ON THE SUBSTITUTION RULE FOR LEBESGUE–STIELTJES INTEGRALS

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Abstract. We show how two change-of-variables formulæ for Lebesgue–Stieltjes integrals generalize when all continuity hypotheses on the integrators are dropped. We find that a sort of "mass splitting phenomenon" arises.

Let $M\colon [a,b] \to \mathbb{R}$ be increasing.[1] Then the measure corresponding to M may be defined to be the unique Borel measure μ on $[a,b]$ such that for each continuous function $f\colon [a,b] \to \mathbb{R}$, the integral $\int_{[a,b]} f\, d\mu$ is equal to the usual Riemann-Stieltjes[2] integral $\int_a^b f(x)\, dM(x)$. Now let $f\colon [a,b] \to \mathbb{R}$ be a bounded[3] Borel function. Then by definition, the Lebesgue-Stieltjes integral $\int_a^b f(x)\, dM(x)$ is equal to $\int_{[a,b]} f\, d\mu$. If $a < c < b$, then of course the equation

$$\int_a^b f(x)\, dM(x) = \int_a^c f(x)\, dM(x) + \int_c^b f(x)\, dM(x)$$

holds but to understand this properly, one should realize that the point c contributes $f(c)\mu(\{c\}) = f(c)\big(M(c+) - M(c-)\big)$ to $\int_a^b f(x)\, dM(x)$ and this contribution is split into a contribution of $f(c)\big(M(c) - M(c-)\big)$ to $\int_a^c f(x)\, dM(x)$ and a contribution of $f(c)\big(M(c+) - M(c)\big)$ to $\int_c^b f(x)\, dM(x)$. This simple kind of splitting was pointed out by Stieltjes himself ([13], pp. J70–J71, item 38; see also [3], pp. 27–28, item 38) and is closely related to the "mass splitting phenomenon" in change-of-variables formulæ alluded to in our abstract.

Now let $N\colon [M(a), M(b)] \to \mathbb{R}$ be increasing and let ν be the measure on $[M(a), M(b)]$ corresponding to N. Let $\Lambda = N \circ M$. Then $\Lambda\colon [a,b] \to \mathbb{R}$ is also increasing. Let λ be the measure on $[a,b]$ corresponding to Λ. It is natural to ask what relations exist between the measures λ, μ, and ν.

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[1]By "increasing," we mean "non-decreasing." Of course, a and b are real numbers with $a < b$.

[2]For an excellent exposition of Riemann–Stieltjes integration, see [1] and [12].

[3]Here and elsewhere in this paper, we have chosen to focus on bounded integrands but our statements may be extended in the usual way to suitable unbounded integrands.

If N is continuous and W is any generalized inverse[4] for the increasing function M, then it is not hard to show that λ is the image of ν under W or equivalently,[5] that for each bounded Borel function $f\colon [a,b]\to\mathbb{R}$, we have

$$\int_a^b f(x)\,dN(M(x)) = \int_{M(a)}^{M(b)} f(W(y))\,dN(y), \tag{1}$$

where $\int_a^b f(x)\,dN(M(x))$ means $\int_a^b f(x)\,d\Lambda(x)$. In the special case where $N(y)\equiv y$, this goes back to Lebesgue [9].

If instead M is continuous, then it is not hard to show that ν is the image of λ under M or equivalently, that for each bounded Borel function $g\colon [M(a),M(b)]\to\mathbb{R}$, we have

$$\int_a^b g(M(x))\,dN(M(x)) = \int_{M(a)}^{M(b)} g(y)\,dN(y). \tag{2}$$

This is standard.[6] In the special case where $N(y)\equiv y$, this is attributed in [4] (Vol. I, Example 3.6.2) to Kolmogorov.

Our aim in this paper is to explain how (1) and (2) generalize when no continuity assumptions are imposed on M and N. As we shall see, a key role is played by the left and right jumps of N at the points of the set

$$H = \{y\in[M(a),M(b)] : M^{-1}[\{y\}] \text{ contains more than one point}\}.$$

We have chosen the letter H for this set because it is the set of all levels at which the graph of M has a *horizontal* portion. Note that $(M^{-1}[\{y\}])_{y\in H}$ is a pairwise disjoint family of non-degenerate intervals in $[a,b]$. Hence H is countable.

