Errata

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Errata

Changes appear in yellow. Line $k+$ (resp., line $k-$) denotes the $k$th line from the top (resp., the bottom) of a page. My thanks go to the following individuals who have contributed to this list: Tobias Wöhrer, Simon Becker, Dennis Cutraro, Mateusz Piorkowski, Laura Kanzler, Mateus Sampaio, Laura Shou, Noema Nicolussi, Andreas Geyer-Schulz, Rene Allerstorfer, Manuel Culqui Rodriguez, Fritz Gesztesy, Marcel Griesemer, Michael Hofacker, Maxim Zinchenko, Jannik Pitt, Jake Fillman, David Damani, Peter Stahlecker, Iryna Karpenko, Michael Putzenlechner, Jacob Shapiro.

Page 12. Line after (0.16): Note that $x \in Y$ if and only if $\text{dist}(x,Y) = 0$.

Page 16. First line: for $a \in \ell^p(\mathbb{N})$, $b \in \ell^q(\mathbb{N})$.

Page 25. Proof of Theorem 0.25: and we can choose $m_2 = \sqrt{\sum_j \|u_j\|^2_1}$.

Page 34. Proof of Lemma 0.36: (if $K_2(x,.)f(.) \notin L^p(Y, dv)$, the inequality is trivially true).

Page 36. Add the following at the end of Lemma 0.39: Moreover, if $u$ and $f$ both have compact support, then $f_k \in C_c^\infty(\mathbb{R}^n)$. 1
Page 36. Proof of Lemma 0.41: \( \varphi_n \in C_c^\infty(\mathbb{R}^n) \) with support inside some open ball \( X \) which converges \( \ldots \) continuous functions \( \varphi_n \) with support in \( X \) which converges to \( g \) \( \ldots \)

Page 54. Proof of Lemma 1.11: (ii) follows from \( \langle \varphi, A^{**}\psi \rangle = \langle A^*\varphi, \psi \rangle = \langle \varphi, A\psi \rangle \).

Page 60. Last sentence in the proof of Theorem 1.16: Since \( f - \varepsilon < f_z \) for all \( z \) we have \( f - \varepsilon < f^E \) and we have found a required function.

Page 61. Problem 1.23: Show that the span of \( \{(t - z)^{-1}|z \in U\} \) is dense in \( C_c^\infty(\mathbb{R}) \).

Page 66. Line after (2.15): measurable function \( A: \mathbb{R}^d \to \mathbb{C} \).

Page 66. Line 4 from the bottom: \(|(Af)(x)|^2 = |A(x)|^2|f(x)|^2 \leq \|A\|^2|f(x)|^2 \)

Page 72. Line 18+: Clearly we have \( \alpha A = \alpha A \) for \( \alpha \in \mathbb{C}\{0\} \) and \( A + B = A + B \) provided \( A \) is closable and \( B \) is bounded (Problem 2.8).

Page 75. Proof of Lemma 2.7:

\[
\|(A - z)\psi\|^2 = \|(A - x)\psi - iy\psi\|^2 \\
(2.46) = \|(A - x)\psi\|^2 + y^2\|\psi\|^2 \geq y^2\|\psi\|^2,
\]

Page 76. Problem 2.8: Suppose that if \( A \) is closable and \( B \) is bounded. Show that \( \alpha\overline{A} = \alpha\overline{A} \) for \( \alpha \in \mathbb{C}\{0\} \) and \( A + B = A + B \).

Page 78. Proof of Lemma 2.11:

\( \mathcal{D}(\tilde{A}) = \{\psi \in \mathcal{S}_A|\exists \tilde{\psi} \in \mathcal{S}: \langle \varphi, \psi \rangle_A = \langle \varphi, \tilde{\psi} \rangle, \forall \varphi \in \mathcal{D}(A)\} = \mathcal{S}_A \cap \mathcal{D}(A^*) \)

as \( \mathcal{D}(A) \subset \mathcal{S}_A \) is dense and \( \langle \varphi, \psi \rangle_A = \langle (A + 1)\varphi, \psi \rangle \) for \( \varphi \in \mathcal{D}(A), \psi \in \mathcal{S}_A \).

Page 82. Proof of Lemma 2.15:

\[
2\operatorname{Re}(\varphi, A\psi) = \frac{1}{2}q(\psi + \varphi) - q(\psi - \varphi) \leq \frac{\|q\|}{2}(\|\psi + \varphi\|^2 + \|\psi - \varphi\|^2) \\
= \|q\|(\|\psi\|^2 + \|\varphi\|^2)
\]

Page 87. Proof of Theorem 2.19:

\( f'(\lambda) = -\|(A - E + \lambda)^{-1}\varphi\|^2 \leq -f(\lambda)^2 \)

Page 88. Problem 2.18: Then so does \( A + B \) if \( \|B\| \ll \|A^{-1}\|^{-1} \).

Page 93. Paragraph after Lemma 2.28: A conjugate linear map \( C: \mathcal{S} \to \mathcal{S} \) is called a conjugation if it satisfies \( C^2 = \mathbb{I} \) and \( \langle C\psi, C\varphi \rangle = \langle \varphi, \psi \rangle \).
Page 94. Proof of Lemma 2.30:
\[
\sum_j \langle \varphi_j(z), \psi_j(z^*) \rangle = \sum_j \langle \psi_j(z), \varphi_j(z^*) \rangle.
\]

Page 116. Proof of Lemma 3.13: To see the converse, note that by Theorem 3.27, the set \(M\) is a support for \(\mu\).

