SINGULAR SCHRÖDINGER OPERATORS AS SELF-ADJOINT EXTENSIONS OF N-ENTIRE OPERATORS

LUIS O. SILVA, GERALD TESCHL, AND JULIO H. TOLOZA

Abstract. We investigate the connections between Weyl–Titchmarsh–Kodaira theory for one-dimensional Schrödinger operators and the theory of n-entire operators. As our main result we find a necessary and sufficient condition for a one-dimensional Schrödinger operator to be n-entire in terms of square integrability of derivatives (w.r.t. the spectral parameter) of the Weyl solution. We also show that this is equivalent to the Weyl function being in a generalized Herglotz–Nevanlinna class. As an application we show that perturbed Bessel operators are n-entire, improving the previously known conditions on the perturbation.

1. Introduction

The Weyl-Titchmarsh-Kodaira theory for a self-adjoint operator H associated with the differential expression

$$\tau := -\frac{d^2}{dx^2} + q(x), \quad -\infty \le a < x < b \le \infty, \tag{1.1}$$

where the potential q is real-valued and satisfies

$$q \in L^1_{loc}(a,b). \tag{1.2}$$

has been an active and alluring subject of research for long time, particularly nowadays. The current interest concerns the case where both endpoints are generically singular. Recent developments show that, under a necessary and sufficient additional condition on $q(x)$ (see Hypothesis 1 below), there exists an entire system of fundamental solutions $\phi(z,x), \theta(z,x)$ of the equation $\tau\varphi = z\varphi$ such that the Wronskian of these two solutions equals one, and one of the solutions (say $\phi(z,x)$) is in the domain of H near the left endpoint. A singular Weyl function $M(z)$ (associated with the left endpoint) is then defined as a function that makes

$$\psi(z,x) := \theta(z,x) + M(z)\phi(z,x) \tag{1.3}$$

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be in the domain of H near the right endpoint (more details are accounted for in the next section). As in the regular case, $M(z)$ encodes all the spectral information related to H. However, contrary to the regular case, there is not a natural choice of normalization for the entire system of fundamental solutions and, therefore, the singular Weyl function $M(z)$ does not generically belong to a particular class of functions as in the regular case.

This indeterminacy has been overcome for the class of perturbed spherical Schrödinger operators (also known as Bessel operators), a class of operators that has attracted considerable interest recently; see for example [1, 4, 5, 3, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19]. For this class of operators, and assuming some mild additional conditions on $q(x)$, a technique for constructing the system of fundamental solutions $\phi(z,x), \theta(z,x)$ based on Frobenius method has been proposed as the natural choice. In particular, it has been proved that in this case, $M(z)$ belongs to a specific class of the generalized Nevanlinna functions N_κ^∞.

In this paper we address the issue of elucidating some properties of $M(z)$ and the singular Weyl solution $\psi(z,x)$ given in (1.3) from a different perspective, although restricted to the cases where H has only discrete spectrum. Here we consider H as a self-adjoint extension of some symmetric, regular (hence completely non-self-adjoint) operator A with deficiency indices $(1,1)$ (for all the technical definitions see Section 3). Among these operators there exists a distinguished class, the so-called n-entire operators [20], defined by the condition

$$\mathcal{H} = \operatorname{ran}(A - zI) \dotplus \operatorname{span}\{\mu_0 + z\mu_1 + \cdots + z^n \mu_n\}, \quad z \in \mathbb{C},$$

for some fixed $\mu_0, \dots, \mu_n \in \mathcal{H}$; here $\mathcal{H}$ stands for the Hilbert space in which A is defined. As discussed in [20], every n-entire operator can be unitarily transformed into the operator of multiplication by the independent variable acting on a de Branges space $\mathcal{B}_A$. Moreover, $\mathcal{B}_A$ is such that the linear manifold $\operatorname{assoc}_n(\mathcal{B}_A)$ of n-associated functions contains a zero-free entire function. On the basis of this setup, we establish in this work an equivalence between (a) the operator A being n-entire, (b) the $(n-1)$-th derivative of the singular Weyl solution $\psi(z,x)$ being square integrable with respect to the spectral measure, and (c) the possibility of choosing the solution $\theta(z,x)$ such that $M(z) \in N_{n-1}^\infty$. Precise formulations of this assertion, tied to the fulfillment of an increasing number of technical conditions, are given in Theorems 4.7, 4.9 and 4.10.

