau_{0,1}$ are both regular at a and b with boundary conditions

$$\cos(\alpha) f_1(a) - \sin(\alpha) f_2(a) = 0, \quad \cos(\beta) f_1(b) - \sin(\beta) f_2(b) = 0. \tag{3.6}$$

Hence we can choose $u_\pm(\lambda, x)$ such that $u_-(\lambda, a) = (\sin(\alpha), \cos(\alpha))$ respectively $u_+(\lambda, b) = (\sin(\beta), \cos(\beta))$. In particular, we may choose

$$\theta_-(\lambda, a) = \alpha \in [0, \pi), \quad -\theta_+(\lambda, b) = \pi - \beta \in [0, \pi). \tag{3.7}$$

Next we introduce

$$\tau_\varepsilon = \tau_0 + \varepsilon(\phi_1 - \phi_0) \tag{3.8}$$

and investigate the dependence with respect to $\varepsilon \in [0, 1]$. If u_ε solves $\tau_\varepsilon u_\varepsilon = 0$, then the corresponding Prüfer angles satisfy

$$\dot\theta_\varepsilon(x) = -\frac{W_x(u_\varepsilon, \dot u_\varepsilon)}{\rho_\varepsilon^2(x)}, \tag{3.9}$$

where the dot denotes a derivative with respect to ε.

Lemma 3.4. *We have*

$$W_x(u_{\varepsilon,\pm}, \dot u_{\varepsilon,\pm}) = \begin{cases} \int_x^b \langle u_{\varepsilon,+}(r), (\phi_0(r) - \phi_1(r)) u_{\varepsilon,+}(r)\rangle dr \\ -\int_a^x \langle u_{\varepsilon,-}(r), (\phi_0(r) - \phi_1(r)) u_{\varepsilon,-}(r)\rangle dr \end{cases}, \tag{3.10}$$

where the dot denotes a derivative with respect to ε and $u_{\varepsilon,\pm}(x) = u_{\varepsilon,\pm}(0, x)$.

Denoting the Prüfer angles of $u_{\varepsilon,\pm}(x) = u_{\varepsilon,\pm}(0,x)$ by $\theta_{\varepsilon,+}(x)$, this result implies for $\phi_0 - \phi_1 \geq 0$,

$$\dot{\theta}_{\varepsilon,+}(x) = -\frac{\int_x^b \langle u_{\varepsilon,+}(r), (\phi_0(r) - \phi_1(r)) u_{\varepsilon,+}(r)\rangle dr}{\rho_{\varepsilon,+}(x)^2} \leq 0,$$

$$\dot{\theta}_{\varepsilon,-}(x) = \frac{\int_a^x \langle u_{\varepsilon,-}(r), (\phi_0(r) - \phi_1(r)) u_{\varepsilon,-}(r)\rangle dr}{\rho_{\varepsilon,-}(x)^2} \geq 0, \tag{3.11}$$

with strict inequalities if $\phi_0 > \phi_1$ on a subset of positive Lebesgue measure of (x,b), respectively (a,x).

Now we are ready to investigate the associated operators H_0 and H_1. In addition, we will choose the same boundary conditions for H_ε as for H_0 and H_1.

Lemma 3.5. *Suppose $\phi_0 - \phi_1 \geq 0$ (resp. $\phi_0 - \phi_1 \leq 0$). Then the eigenvalues of H_ε are analytic functions with respect to ε and they are decreasing (resp. increasing).*

In particular, this implies that $\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_\varepsilon)$ is continuous from below (resp. above) in ε if $\phi_0 - \phi_1 \geq 0$ (resp. $\phi_0 - \phi_1 \leq 0$).

Now the proof of Theorem 3.3 follows literally as in [6].

4. Relative oscillation criteria

As in the previous sections, we will consider two Dirac operators τ_j, $j = 0,1$, and corresponding self-adjoint operators H_j, $j = 0,1$. Now we want to answer the question, when a boundary point E of the essential spectrum of H_0 is an accumulation point of eigenvalues of H_1. By Theorem 2.6 we need to investigate if $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ or not, that is, if the difference of Prüfer angles $\Delta_{1,0} = \theta_1 - \theta_0$ is bounded or not.