Let X and Ξ be the left-continuous and right-continuous generalized inverses for M. These are the functions from $[M(a),M(b)]$ to $[a,b]$ defined respectively by

$$X(y) = \inf\{x\in[a,b] : y\le M(x)\} \quad\text{and}\quad \Xi(y) = \sup\{x\in[a,b] : M(x)\le y\}$$

for all y in $[a,b]$. On $[M(a),M(b)]\setminus H$, we have $X=\Xi$, while for each y in the range of M, $X(y)$ is the left endpoint of the interval $M^{-1}[\{y\}]$ and $\Xi(y)$ is its right endpoint. It is easy to check that a function $W\colon [M(a),M(b)]\to\mathbb{R}$ is a generalized

[4]To say that W is a generalized inverse for the increasing function M means that W is an increasing function from $[M(a),M(b)]$ to $[a,b]$ and for each y in the range of M, $W(y)$ is in the closure of the interval $M^{-1}[\{y\}]$. This concept, with or without this name, is well-established in the literature. For further information, see [6].

[5]This equivalence is a standard result about images of measures under measurable mappings. See for instance [5], Theorem 1.6.9. It is stated there for probability measures but that restriction is inessential.

[6]See for example [11], Chapter 1, §4, Proposition (4.10). Attention is restricted there to the case where N is right-continuous but this is not essential. In fact, if M and g are continuous, then (2) is obvious by considering Riemann-Stieltjes sums, for then each upper Riemann-Stieltjes sum for $\int_{M(a)}^{M(b)} g(y)\,dN(y)$ is equal in value to one of the upper Riemann-Stieltjes sums for $\int_a^b g(M(x))\,dN(M(x))$, and similarly for lower Riemann-Stieltjes sums, so the upper and lower Riemann-Stieltjes integrals corresponding to $\int_a^b g(M(x))\,dN(M(x))$ lie between those corresponding to $\int_{M(a)}^{M(b)} g(y)\,dN(y)$, so the Riemann-Stieltjes integrals $\int_a^b g(M(x))\,dN(M(x))$ and $\int_{M(a)}^{M(b)} g(y)\,dN(y)$ are equal. It follows that if M is continuous and g is a bounded Borel function, then the Lebesgue-Stieltjes integrals $\int_a^b g(M(x))\,dN(M(x))$ and $\int_{M(a)}^{M(b)} g(y)\,dN(y)$ are equal.

We would like to mention that change-of-variables formulæ for certain other types of integrals are given in [8] and [10].

inverse for M if and only if $X \leq W \leq \Xi$. In particular, X and Ξ are indeed generalized inverses for M.

Proposition 1. *Suppose N is right-continuous*[7] *at y for each y in H. Then λ is the image of ν under X and for each bounded Borel function $f\colon [a,b] \to \mathbb{R}$, we have*