Reciprocally, the connection between the Weyl–Titchmarsh–Kodaira theory for one-dimensional Schrödinger operators and the theory of n-entire operators can be exploited in another direction, as it allows to broaden the classes of differential operators that are known to be n-entire. A first investigation of this matter has been done in [21], where it is shown that perturbed Bessel operators on a finite interval are n-entire, with n given in terms of the angular momentum number, provided that $q(x)$ obeys a certain rather restrictive technical condition that arises from the perturbation arguments used in that paper. As an application of the results obtained in this work, we generalize the classes addressed in [21] by lifting this technical restriction; this is asserted in Theorem 4.11.

We conclude this introduction with an outline of this paper. All the relevant aspects of the Weyl–Titchmarsh–Kodaira theory are reviewed in Section 2. The theory of n-entire operators as well as their connection with the theory of de Branges spaces is briefly recalled in Section 3. Finally, Section 4 contains the main results of this work.

2. Singular Weyl–Titchmarsh–Kodaira theory

One of our fundamental ingredients will be singular Weyl–Titchmarsh–Kodaira theory and hence we begin by recalling the necessary facts from [13]. Consider one-dimensional Schrödinger operators on $L^2(a,b)$ with $-\infty \leq a < b \leq \infty$ associated with the differential expression (1.1) with potential (1.2). We use H to denote a self-adjoint operator given by τ with separated boundary conditions at a and/or b. For further background we refer to [22, Chap. 9] or [23].

As mentioned in the Introduction, to define the singular Weyl function at the, in general singular, endpoint a, we need a fundamental system of solutions $\theta(z,x)$ and $\phi(z,x)$ of the equation $\tau\varphi = z\varphi$ which are entire with respect to z and such that $\phi(z,x)$ lies in the domain of H near a and the Wronskian

$$W(\theta(z),\phi(z)) := \theta(z,x)\phi'(z,x) - \theta'(z,x)\phi(z,x) \equiv 1. \tag{2.1}$$

Recall that the Wronskian does not depend on x when its arguments are solutions of the same equation. Thus, (2.1) tells us that the function of z on the l. h. s is identically 1.

Denote the restriction of H to (a,c) with a Dirichlet boundary condition at c by $H^D_{(a,c)}$, i.e., $\operatorname{dom}(H^D_{(a,c)})$ consists of all functions which are restrictions of functions from $\operatorname{dom}(H)$ to (a,c) and satisfy $f(c)=0$.

Lemma 2.1 ([13]). *The following properties are equivalent:*

(i) The spectrum of $H^D_{(a,c)}$ is purely discrete for some $c \in (a,b)$.
(ii) There is a real entire solution $\phi(z,x)$, which is nontrivial and lies in the domain of H near a for each $z \in \mathbb{C}$.
(iii) There exist real entire solutions $\phi(z,x)$, $\theta(z,x)$ with $W(\theta(z),\phi(z)) \equiv 1$, such that $\phi(z,x)$ lies in the domain of H near a for each $z \in \mathbb{C}$.

Thus, for dealing with the singular Weyl function $M(z)$ as defined in the Introduction it is necessary and sufficient that item (i) holds. This will be our first hypothesis.

Hypothesis 1. *Suppose that the spectrum of $H^D_{(a,c)}$ is purely discrete for one (and hence for all) $c \in (a,b)$.*

Note that this hypothesis is for example satisfied if $q(x) \to +\infty$ as $x \to a$ (cf. Problem 9.7 in [22]).

Remark 2.2. It is important to point out that a fundamental system satisfying the conditions we have imposed on $\phi(z,x)$ and $\theta(z,x)$ is not unique and any other such system is given by

$$\widetilde{\theta}(z,x) = \mathrm{e}^{-g(z)}\theta(z,x) - f(z)\phi(z,x), \qquad \widetilde{\phi}(z,x) = \mathrm{e}^{g(z)}\phi(z,x),$$

where $f(z)$, $g(z)$ are entire functions with $f(z)$ real and $g(z)$ real modulo $\mathrm{i}\pi$. The singular Weyl functions are related via

$$\widetilde{M}(z) = \mathrm{e}^{-2g(z)}M(z) + \mathrm{e}^{-g(z)}f(z).$$

The singular Weyl function $M(z)$ is by construction analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies $M(z) = M(z^*)^*$. Recall also from [13, Lemma 3.3] that associated with $M(z)$ there is a spectral measure ρ given by the Stieltjes–Livšić inversion formula

$$\frac{1}{2}\left(\rho\big((x_0,x_1)\big) + \rho\big([x_0,x_1]\big)\right) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{x_0}^{x_1} \mathrm{im}\big(M(x+\mathrm{i}\varepsilon)\big)dx. \tag{2.2}$$

In all assertions of this section Hypothesis 1 is assumed.