Hence the first step is to derive an ordinary differential equation for $\Delta_{1,0}$. While this can easily be done by subtracting the differential equations for θ_1 and θ_0, the result turns out to be not very effective for our purpose. However, since the number of weighted sign flips $\#_{(c,d)}(u_0,u_1)$ is all we are eventually interested in, any *other* Prüfer angle which gives the same result will be as good:

Definition 4.1. *We will call a continuous function ψ a Prüfer angle for the Wronskian $W(u_0,u_1)$, if $\#_{(c,d)}(u_0,u_1) = \lceil \psi(d)/\pi \rceil - \lfloor \psi(c)/\pi \rfloor - 1$ for any $c,d \in (a,b)$.*

Hence we will try to find a more effective Prüfer angle ψ than $\Delta_{1,0}$ for the Wronskian of two solutions. The right choice for Sturm–Liouville equations was found by Rofe-Beketov [8] (see also the recent monograph [13]) and it turns out the analogous definition is also the right one for Dirac operators [16]:

Let u_0, v_0 be two linearly independent solutions of $(\tau_0 - \lambda)u = 0$ with $W(u_0,v_0) = 1$ and let u_1 be a solution of $(\tau_1 - \lambda)u = 0$. Define ψ via

$$W(u_0,u_1) = -R\sin(\psi), \qquad W(v_0,u_1) = -R\cos(\psi). \tag{4.1}$$

Since $W(u_0,u_1)$ and $W(v_0,u_1)$ cannot vanish simultaneously, ψ is a well-defined absolutely continuous function, once one value at some point x_0 is fixed.

Lemma 4.2. *The function ψ defined in* (4.1) *is a Prüfer angle for the Wronskian $W(u_0,u_1)$.*

Proof. Since $W(u_0,u_1) = -R\sin(\psi) = -\rho_{u_0}\rho_{u_1}\sin(\Delta_{1,0})$ it suffices to show that $\psi = \Delta_{1,0} \bmod 2\pi$ at each zero of the Wronskian. Since we can assume $\theta_{v_0} -$

$\theta_{u_0} \in (0,\pi)$ (by $W(u_0,v_0)=1$), this follows by comparing signs of $R\cos(\psi) = \rho_{v_0}\rho_{u_1}\sin(\theta_{u_1}-\theta_{v_0})$. □

Lemma 4.3. *Let u_0, v_0 be two linearly independent solutions of $(\tau_0-\lambda)u=0$ with $W(u_0,v_0)=1$ and let u_1 be a solution of $(\tau_1-\lambda)u=0$.*

Then the Prüfer angle ψ for the Wronskian $W(u_0,u_1)$ defined in (4.1) *obeys the differential equation*

$$\psi' = -\langle u_0\cos(\psi) - v_0\sin(\psi), \Delta\phi(u_0\cos(\psi)-v_0\sin(\psi))\rangle, \tag{4.2}$$

where

$$\Delta\phi = \phi_1 - \phi_0.$$

Proof. Observe $R\psi' = -W(u_0,u_1)'\cos(\psi) + W(v_0,u_1)'\sin(\psi)$ and use (3.4), (4.1) to evaluate the right-hand side. □

To proceed we will need the following formula for a second solution of a Dirac equation which can be verified by a straightforward calculation:

Lemma 4.4 ([12], [16, Lem. 1])**.** *Let u be a nontrivial solution of $\tau u = zu$ and choose $x_0 \in I$. Then*

$$v(x) = \left(2\int_{x_0}^{x} \frac{\langle u(r), \hat{\phi}(r)u(r)\rangle}{|u(r)|^4} dr - \mathrm{i}\frac{\sigma_2}{|u(x)|^2}\right) u(x), \tag{4.3}$$

where

$$\hat{\phi}(x) = (m+\phi_{\mathrm{sc}}(x))\sigma_3 + \phi_{\mathrm{am}}(x)\sigma_1, \tag{4.4}$$

is a second linearly independent solution satisfying $W(u,v)=1$.

Now we will choose v_0 to be given by (4.3) and, following Schmidt [16], perform a Kepler transformation

$$\cot(\varphi(x)) = \frac{1}{x}\left(\cot(\psi(x)) - 2\int_a^x \frac{\langle u_0(r), \hat{\phi}_0(r)u_0(r)\rangle}{|u(r)|^4} dr\right) \tag{4.5}$$

to obtain

$$\begin{aligned}\varphi'(x) =& \frac{1}{x}\Big(2\frac{\langle u_0(x), \hat{\phi}_0(x)u_0(x)\rangle}{|u_0(x)|^4}\sin^2(\varphi(x)) + \sin(\varphi(x))\cos(\varphi(x)) - \\ &\Big\langle\Big(\cos(\varphi(x)) - \mathrm{i}\frac{\sin(\varphi(x))}{|u_0(x)|^2}\sigma_2\Big)u_0(x), \\ &x^2\Delta\phi(x)\Big(\cos(\varphi(x)) - \mathrm{i}\frac{\sin(\varphi(x))}{|u_0(x)|^2}\sigma_2\Big)u_0(x)\Big\rangle\Big).\end{aligned} \tag{4.6}$$

