Errata
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Errata

Changes appear in yellow. Line \( k^+ \) (resp., line \( k^- \)) denotes the \( k \)th line from
the top (resp., the bottom) of a page. My thanks go to the following individuals
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Page 6. 7+: Furthermore, \( K^{j+1} \in C^k(U_j, M) \) for any

Page 19. Problem 1.26, second item: \( y(x_0 + x) = y(x_0 - x) \)

Page 45. 10+: \( \Delta(t) = |\phi(t, t_0, y_0) - \phi(t, s_0, y_0)| \) and use . . .

Page 51. Theorem 2.13: Suppose the IVP (2.10) has a unique local solution
for every \( (t_0, x_0) \in U \)

Page 57. 4+: Taking \( m \to \infty \) we finally obtain

Page 62.

\[
\exp(tJ) = \exp(\alpha t) \exp(tN) = e^{\alpha t} \sum_{j=0}^{k-1} \frac{t^j}{j!} N^j.\quad (3.18)
\]
Problem 3.13: It should read $\deg(q(t)) \leq \deg(p(t)) + s$ where $s$ is the size of the largest Jordan block corresponding to the eigenvalue $\beta$. Moreover, here is an extended hint:

(Hint: Investigate (3.48) using the following fact: $\int^t (t-s)^m p(s) e^{\beta s} ds = q(t)e^{\beta t}$, where $q(t)$ is a polynomial of degree $\deg(q) = \deg(p) + m$ if $\beta \neq 0$ and $\deg(q) = \deg(p) + m + 1$ if $\beta = 0$. To see this differentiate $\int^t s^k e^{\beta s} ds$ with respect to $\beta$.)

Page 84. Line before (3.98): Let $X(t)$ be the identity matrix with the first column replaced by $\phi_1(t)$,

Page 89. Example:

$$(e^t \ddot{c}(t) + 4t e^t \dot{c}(t) + (2 + 4t^2) e^t c(t)) - 2t (e^t \ddot{c}(t) + 2te^t c(t)) - 2e^t c(t)$$

Page 90. Problem 3.34: Consider the equation $\ddot{x} + q_0(t)x = 0$.

Page 90. Problem 3.37:

$$y(t) = Q(t)^{-1} x(t), \quad Q(t) = e^{\frac{1}{\alpha} \int^t q_{n-1}(s) ds}.$$

$$y^{(n)} + Q(t)^{-1} \sum_{k=0}^{n-2} \sum_{j=k}^{n} \binom{j}{k} q_j(t) Q^{(j-k)}(t) y^{(k)} = 0.$$  

Page 90. Problem 3.38:

$$\dot{y} + e^{-Q(t)} y^2 + e^{-Q(t)} q(t) = 0.$$

Page 100. Paragraph after the proof of Theorem 3.23: . . . is constant. To this end recall that Corollary 3.5 tells us when the system corresponding to $B(t) = 0$ is stable. Moreover, . . .

Page 101. Theorem 3.26:

$$|x(t)| \leq Ce^{-(\alpha-b_0C)t} |x_0|, \quad |x_0| < \frac{\delta}{C}, \quad t \geq 0. \quad (3.168)$$

Page 117. Lemma 4.4: In this case we have $\alpha = -\lim_{z \to 0} z p(z)$ and the radius of convergence for the Laurent series of $p(z)$ equals the radius of the largest ball on which $h(z)$ is analytic and nonzero.

Page 120. Proof of Theorem 4.5: An argument that $h_2(z)$ has the same radius of convergence is missing:

If there is a second solution of the form $u_2(z) = z^{\alpha_2} h_2(z)$ the same argument can be used for $h_2(z)$. Otherwise one has to use (4.56) below in place of (4.39).

It is also possible to use $u_2(z) = c(z) u_1(z)$ with

$$c'(z) = z^{-2\alpha_1 - p_0} h_1(z)^{-2} \exp \left( - \int^z \tilde{p}(z) \right) = u_1(z)^{-2} \exp \left( - \int^z p(z) \right), \quad (2)$$
where \( \tilde{p}(z) = p(z) - \frac{p_0}{z} \) is the analytic part of \( p(z) \) near 0. However the primitive \( c(z) \) might have logarithmic terms at a zero \( z_0 \) of \( u_1(z) \). A computation shows

\[
c'(z) = \exp \left( -\int_{z_0}^{z} p(z') \, dz' \right) \left( \frac{1}{(z - z_0)^2} + \frac{u''(z_0) + p(z_0)u'(z_0)}{z - z_0} + \cdots \right)
\]

and since \( u''(z_0) + p(z_0)u'(z_0) = 0 \) we conclude that \( c \) has a first order pole at \( z_0 \) which will be removed by \( u_1(z) \).