Theorem 2.3 ([10])**.** *Define*

$$\widehat{f}(\lambda) = \lim_{c\uparrow b}\int_a^c \phi(\lambda,x)f(x)dx,$$

where the right-hand side is to be understood as a limit in $L^2(\mathbb{R},d\rho)$*. Then the map*

$$U : L^2(a,b) \to L^2(\mathbb{R},d\rho), \qquad f \mapsto \widehat{f}, \tag{2.3}$$

is unitary and its inverse is given by

$$f(x) = \lim_{r\to\infty}\int_{-r}^{r}\phi(\lambda,x)\widehat{f}(\lambda)d\rho(\lambda),$$

where again the right-hand side is to be understood as a limit in $L^2(a,b)$*. Moreover,* U *maps* H *to multiplication by* λ*.*

Remark 2.4. We have seen in Remark 2.2 that $M(z)$ is not unique. However, given $\widetilde{M}(z)$ as in Remark 2.2, the spectral measures are related by

$$d\widetilde{\rho}(\lambda) = \mathrm{e}^{-2g(\lambda)}d\rho(\lambda).$$

Hence the measures are mutually absolutely continuous and the associated spectral transformations just differ by a simple rescaling with the positive function $\mathrm{e}^{-2g(\lambda)}$.

Finally, $M(z)$ can be reconstructed from ρ up to an entire function via the following integral representation.

Theorem 2.5 ([13])**.** *Let* $M(z)$ *be a singular Weyl function and* ρ *its associated spectral measure. Then there exists an entire function* $g(z)$ *such that* $g(\lambda) \geq 0$ *for* $\lambda \in \mathbb{R}$ *and* $\mathrm{e}^{-g(\lambda)} \in L^2(\mathbb{R},d\rho)$*.*

Moreover, for any entire function $\widehat{g}(z)$ *such that* $\widehat{g}(\lambda) > 0$ *for* $\lambda \in \mathbb{R}$ *and* $(1+\lambda^2)^{-1}\widehat{g}(\lambda)^{-1} \in L^1(\mathbb{R}, d\rho)$ *(e.g.* $\widehat{g}(z) = \mathrm{e}^{2g(z)}$*) we have the integral representation*

$$M(z) = E(z) + \widehat{g}(z) \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \frac{d\rho(\lambda)}{\widehat{g}(\lambda)}, \qquad z \in \mathbb{C}\backslash\sigma(H),$$

where $E(z)$ *is a real entire function.*

As a consequence one obtains a criterion when the singular Weyl function is a generalized Nevanlinna function with no nonreal poles and the only generalized pole of nonpositive type at ∞. We will denote the set of all such generalized Nevanlinna functions by N_κ^∞ (see Appendix C [13] for a definition and further references).

Theorem 2.6 ([13])**.** *Fix the solution* $\phi(z,x)$*. Then there exists a corresponding solution* $\theta(z,x)$ *such that* $M(z) \in N_\kappa^\infty$ *for some* $\kappa \le k$ *if and only if* $(1+\lambda^2)^{-k-1} \in L^1(\mathbb{R}, d\rho)$*. Moreover,* $\kappa = k$ *if* $k = 0$ *or* $(1+\lambda^2)^{-k} \notin L^1(\mathbb{R}, d\rho)$*.*

In order to identify possible values of k one can try to bound λ^{-k} by a linear combination of $\phi(\lambda,x)^2$ and $\phi'(\lambda,x)^2$ which are in $L^1(\mathbb{R}, (1+\lambda^2)^{-1}d\rho)$ by [13, Lemma 3.6].

Remark 2.7. Choosing a real entire function $g(z)$ such that $\exp(-2g(\lambda))$ is integrable with respect to $d\rho$, we see that

$$M(z) = \mathrm{e}^{2g(z)} \int_{\mathbb{R}} \frac{1}{\lambda - z} \mathrm{e}^{-2g(\lambda)} d\rho(\lambda) - E(z).$$

Hence if we choose $f(z) = \exp(-g(z))E(z)$ and switch to a new system of solutions as in Remark 2.2, we see that the new singular Weyl function is a Herglotz–Nevanlinna function

$$\widetilde{M}(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} \mathrm{e}^{-2g(\lambda)} d\rho(\lambda).$$

3. N-entire operators and de Branges spaces

In a separable Hilbert space, let A be a closed symmetric operator with deficiency indices $(1,1)$ such that, for every $z \in \mathbb{C}$, there is a positive constant c_z for which

$$\|(A - zI)f\| \ge c_z \|f\|, \quad \forall f \in \operatorname{dom}(A). \tag{3.1}$$

The operator A is said to be n-entire ($n \in \mathbb{Z}^+$) if moreover there exist $n+1$ vectors $\mu_0, \dots, \mu_n \in \mathcal{H}$ such that

$$\mathcal{H} = \operatorname{ran}(A - zI) \dotplus \operatorname{span}\{\mu_0 + z\mu_1 + \cdots + z^n \mu_n\} \tag{3.2}$$

for all $z \in \mathbb{C}$ [20]. The operator A is called minimal n-entire whenever there is no smaller n with this property. Notice that a necessary but not sufficient condition for A to be minimal n-entire is that $\mu_n \ne 0$.