Here we assume that $a>0$ is regular and $b=\infty$ without loss of generality. Under the further assumption that $|u_0(x)|$, $|u_0(x)|^{-1}$, and $x^2\Delta\phi(x)$ are bounded this simplifies to

$$\varphi'(x) = \frac{1}{x}\Big(A(x)\sin^2(\varphi(x)) + \sin(\varphi(x))\cos(\varphi(x)) + B(x)\cos^2(\varphi(x))\Big) + O(x^{-2}), \tag{4.7}$$

where

$$A(x) = 2\frac{\langle u_0(x), \hat{\phi}_0(x)u_0(x)\rangle}{|u_0(x)|^4} \quad \text{and} \quad B(x) = -\langle u_0(x), x^2\Delta\phi(x)u_0(x)\rangle. \tag{4.8}$$

Now we turn to the case where $\phi_0(x)$ is periodic with period $\alpha > 0$ and choose u_0 to be the (anti-)periodic solution at a band edge. Taking averages

$$\overline{\varphi}(x) = \frac{1}{\alpha}\int_x^{x+\alpha} \varphi(r)dr \tag{4.9}$$

the above differential equation turns into (see [5, Section 5])

$$\overline{\varphi}'(x) = \frac{1}{x}\Big(\overline{A}\sin^2(\overline{\varphi}(x)) + \sin(\overline{\varphi}(x))\cos(\overline{\varphi}(x)) + \overline{B}(x)\cos^2(\overline{\varphi}(x))\Big) + O(x^{-2}), \tag{4.10}$$

where

$$\overline{A} = \frac{2}{\alpha}\int_0^\alpha \frac{\langle u_0(x), \hat{\phi}_0(x) u_0(x)\rangle}{|u_0(x)|^4} dx,$$
$$\overline{B}(x) = -\frac{1}{\alpha}\int_x^{x+\alpha} \langle u_0(r), r^2\Delta\phi(r) u_0(r)\rangle dr. \tag{4.11}$$

Moreover, if $\phi_1(x)$ is given by (1.15) then one computes

$$\overline{B}(x) = -\frac{1}{4}\sum_{k=0}^n \frac{x^2}{L_k(x)^2} B_k + o(\frac{x^2}{L_n(x)^2}), \tag{4.12}$$

with B_k defined in (1.16). Now we use the following result:

Lemma 4.5 ([5, Lemma 4.7]). *Fix some $n \in \mathbb{N}_0$, let Q be locally integrable on (a, ∞) and abbreviate*

$$Q_n(x) = -\frac{1}{4}\sum_{j=0}^{n-1} \frac{1}{L_j(x)^2}.$$

Then all solutions of the differential equation

(4.13)

$$\varphi'(x) = \frac{1}{x}\left(\sin^2(\varphi(x)) + \sin(\varphi(x))\cos(\varphi(x)) - x^2 Q(x)\cos^2(\varphi(x))\right) + o\Big(\frac{x}{L_n(x)^2}\Big)$$

tend to ∞ if

$$\limsup_{x\to\infty} L_n(x)^2\left(Q(x) - Q_n(x)\right) < -\frac{1}{4}$$

and are bounded from above if

$$\liminf_{x\to\infty} L_n(x)^2\left(Q(x) - Q_n(x)\right) > -\frac{1}{4}.$$

In the last case all solutions are bounded under the additional assumption $Q = Q_n(x) + O(L_n(x)^{-2})$.

Now this lemma implies Theorem 1.5 if $\overline{A} = 1$. However, if $\overline{A} > 0$ we can easily reduce it to the case $\overline{A} = 1$ by the simple scaling $u_0(x) \to (\overline{A})^{1/2} u_0(x)$. which renders $\overline{A} \to 1$ and $B_k \to \overline{A} B_k$. Similarly, if $\overline{A} < 0$ we can reduce it to the case $\overline{A} > 0$ via the transformation $\varphi \to -\varphi$ which renders $\overline{A} \to -\overline{A}$, $B_k \to -B_k$. Finally, in the case $\overline{A} = 0$ the result follows by using Proposition 1 from [16] (Lemma 5.1 in [5]) in place of the above lemma.

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