Alternatively, note that the present theorem will also follow as a special case of Theorem 4.13.

Page 126. Problem 4.5: \( \Gamma(z) = (z^n + 1) + O(1) \).

Page 126. Problem 4.8: \( h_j = -\frac{1}{j} \sum_{k=0}^{j-1} p_{j-k} h_k \).

Page 126. Problem 4.10: For (i) you can use that \( \Gamma(z) \) has no zeros and hence \( \Gamma(z)^{-1} \) is an entire function.

Page 126. Problem 4.11 (iii): \( J_{\nu+1}(z) - J_{\nu-1}(z) = -2J_{\nu}'(z) \).

Page 145. Problem 5.3: We have \( x \in [0,1] \) and there is only one boundary condition \( u(t,1) = 0 \).

Page 145. Problem 5.5: \( m \frac{d^2}{dt^2} u(t,n) = -k(u(t,n+1) - u(t,n)) - k(u(t,n-1) - u(t,n)) \).

Page 154.

\[
y(x) = y(x_0)c(z,x,x_0) + p(x_0)y'(x_0)s(z,x,x_0) - \int_{x_0}^{x} s(z,x,t)g(t)r(t)dt. \quad (5.50)
\]

\[
s(z,x,x_0) = \frac{u(x)v(x_0) - u(x_0)v(x)}{W(u,v)}. \quad (5.51)
\]

Page 155. Problem 5.13: \( Q(y) = \frac{q(x)}{r(x)} - \frac{(p(x)y'(x))^{1/4}}{r(x)(p(x)y)r(x(y))^{1/4}} \).

Page 162.

\[
\min_{x \in [a,b]} \frac{q(x)}{r(x)} \leq E_0. \quad (5.77)
\]

Page 166. Problem 5.22: with \( f^{(2j)}(0) = f^{(2j)}(1) = 0 \) for \( 0 \leq j \leq k \).
In fact, $\theta_b(\lambda, x)$ as defined in (5.89) is the Prüfer angle for $-u_b(\lambda, x)$, but this will be of no importance for our purpose.

Moreover, note that we also have:

\[
\#(\lambda, L) = \left\lceil \frac{\theta_a(\lambda, b) - \beta}{\pi} \right\rceil = \left\lceil \frac{\alpha - \theta_b(\lambda, a)}{\pi} \right\rceil - 1, \quad (5.91)
\]

Moreover, note that we also have:

\[
\#(\lambda, L) = \left\lfloor \frac{\theta_a(\lambda, b) - \beta}{\pi} \right\rfloor + 1 = \left\lfloor \frac{\alpha - \theta_b(\lambda, a)}{\pi} \right\rfloor,
\]

Replace “As in the case of Theorem 5.18 one proves” by ”As an immediate consequence of Lemma 5.16 we obtain”

Problem 5.29: suppose $0 \leq \alpha_2 < \alpha_1 < \pi$ and show

$$W'(u_0, u_1) = (q_1 - \lambda t_1 q_1 - g_0 + \lambda_0 t_0 g_0) u_0 u_1 + \left( \frac{1}{p_0} - \frac{1}{p_1} \right) p_0 u_0' p_1 u_1'. \quad (5.111)$$

Problem 5.31:

$$u'(x) = a \left( \cos(x + b) + \frac{(-3/4 - \nu^2) \sin(x + b)}{2x^{3/2}} + O(x^{-5/2}) \right).$$

Problem 5.33:

$$G_{\pm}(z, x, y) = \frac{W(z)^{-1}}{1 \mp \rho_+(z)} \begin{cases} u_+(z, x) u_-(z, y) \mp \rho_+(z) u_-(z, x) u_+(z, y), & y < x, \\ u_-(z, y) u_+(z, x) \mp \rho_+(z) u_+(z, y) u_-(z, x), & y > x, \end{cases}$$

with $W(z) = W(u_+(z), u_-(z)).$

Problem 5.35: (iii) $c(z, \ell) = \mathcal{G}(z, \ell)$.

$$\text{sign}(x_1) \int_{x_0}^x \frac{d\xi}{\sqrt{2(E - U(\xi))}} = t, \quad E = \frac{y_1^2}{2} + U(x_0). \quad (6.47)$$

Problem 6.24:

$$\frac{\partial}{\partial t} u(t, x) + \frac{\partial^3}{\partial x^3} u(t, x) + 6u(t, x) \frac{\partial}{\partial x} u(t, x) = 0$$

Problem 6.25 should be changed according to:
Show that if \( \{ (\dot{x}, x) \in M | E(\dot{x}, x) = E_0 \} \) is compact and contains no fixed point, then it corresponds to a periodic orbit. Conclude that all solutions are periodic if \( \lim_{|x| \to \infty} U(x) = +\infty \) and \( U \) has a unique minimum.