To any closed symmetric operator A with deficiency indices $(1,1)$ satisfying (3.1), there corresponds a de Branges space $\mathcal{B}_A$ in which the operator becomes multiplication by the independent variable [20]. Recall that a de Branges Hilbert space is a linear manifold given by

$$\mathcal{B} := \{F(z) \text{ entire} : F(z)/E(z), F^{\#}(z)/E(z) \in H^2(\mathbb{C}^+)\} \tag{3.3}$$

with the inner product

$$\langle G, F\rangle_{\mathcal{B}} := \frac{1}{\pi}\int_{\mathbb{R}} \frac{G^*(x)F(x)}{|E(x)|^2}dx\,, \tag{3.4}$$

where $E(z)$ is in the Hermite–Biehler class, that is, an entire function such that $|E(z)| > |E(z^*)|$ for all $z \in \mathbb{C}^+$. Above we have used the notation $F^{\#}(z) := F(z^*)^*$. A de Branges space is a reproducing kernel Hilbert space with reproducing kernel

$$K(z,w) = \begin{cases} \dfrac{E^{\#}(z)E(w^*) - E(z)E^{\#}(w^*)}{2\pi\mathrm{i}(z-w^*)}, & w \neq z^*, \\ \frac{1}{2\pi\mathrm{i}}\left[E^{\#\prime}(z)E(z) - E'(z)E^{\#}(z)\right], & w = z^*. \end{cases}$$

For further details we refer to [6].

Given $n \in \mathbb{Z}^+$, the set of n-associated functions is defined by

$$\operatorname{assoc}_n(\mathcal{B}) := \mathcal{B} + z\mathcal{B} + \cdots + z^n\mathcal{B}\,. \tag{3.5}$$

Clearly, $\operatorname{assoc}_n(\mathcal{B}) \subset \operatorname{assoc}_{n+1}(\mathcal{B})$ for $n \in \mathbb{Z}^+$. Also, it is straightforward to verify that $E(z)$ is in $\operatorname{assoc}_1(\mathcal{B})$ but not in $\mathcal{B}$ [6].

Lemma 3.1 ([20])**.** *The operator A is n-entire if and only if $\operatorname{assoc}_n(\mathcal{B}_A)$ contains a zero-free function. Moreover, A is minimal n-entire if and only if, additionally, no zero-free function lies in $\operatorname{assoc}_m(\mathcal{B}_A)$ for every $m < n$.*

Theorem 3.2 ([20])**.** *The following statements are equivalent:*

(i) The operator A is n-entire.

(ii) Let A_{β_1} and A_{β_2}, $\beta_1 \neq \beta_2$, be canonical self-adjoint extensions of A (that is, self-adjoint restrictions of A^*). Set $\{x_j\}_{j\in\mathbb{N}} = \{x_j^+\}_{j\in\mathbb{N}} \cup \{x_j^-\}_{j\in\mathbb{N}} = \sigma(A_{\beta_1})$, where $\{x_j^+\}_{j\in\mathbb{N}}$ and $\{x_j^-\}_{j\in\mathbb{N}}$ are the sequences of positive, respectively nonpositive, elements of $\sigma(A_{\beta_1})$, arranged according to increasing modulus. Then the following assertions hold true:

(C1) The limit $\displaystyle\lim_{r\to\infty}\sum_{0<|x_j|\le r}\frac{1}{x_j}$ exists.

(C2) $\displaystyle\lim_{j\to\infty}\frac{j}{x_j^+} = -\lim_{j\to\infty}\frac{j}{x_j^-} < \infty$.

(C3) Assuming that $\{b_j\}_{n\in\mathbb{N}} = \sigma(A_\beta)$, define

$$h_\beta(z) := \begin{cases} \displaystyle\lim_{r\to\infty}\prod_{|b_j|\le r}\left(1-\frac{z}{b_j}\right) & \text{if } 0 \notin \sigma(A_\beta), \\ \displaystyle z\lim_{r\to\infty}\prod_{0<|b_j|\le r}\left(1-\frac{z}{b_j}\right) & \text{otherwise.} \end{cases}$$

$$\textit{The series } \sum_{x_j \neq 0} \left| \frac{1}{x_j^{2n} h_{\beta_2}(x_j) h'_{\beta_1}(x_j)} \right| \textit{ is convergent.}$$

4. Main results

We begin this section by introducing and discussing a hypothesis that is used to obtain the auxiliary results leading to the main ones.