$$\int_a^b f(x)\,dN(M(x)) = \int_{M(a)}^{M(b)} f(X(y))\,dN(y). \tag{3}$$

Proof. It is easy to check that for each x in $[a,b)$ and each y in $[M(a), M(b)]$, we have $X(y) \leq x$ if and only if $y \leq M(x+)$. Let G be the set of all x in $[a,b)$ such that M and Λ are both right-continuous at x. Then $[a,b] \setminus G$ is countable. Hence G is dense in $[a,b]$. Let x be in G. Then $\nu\big(X^{-1}\big[[a,x]\big]\big) = \nu([M(a), M(x+)]) = \nu([M(a), M(x)]) = N(M(x)+) - N(M(a))$. Now either for each x' in $(x,b]$, we have $M(x) < M(x')$, or there exists x' in $(x,b]$ such that $M(x) = M(x')$. Consider the case where for each x' in $(x,b]$, we have $M(x) < M(x')$. Then since x is in G, $M(x) < M(x') \to M(x)$ as $x' \downarrow x$, so $N(M(x')) \to N(M(x)+)$ as $x' \downarrow x$. But again, since $x \in G$, $N(M(x')) = \Lambda(x') \to \Lambda(x) = N(M(x))$ as $x' \downarrow x$. Hence $N(M(x)+) = N(M(x))$. Now consider the case where there exists x' in $(x,b]$ such that $M(x) = M(x')$. Then $M = M(x)$ on $[x,x']$, so $M(x)$ is in H, so $N(M(x)+) = N(M(x))$ by assumption. Thus in any case, $N(M(x)+) = N(M(x))$. Therefore $\nu\big(X^{-1}\big[[a,x]\big]\big) = N(M(x)) - N(M(a)) = \Lambda(x) - \Lambda(a)$. But since x is in G, $\Lambda(x) - \Lambda(a) = \lambda([a,x])$. Thus $\lambda([a,x]) = \nu\big(X^{-1}\big[[a,x]\big]\big)$. This holds for each x in G. Let $\mathcal{P}$ be the set of all intervals of the form $[a,x]$ with $x \in G$. Then $\mathcal{P}$ is a π-system on $[a,b]$ and since G is dense in $[a,b]$, $\mathcal{P}$ generates the Borel σ-field on $[a,b]$. As we've just seen, $\mathcal{P}$ is contained in the set $\mathcal{L}$ of all Borel sets $E \subseteq [a,b]$ such that $\lambda(E) = \nu(X^{-1}[E])$. Note that $[a,b] \in \mathcal{L}$ because $\lambda([a,b]) = \Lambda(b) - \Lambda(a) = N(M(b)) - N(M(a)) = \nu([M(a), M(b)]) = \nu(X^{-1}[[a,b]])$. Hence $\mathcal{L}$ is a λ-system on $[a,b]$. (The λ in λ-system does not refer to our measure λ.) It follows that for each Borel set $E \subseteq [a,b]$, $\lambda(E) = \nu(X^{-1}[E])$, by the π-λ theorem. (See, for instance, [5], Theorem A.1.4.) In other words, λ is the image of ν under X, as claimed. Equation (3) follows from this. $\square$

Similarly, we have:

Proposition 2. *Suppose N is left-continuous*[8] *at y for each y in H. Then λ is the image of ν under Ξ and for each bounded Borel function $f\colon [a,b] \to \mathbb{R}$, we have*

$$\int_a^b f(x)\,dN(M(x)) = \int_{M(a)}^{M(b)} f(\Xi(y))\,dN(y). \tag{4}$$

When no continuity condition is imposed on N, then λ need not be the image of ν under any point mapping. Instead, for each y in H, the mass that ν assigns to $\{y\}$ is split in λ between the singletons $\{X(y)\}$ and $\{\Xi(y)\}$. This was alluded to above in our abstract and is explained in detail in our main result:

Theorem 3. *Let N_1 be the increasing function that is obtained from N by removing the jumps that N has at the points of H. For each y in H, let*

$$\Delta N(y,-) = N(y) - N(y-) \quad \textit{and} \quad \Delta N(y,+) = N(y+) - N(y)$$

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[7] By convention, we consider N to be right-continuous at $M(b)$ and we consider $N(M(b)+)$ to be $N(M(b))$.

[8] By convention, we consider N to be left-continuous at $M(a)$ and we consider $N(M(a)-)$ to be $N(M(a))$.

be the left and right jumps of N at y respectively. Then for each bounded Borel function $f\colon [a,b] \to \mathbb{R}$, we have

$$\begin{aligned}\int_a^b f(x)\,dN(M(x)) = &\int_{M(a)}^{M(b)} f(X(y))\,dN_1(y)\\ &+\sum_{y\in H} f(X(y))\Delta N(y,-)\\ &+\sum_{y\in H} f(\Xi(y))\Delta N(y,+).\end{aligned} \tag{5}$$

Furthermore, X may be replaced by Ξ in the first term on the right in (5)*.*

Proof. For each y in H, observe that $\Delta N(y,+) \ge 0$ and $\Delta N(y,-) \ge 0$, let

$$N_-^y = \Delta N(y,-)\mathbb{1}_{[y,M(b)]} \qquad \text{and} \qquad N_+^y = \Delta N(y,+)\mathbb{1}_{(y,M(b)]},$$

and observe that N_-^y is right-continuous and N_+^y is left-continuous. Let $N_2 = \sum_{y\in H} N_-^y$ and $N_3 = \sum_{y\in H} N_+^y$. Note that these series converge uniformly on $[M(a), M(b)]$, because $\sum_{y\in H}[\Delta N(y,-) + \Delta N(y,+)] = \nu(H) < \infty$. By definition,