Page 213. Proof of Theorem 7.4: Exchange the definitions of \( Q_3 \) and \( Q_4 \).

Page 214. Problem 7.4: The trajectory enters \( Q_3 \) and satisfies \( x(t) < x_0 \) in \( Q_3 \) since \( y(t) \) decreases, implying \( x(t) x_1 = \frac{1-\mu}{\lambda} \) when \( \ldots \). If \( y_2 \geq y_0 \), that is, if
\[
\lambda \mu^2 ((\mu \lambda)^2 - 1) (x_0 - \frac{1+\mu}{1+\mu \lambda}) > 0.
\]

Page 222. The proof of Lemma 7.13 only shows that \( \omega_\sigma(x) \) contains a regular periodic orbit. However, the claim follows from Lemma 7.14 if \( \omega_\sigma(x) \) is connected. Moreover, connectedness follows from compactness as in the proof of Lemma 6.6. In particular, the connectedness assumption in Theorem 7.16 is superfluous.

Page 225. First equation in the example:
\[
f(x, y) = \left( -\eta E(x, y)^2 y - U'(x) \right),
\]

Page 231. Sentence before (8.9): Moreover, for any positively invariant neighborhood \( \tilde{U} \subseteq W^+(\Lambda) \) we have

Page 233. Problem 8.2: Let \( V_R = \{ x \in M | L(x) \leq R \} \) be a relatively compact set

Page 235. Replace the last sentence by: Using the Routh–Hurwitz criterion one can show that the two new fixed points are asymptotically stable for \( 1 < r < \frac{\sigma(3+b+\sigma)}{\sigma-b-1} \) if \( 1+b < \sigma \) and \( 1 < r \) if \( 1+b \geq \sigma \). For the classical values \( \sigma = 10, b = 8/3 \) this gives \( 1 < r < 470/19 = 24.74 \).

Page 240.
\[
\frac{d}{dt} \left( \frac{p}{q} \right) = \frac{1}{p} \text{grad}_s H(p, q),
\]

Page 242. Problem 8.12 should be changed to:

The Lagrangian of a relativistic particle in an external force field is given by
\[
L(v, q) = -mc^2 \sqrt{1-\frac{v^2}{c^2}} - U(q),
\]
where \( c \) is the speed of light, \( m \) the (rest) mass of the particle, and \( U \) the potential of the force field. Derive the equation of motions from Hamilton’s principle. Derive the corresponding Hamilton equations.
Page 246.

\[ H(p, q) = \frac{1}{2} (pM^{-1}p + qWq) \]  

(8.77)

Page 246. Line before equation (8.78). Then the symplectic transform \((P, Q) = (V^T M^{-1/2}p, V^T M^{-1/2}q)\) (Problem 8.15) gives the decoupled system

\[ H(p, q) = \sum_{j=1}^{n} \frac{p_j}{2m} + \sum_{j=0}^{n} U_0(q_{j+1} - q_j), \quad q_0 = q_{n+1} = 0, \]  

(8.81)

Page 246. In order to better explain what Problem 8.18 is about the text between “If we assume that the particles . . . of the Jacobian matrix of the potential.” should be replaced by:

If we assume that the particles are coupled by springs, the potential would be \(U_0(x) = \frac{k}{2}x^2\), where \(k > 0\) is the so called spring constant, and we have a harmonic oscillator with

\[
M = mI, \quad W = k \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & \ddots & \\
& \ddots & \ddots & \ddots \\
& & 2 & -1 \\
& & & 1
\end{pmatrix}.
\]

The motion is decomposed into \(n\) modes corresponding to the eigenvectors of \(W\), which are given by

\[ v^j = \sqrt{\frac{2}{n+1}} \begin{pmatrix}
\sin(\eta_j) \\
\sin(2\eta_j) \\
\vdots \\
\sin(n\eta_j)
\end{pmatrix}, \quad \eta_j = \frac{\pi j}{n+1}. \]

The corresponding eigenvalues are \(m\omega^2_j\), where \(\omega^2_j = \frac{2k}{m}(1 - \cos(\eta_j)) = \frac{4k}{m} \sin^2(\eta_j)\).