Hypothesis 2. *(i) Suppose H has purely discrete spectrum and let $\phi(z,x)$, $\chi(z,x)$ be entire solutions such that $\phi(z,x)$ is in the domain of H near a and $\chi(z,x)$ is in the domain of H near b. Abbreviate*

$$W(z) := W(\phi(z), \chi(z)) \tag{4.1}$$

which is of course also entire.

(ii) For every compact subset K of $\mathbb{C} \times \rho(H)$, there exists $F \in L^1(a,b)$ such that

$$|\phi(w,x)\chi(z,x)| \le F(x)$$

for every $(w,z) \in K$.

(iii) We have

$$\lim_{x \downarrow a} W_x(\phi(w), \chi(z)) = W(z) \quad \textit{and} \quad \lim_{x \uparrow b} W_x(\phi(w), \chi(z)) = W(w),$$

where the Wronskian here depends on x since $\phi(w,x)$ and $\chi(z,x)$ are solutions of equations with different spectral parameters.

Item (i) above amounts to assume that $M(z)$, hence the Weyl solution $\psi(z,x)$, is a meromorphic function with (necessarily simple) poles at $\sigma(H)$. Thus, given an entire function $W(z)$ whose zero set includes $\sigma(H)$,

$$\chi(z,x) = W(z)\psi(z,x) \tag{4.2}$$

is the entire solution that obeys (4.1). Note by the way that this item implies Hypothesis 1.

The reason for Hypothesis 2 — in particular, items (ii) and (iii) — will become clear later. For now we just point out that it holds for example if both endpoints are in the limit circle case as can be seen from Appendix A in [13] (for item (ii) see the proof of Lemma A.3 and for (iii) use $\chi(z,x) = W(\chi(z),\phi(z))\theta(z,x) - W(\chi(z),\theta(z))\phi(z,x)$ plus Corollary A.4). In the general case we first show that the items in the above hypothesis are not independent. Our first result exploits the fact that

$$G(z,x,x) = \frac{\phi(z,x)\chi(z,x)}{W(z)}$$

is the diagonal of the kernel of the resolvent of H.

Lemma 4.1. *Consider the condition*

$$\int_a^b |\phi(z,x)\chi(z,x)| dx < \infty \tag{4.3}$$

for some $z \in \mathbb{C}$.

(i) *Assume that H is bounded from below. The inequality* (4.3) *holds for one (and hence for all) $z < \inf\sigma(H)$ if and only if $(H-z)^{-1}$ is trace class. In this case we have*
$$\frac{1}{W(z)}\int_a^b \phi(z,x)\chi(z,x)dx = \operatorname{tr}\big((H-z)^{-1}\big), \qquad z\in\rho(H).$$
(ii) *If* (4.3) *holds for one $z\in\mathbb{C}\setminus\mathbb{R}$ then it holds for all $z\in\rho(H)$ and $(H-z)^{-1}$ is Hilbert–Schmidt.*

Proof. (i) For $z < \inf\sigma(H)$ the resolvent is a positive operator and hence the claim follows from the lemma on page 65 in [18, Section XI.4]. Conversely, if $(H-z)^{-1}$ is trace class, then the above equality holds for all $z\in\rho(H)$ by Theorem 3.1 from [2].

(ii) That $(H-z)^{-1}$ is Hilbert–Schmidt follows from the proof of Lemma 9.12 in [22]. The rest follows from the first resolvent formula which implies
$$G(z,x,y) - G(w,x,y) = (z-w)\int_a^b G(z,x,t)G(w,t,y)dt. \qquad \square$$

Corollary 4.2. *Assume that H is bounded from below. Then assertion* (4.3) *holds for one (and hence for all) $z < \inf\sigma(H)$ if and only if $\sigma(H)$ obeys condition (C1) of Theorem 3.2.*