$$N_1 = N - N_2 - N_3,$$

so $N = N_1 + N_2 + N_3$. Now N_1, N_2, and N_3 are increasing on $[M(a), M(b)]$, N_2 is right-continuous, N_3 is left-continuous, and for each $y \in H$, N_1 is continuous at y. Let ν_1, ν_2, and ν_3 be the measures corresponding to N_1, N_2, and N_3 respectively. Let $H^c = [M(a), M(b)] \setminus H$. Then $X = \Xi$ on H^c. Also, for each Borel set $E \subseteq [M(a), M(b)]$, we have $\nu(H^c \cap E) = \nu_1(E)$ and $\nu(H \cap E) = \nu_2(E) + \nu_3(E)$. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded Borel function. By (3) and (4),

$$\int_a^b f(x)\,dN_1(M(x)) = \int_{M(a)}^{M(b)} f(X(y))\,dN_1(y) = \int_{M(a)}^{M(b)} f(\Xi(y))\,dN_1(y).$$

By (3),

$$\int_a^b f(x)\,dN_2(M(x)) = \int_{M(a)}^{M(b)} f(X(y))\,dN_2(y) = \sum_{y\in H} f(X(y))\Delta N(y,-).$$

By (4),

$$\int_a^b f(x)\,dN_3(M(x)) = \int_{M(a)}^{M(b)} f(\Xi(y))\,dN_3(y) = \sum_{y\in H} f(\Xi(y))\Delta N(y,+).$$

The result follows by addition. □

Corollary 4. *Equation* (1) *still holds if N is just continuous at each point of H. In particular, if M is strictly increasing, then* (1) *holds with no continuity assumption on N.*

Proof. If N is continuous at each point of H, then the two sums on the right in (5) vanish, $N_1 = N$, $\nu(H) = 0$, and if W is any generalized inverse for M, then $X \le W \le \Xi$, with equality on $[M(a), M(b)] \setminus H$. If M is strictly increasing, then H is empty, so it is vacuously true that N is continuous at each point of H. □

Corollary 5. *For each bounded Borel function g on the range of M, we have*

$$
\begin{aligned}
\int_a^b g(M(x))\,dN(M(x)) = {} & \int_{M(a)}^{M(b)} g(M(X(y)))\,dN_1(y) \\
& + \sum_{y\in H} g(M(X(y)))\Delta N(y,-) \\
& + \sum_{y\in H} g(M(\Xi(y)))\Delta N(y,+),
\end{aligned}
\tag{6}
$$

where the notation is as in the theorem. Furthermore, X may be replaced by Ξ in the first term on the right in (6).

Proof. Let $f = g \circ M$ in (5). □

Note that (6) is a generalization of (2), because in the special case where M is continuous, it is clear that $M(X(y)) = y = M(\Xi(y))$ for each y in $[M(a), M(b)]$.

Since equations (5) and (6) are a bit complicated, it is worth noting that they yield some simpler-looking inequalities when f and g are monotone. For each increasing function $f\colon [a,b] \to \mathbb{R}$ and for each y in H, we have $f(X(y)) \le f(\Xi(y))$, so by (5),

$$
\int_{M(a)}^{M(b)} f(X(y))\,dN(y) \le \int_a^b f(x)\,dN(M(x)) \le \int_{M(a)}^{M(b)} f(\Xi(y))\,dN(y). \tag{7}
$$

Let $g\colon [M(a), M(b)] \to \mathbb{R}$ be increasing and let f be the increasing function $g \circ M$. If M is left-continuous, then for each y in $[M(a), M(b)]$, we have $M(\Xi(y)) \le y$, so from the right-hand inequality in (7), we get

$$
\int_a^b g(M(x))\,dN(M(x)) \le \int_{M(a)}^{M(b)} g(y)\,dN(y). \tag{8}
$$

If instead M is right-continuous, then for each y in $[M(a), M(b)]$, we have $y \le M(X(y))$, so from the left-hand inequality in (7), we get

$$
\int_{M(a)}^{M(b)} g(y)\,dN(y) \le \int_a^b g(M(x))\,dN(M(x)). \tag{9}
$$

If g is decreasing rather than increasing, then the inequalities (8) and (9) must be reversed. To see this, just replace g by $-g$.

A related inequality, in the special case where $g(x) \equiv x^n$, is established by a different method in [2], where it is applied to prove a Gronwall lemma for Lebesgue–Stieltjes integrals. An application of (6) can be found in [7].

Our results can easily be extended, with appropriate modifications, to the case where $[a,b]$ is replaced by any interval I and $[M(a), M(b)]$ is replaced by the smallest interval J containing the range of M.

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