Consequently the \(j\)’th mode corresponds to the initial condition \((p(0), q(0)) = (0, v^j)\) and is given by

\[ q^j(t) = \cos(\omega_j t)v^j, \quad p^j(t) = -m\omega_j \sin(\omega_j t)v^j. \]

The energy of the \(j\)’th mode is \(H(p^j, q^j) = \frac{m\omega^2_j}{2}\).

Page 247. Problem 8.16: Ignore the hint.

Page 257ff. The requirement \(x \in U(x_0)\) could be made part of the definition of \(M^{x_0, \alpha}\). Then the intersection with \(U(x_0)\) can be dropped at various later points.
Theorem 9.3: there are neighborhoods \( U(x_0) \) of \( x_0 \) and \( U \) of 0 and a function \( h^{+,-} \in C^k(E^{+,-} \cap U, E^{0+,-}) \) such that

\[
\text{Page 261. Theorem 9.4: there are neighborhoods } U(x_0) \text{ of } x_0 \text{ and } U \text{ of 0 and functions } h^{\pm} \in C^k(E^{\pm} \cap U, E^{0+,-}) \text{ such that}
\]

Note that while we always have \( M^{\pm}(x_0) \subseteq W^{\pm}(x_0) \cap U(x_0) \), equality might not hold even in the case of a hyperbolic fixed point. In fact, \( W^{\pm}(x_0) \) might also contain points from \( M^{-}(x_0) \) as the example of a homoclinic orbit shows.

Theorem 9.5: Delete equation (9.22) from the statement and add the following remark after the theorem:

\[
\text{Page 261. Theorem 9.5: Delete equation (9.22) from the statement and add the following remark after the theorem:}
\]

\[
\text{Page 266. Proof of Lemma 9.7: The equation } A^{-1} \circ \varphi \circ \vartheta = \varphi \circ \vartheta \circ A^{-1} \text{ should read } f \circ \varphi \circ \vartheta = \varphi \circ \vartheta \circ f. \text{ This last equation implies } \varphi \circ \vartheta = I + l, \text{ where } l \text{ is a solution of } Ll(x) = g(x) - g(x + l(x)). \text{ Using the estimates for the inverse of } L \text{ and for } g \text{ one obtains } l \equiv 0 \text{ and thus } \varphi \text{ is a homeomorphism.}
\]

\[
\text{Page 268. The very last equation on the bottom of the page is only true if } \Phi_t \text{ is linear. Set } \Phi_t = e^{tA} + G_t \text{, where } G_t \text{ is bounded, and replace this equation by }
\]

\[
\varphi(x) = (x_1 + x_2^2, x_2).
\]

Line before (10.19) which is zero at \( p \), positive for \( x \neq p \), whose level sets \( S_\delta = \{ x \in U(p) | L(x) \leq \delta \} \) are connected for sufficiently small \( \delta \), and such that \( x \in U(p) \) implies \( f^n(x) \in U(p) \) and

\[
x(m) = \Pi(m, m_0)x_0 + \sum_{j=m_0}^{m-1} \Pi(m, j+1)g(j), \quad m < m_0.
\]

Second sentence after (11.10): Change \( W^s \) to \( W^+ \).

Third paragraph: A map \( f \) as above is called topologically transitive if for any given nonempty open sets \( U, V \subseteq M \) there is an \( n \in \mathbb{N} \) such that \( f^n(U) \cap V \neq \emptyset \).

Paragraph after (3.19): Moreover, since the endpoints of the subintervals of \( \Lambda_n \) are just given by \( T^{-n}_\mu(0,1) \), we see . . .
Lemma 11.10: the number of periodic points of period at most \( l \) is equal to \( \text{tr}(A^l) \).

Problem 11.10: the number of periodic orbits of period at most \( n \).

Theorem 11.20: the Hausdorff dimension of the repeller \( \Lambda \) of the tent map \( T_\mu \) for \( \mu > 2 \) is

Problem 11.13: It seems too difficult to give a counterexample. The problem should be ignored.

Page 333. Change \( W_s \) to \( W^+ \) and \( W_u \) to \( W^- \) in the picture and the text before the picture.

Problem 13.3:

\[
\dot{q} = p,
\dot{p} = -\sin(q) + \varepsilon \sin(t).
\]

Page 339.

\[
M(t) = \int_{-\infty}^{\infty} p_0(s) \left( \delta p_0(s) + \gamma \cos(\omega(s-t)) \right) ds
= \frac{4\delta}{3} - \sqrt{2\pi\gamma\omega} \operatorname{sech}(\frac{\pi\omega}{2}) \sin(\omega t). \tag{13.26}
\]

Thus the perturbed Duffing equation is chaotic for \( \varepsilon \) sufficiently small provided