The Lagrange identity implies
$$(w-z)\int_c^d \phi(w,x)\chi(z,x)dx = W_c(\phi(w,x),\chi(z,x)) - W_d(\phi(w,x),\chi(z,x))$$
for arbitrary $a<c<d<b$. By item (ii) of Hypothesis 2, we can take limits $c\downarrow a$ and $d\uparrow b$ to obtain
$$(w-z)\int_a^b \phi(w,x)\chi(z,x)dx = W_a(\phi(w,x),\chi(z,x)) - W_b(\phi(w,x),\chi(z,x))$$
with both limiting Wronskians being entire functions of both z and w. Moreover, note that $W_a(\phi(w,x),\chi(z,x))$ has the same zeros as $W(z)$, and also $W_b(\phi(w,x),\chi(z,x))$ has the same zeros as $W(w)$. However, it is not immediate that there is always equality and hence we have imposed item (iii) of Hypothesis 2 which finally yields
$$\int_a^b \phi(w,x)\chi(z,x)dx = -\frac{W(z)-W(w)}{z-w}. \tag{4.4}$$
In the limit $w\to z$ this gives
$$\int_a^b \phi(z,x)\chi(z,x)dx = -\frac{d}{dz}W(z).$$
Next we want to relate this to Weyl–Titchmarsh–Kodaira theory from Section 2. Of course $\chi(z,x)$ is related to the Weyl solution via (4.2) and we obtain the following formula which will be crucial for us.

Lemma 4.3. *Assume Hypothesis 2 and abbreviate*

$$\psi^{(j)}(z,x) := \frac{\partial^j}{\partial z^j}\psi(z,x).$$

Then

$$(4.5)\quad \frac{(w-z)^{j+1}}{j!}\int_a^b \phi(w,x)\psi^{(j)}(z,x)dx = 1 - \sum_{k=0}^{j}\frac{(w-z)^k}{k!}W_b(\phi(w,x),\psi^{(k)}(z,x)).$$

Proof. The case $j=0$ follows from (4.4) upon using (4.2). The case $j \geq 1$ follows from induction by differentiation with respect to z. Note that by Cauchy's integral formula the derivatives w.r.t. z of $\chi(z,x)$ also satisfy item (ii) of Hypothesis 2. □

Note that when λ is an eigenvalue we obtain:

Corollary 4.4. *Assume Hypothesis 2. If $\lambda \in \sigma(H)$ and $z \in \rho(H)$, then*

$$(4.6)\quad \int_a^b \phi(\lambda,x)\psi^{(j)}(z,x)dx = \frac{j!}{(\lambda-z)^{j+1}}.$$

Proof. The assertion for $j=0$ follows from (4.5) by taking into account that $W_b(\phi(\lambda,x),\psi(z,x)) = 0$ whenever $\lambda \in \sigma(H)$. Now use induction as before. □

The next assumption will allow us to associate H with a certain symmetric non self-adjoint operator. Also, in combination with Hypothesis 1, it will imply item (i) of Hypothesis 2.

Hypothesis 3. *The endpoint b is in the limit circle case.*

Let A be the closure of the restriction of H to functions vanishing in a neighborhood of b. By Hypotheses 1 and 3, this operator has deficiency indices $(1,1)$ and satisfies (3.1). One way of constructing the corresponding de Branges space $\mathcal{B}_A$ is the following. Fix two real-valued solutions $c(x)$ and $s(x)$ corresponding to the same spectral parameter with $W(c,s)=1$. Now introduce the entire function

$$(4.7)\quad E(z) = W_b(c,\phi(z)) + \mathrm{i}W_b(s,\phi(z)).$$

Note that by our limit circle assumption the limit of the Wronskians exist at b and are indeed entire with respect to z (cf. Appendix A in [13]). Moreover, an analogous computation as before verifies

$$\frac{E(z)E^{\#}(w^*) - E(w^*)E^{\#}(z)}{2\mathrm{i}(w^*-z)} = \int_a^b \phi(w,x)^*\phi(z,x)dx, \quad w,z\in\mathbb{C}.$$

In particular, taking $w=z$ this shows that $E(z)$ is a Hermite–Biehler function. Moreover, note that $E(z)$ does not have any real zero, since otherwise both, $W_b(c,\phi(z))$ and $W_b(s,\phi(z))$ would vanish, contradicting $W(c,s)=1$.

Now, $\mathcal{B}_A$ is the de Branges space generated by $E(z)$ as specified in (3.3). The reproducing kernel of this space is given by

$$K(w,z) = \int_a^b \phi(w,x)^* \phi(z,x) dx, \quad w,z \in \mathbb{C}. \tag{4.8}$$

This also shows that the de Branges norm equals the spectral norm,

$$\langle F, G \rangle_{\mathcal{B}_A} = \int_{\mathbb{R}} F(x)^* G(x) d\rho(x). \tag{4.9}$$

Remark 4.5. (a) Identities (4.8) and (4.9) imply that $\mathcal{B}_A$ and $L^2(\mathbb{R}, d\rho)$ are unitarily equivalent in the sense that, the restriction to $\mathbb{R}$ of every function in $\mathcal{B}_A$ belongs to $L^2(\mathbb{R}, d\rho)$ while for every function in $L^2(\mathbb{R}, d\rho)$ there exist one, and only one, function in $\mathcal{B}_A$ whose restriction to $\mathbb{R}$ belongs to the same equivalence class (with respect to the measure ρ).

(b) Since $\operatorname{assoc}_n \mathcal{B}(E) = \mathcal{B}(E_n)$ with $E_n(z) := (z+\mathrm{i})^n E(z)$ (as sets) [16], one easily obtains

$$\operatorname{assoc}_n \mathcal{B}(E) \cong L^2\left(\mathbb{R}, \tfrac{d\rho}{(x^2+1)^n}\right),$$

where the isomorphism is in the sense given in (a).

(c) Since $E(z) \in \operatorname{assoc}_1(\mathcal{B}_A) \setminus \mathcal{B}_A$ and A is densely defined, the functions

$$W_b(c, \phi(z)) = \frac{E(z) + E^\#(z)}{2} \quad \text{and} \quad W_b(s, \phi(z)) = \frac{E(z) - E^\#(z)}{2\mathrm{i}}$$

also belong to $\operatorname{assoc}_1(\mathcal{B}_A) \setminus \mathcal{B}_A$ [6].

Theorem 4.6. *Assume Hypotheses 2 and 3 and let A be the operator defined above. If there is $z \in \rho(H)$ such that $\psi^{(n-1)}(z,x) \in L^2(a,b)$, then the operator A is n-entire.*

Proof. Our assumption implies, by letting $j := n-1$ in (4.5), that the left-hand side in (4.5) is in $\operatorname{assoc}_n(\mathcal{B}_A)$. The same is true for the sum on the right-hand side which is a sum of a polynomial in w of degree $n-1$ times $W(c,\phi(w))$ and $W(s,\phi(w))$ since

$$W_b(\phi(w), \psi^{(j)}(z)) = W_b(s, \psi^{(j)}(z)) W_b(c, \phi(w)) - W_b(c, \psi^{(j)}(z)) W_b(s, \phi(w)).$$

In view of item (c) of Remark 4.5 one has that $1 \in \operatorname{assoc}_n(\mathcal{B}_A)$ which in turn implies the assertion by Lemma 3.1. □

Theorem 4.7. *Let the assumptions of Theorem 4.6 hold. Then, the following are equivalent:*

(i) The operator A is n-entire.
(ii) There is a choice of the entire solution $\theta(z,x)$ such that $M(z) \in N_\kappa^\infty$ for $\kappa \le n-1$.

Proof. (i) ⇒ (ii). Since A is n-entire, there exists a zero-free function in $\operatorname{assoc}_n(\mathcal{B}_A)$. Without loss of generality we can assume this function to be equal to 1. By item (b) of Remark 4.5 one has that $(1+\lambda^2)^{-n}$ is in $L^1(\mathbb{R}, d\rho)$. Hence Theorem 2.6 yields (ii).

(ii) $\Rightarrow$ (i). By Theorem 2.6 (ii) implies that $(1+\lambda^2) \in L^1(\mathbb{R}, d\rho)$. The claim follows now from item (b) of Remark 4.5 and Lemma 3.1. $\square$

Corollary 4.8. *Let the assumptions of Theorem 4.6 hold. Suppose moreover that one of the following holds true:*

(a) There is a choice of the entire solution $\theta(z,x)$ such that $M(z) \in N_\kappa^\infty$ for $\kappa \le n-1$.

(b) There exists $z \in \rho(H)$ such that $\psi^{(n-1)}(z,x) \in L^2(a,b)$.

Then there is another self-adjoint extension H' of A such that $\sigma(H)$ and $\sigma(H')$ satisfy (C1), (C2), (C3) of Theorem 3.2.

Proof. The claim is obtained immediately from Theorems 4.6 and 4.7 in combination with Theorem 3.2. $\square$

Hypothesis 4. *Let $\phi(z,x)$ and $\psi(z,x)$ be such that if*

$$\int_a^b \phi(\lambda,x)\psi^{(j)}(z,x)dx \in L^2(\mathbb{R}, d\rho)$$

for some $j \in \mathbb{N}$ and $z \in \rho(H)$, then

$$\psi^{(j)}(z,x) = \lim_{r\to\infty} \int_{-r}^{r} \phi(\lambda,x)\left(\int_a^b \phi(\lambda,y)\psi^{(j)}(z,y)dy\right) d\rho(\lambda)\,,$$

where the limit is understood as a limit in $L^2(a,b)$.

Theorem 4.9. *Let the assumptions of Theorem 4.6 hold and assume Hypothesis 4. Then, the following are equivalent:*

(i) The operator A is n-entire.

(ii) There is a choice of the entire solution $\theta(z,x)$ such that $M(z) \in N_\kappa^\infty$ for $\kappa \le n-1$.

(iii) $\psi^{(n-1)}(z,x) \in L^2(a,b)$ for one (and hence for all) $z \in \rho(H)$.

Proof. In view of Theorems 4.6 and 4.7, one only has to show that (ii) $\Rightarrow$ (iii). By Theorem 2.6, (ii) implies that $(1+\lambda^2)^{-n}$ is in $L^1(\mathbb{R}, d\rho)$, so the function $(n-1)!(\lambda-z)^{-n}$ is in $L^2(\mathbb{R}, d\rho)$. Therefore there is a function $\eta(z,x) \in L^2(a,b)$ such that $\eta(z,x) = U^{-1}\left(\frac{(n-1)!}{(\lambda-z)^n}\right)$. By Corollary 4.4, Hypothesis 4 implies that, at least for one $z \in \rho(H)$, $\eta(z,x) = \psi^{(n-1)}(z,x)$. $\square$

The proof of the previous assertion can be complemented to obtain the following sharpened version of it.

Theorem 4.10. *Under the assumptions of Theorem 4.9, the following are equivalent:*

(i) The operator A is minimal n-entire.

(ii) There is a choice of the entire solution $\theta(z,x)$ such that $M(z) \in N_{n-1}^\infty$.

(iii) $\psi^{(n-1)}(z,x) \in L^2(a,b)$ but $\psi^{(n-2)}(z,x) \notin L^2(a,b)$, for one (and hence for all) $z \in \rho(H)$.

A class of operators attracting attention nowadays and for which Hypotheses 2, 3, and 4 are satisfied is the class of spherical Schrödinger operators.

Theorem 4.11. *Fix $l \geq -\frac{1}{2}$ and $b > 0$. Suppose*

$$\tau = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \quad x \in (0,b), \tag{4.10}$$

where

$$\begin{cases} xq(x) \in L^1(0,b), & l > -\frac{1}{2}, \\ x(1-\log(x/b))q(x) \in L^1(0,b), & l = -\frac{1}{2}. \end{cases} \tag{4.11}$$

If τ is limit circle at $a = 0$ we impose the usual boundary condition (corresponding to the Friedrichs extension; see also [3], [9]*)*

$$\lim_{x\to 0} x^l((l+1)f(x) - xf'(x)) = 0, \qquad l \in [-\frac{1}{2}, \frac{1}{2}). \tag{4.12}$$

Then the assumptions of Theorem 4.9 are satisfied and, whenever $n \in \mathbb{Z}^+$ obeys $2n \geq \lfloor l + \frac{5}{2} \rfloor$ (equivalently, $n > \frac{l}{2} + \frac{3}{4}$), the corresponding operator A is n-entire.

Proof. Item (ii) of Hypothesis 2 follows from Lemma 2.2 and 2.6 in [12] and item (iii) follows from Corollary 3.12 in [15]. The first part of Lemma 4.4 in [15] implies that Hypothesis 4 is satisfied. Moreover, that $\psi^{(n-1)}(z,x) \in L^2(a,b)$ for the proposed values of n is shown in Lemma 4.4 of [15]. □

In particular, this generalizes Theorem 4.3 from [21]. Note that we could even allow a nonintegrable singularity at b as long as τ is limit circle at b. Of course this also generalizes Corollary 4.4 from [21]:

Corollary 4.12. *Under the assumptions of Theorem 4.11, the spectra of two canonical self-adjoint extensions H_1, H_2 of A satisfy conditions (C1), (C2) and (C3) of Theorem 3.2 whenever $2n \geq \lfloor l + \frac{5}{2} \rfloor$.*

Finally, one has the following consequence of Theorem 4.10.

Corollary 4.13. *Under the assumptions of Theorem 4.11, the underlying operator A is minimal $\lfloor l + \frac{5}{2} \rfloor$-entire.*

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Departamento de Física Matemática, Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, C.P. 04510, México D.F.
E-mail address: silva@iimas.unam.mx

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/

CONICET, and Centro de Investigación en Informática para la Ingeniería, Universidad Tecnológica Nacional – Facultad Regional Córdoba, Maestro López s/n, X5016ZAA Córdoba, Argentina
E-mail address: jtoloza@scdt.frc.utn.edu.ar