Page 126. Proof of Theorem 3.26: Clearly the second part can be estimated by
\[
\int_{\mathbb{R}\setminus I} K_t(t-\lambda)d\mu(t) \leq \varepsilon \int_{\mathbb{R}\setminus I} \frac{d\mu(t)}{(t-\lambda)^2}.
\]

Page 127. Proof of Lemma 3.30: First of all, note that we can split \(F(z) = F_1(z) + F_2(z)\) according to
\[
d\mu = \chi_{[-1,1]}d\mu + (1 - \chi_{[-1,1]})d\mu.
\]

Page 135. Problem 4.11:
\[
\chi_{\Omega}(A) = -\frac{1}{2\pi i} \int_{\Gamma} R_A(z)dz,
\]

Page 138.
\[
(4.31) \quad \|A\psi\|^2 = \langle \psi, A^2\psi \rangle = \langle \psi, A^*A\psi \rangle = \|A\psi\|^2, \quad \psi \in \mathcal{D}(|A|) = \mathcal{D}(A),
\]

Page 139:
\[
(4.34) \quad U^*U = P_{\ker(A)}I, \quad UU^* = P_{\ker(A^*)}I,
\]

Page 139: Last line of Theorem 4.10: \(\ker(U) = \ker(A)\)

Page 141: (ii) We have
\[
(4.40) \quad \inf_{\psi \in U(\varphi_1,\ldots,\varphi_{n-1})} \langle \psi, A\psi \rangle \geq E_n
\]

Page 141: Corollary 4.13: Suppose \(A\) and \(B\) are self-adjoint operators with \(\mathcal{D}(A) = \mathcal{D}(B)\) and \(A \geq B\) (i.e., \(A - B \geq 0\)).

Page 143: Proof of Theorem 4.16: Thus
\[
\langle \psi, (A-\lambda_2)(A-E)\psi \rangle = \|A\psi\|^2 + \lambda_2E \geq 0 \quad \text{and} \quad \ldots
\]

Page 144: Proof of Theorem 4.17: Consequently, \((P - \lambda)\psi_k \to 0\), where \(\psi_k = \psi_{1,k} \otimes \cdots \otimes \psi_{n,k}\) and hence \(\lambda \in \sigma(P)\).
Page 146: Proof of Lemma 5.2:
\[
\frac{d}{dt} \varphi(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \left( U(-t-\varepsilon) - U(t) \right) \left( \psi(t) - \varepsilon i A \psi(t) + o(\varepsilon) \right) + U(t) \left( \psi(t+\varepsilon) - \psi(t) \right) \right) = 0.
\]

Page 148: Top of page: Let \( \psi \in \mathcal{D}(A) \) and abbreviate \( \psi(t) = (U(t) - V(t)) \psi \).
Then
\[
\lim_{\varepsilon \to 0} \frac{\psi(t+\varepsilon) - \psi(t)}{\varepsilon} = -i A \psi(t)
\]
and hence \( \frac{d}{dt} \| \psi(t) \|^2 = -2 \text{Re} \langle \psi(t), i A \psi(t) \rangle = 0. \)

Page 152: Proof of Theorem 5.7: Since \( K(A-i)^{-1} \) is compact by assumption,
Page 154: Proof of Theorem 5.9: We will assume that \( K \) is compact.
Page 155: Problem 5.7:
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \varphi, e^{it(A-\lambda_0)} \psi \rangle dt = \langle \varphi, P_A(\{\lambda_0\}) \psi \rangle
\]
Page 155: Problem 5.9:
\[
(5.27) \quad \mathcal{H}_{rc} = \{ \psi \in \mathcal{D} \mid \lim_{t \to \infty} \langle \psi, e^{-itA} \psi \rangle = 0 \} \supseteq \mathcal{H}_{ac},
\]
Page 156: Proof of Theorem 5.11:
\[
\left\| \left( e^{iA} e^{iB} \right)^n - e^{it(A+B)} \psi \right\| \leq |t| \max \left| s \right| F_r(s),
\]
Page 159. Theorem 6.4:
\[
(6.4) \quad \gamma - \max \left( a|\gamma| + b, \frac{b}{1-a} \right).
\]
Page 159. Proof of Theorem 6.4: last sentence: The explicit bound (6.4) follows since this condition implies \( \| BR_A(-\lambda) \| < 1 \) by virtue of (6.2) from the proof of the previous lemma.
Page 161. Lemma 6.8:
\[
(6.9) \quad s_n(K) = \min_{\psi_1,\ldots,\psi_{n-1}} \sup_{\psi \in U(\psi_1,\ldots,\psi_{n-1})} \| K \psi \|,
\]
Page 162. Proof of Lemma 6.9: last formula
\[
\gamma_n = \| K - K_n \| = \sup_{\| \psi \|=1} \| K(\psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle \varphi_j) \|.
\]
Proof of Lemma 6.10: Conversely, choose \( \varphi_i = \hat{\varphi}_i \).

Theorem 6.25: add for \( \lambda > \frac{b}{2} - \gamma \), after (6.44)

Line before equation (6.45): Furthermore, we can define \( C_q(\lambda) \) for all \( z \in \rho(A) \), using

Problem 6.18: Suppose \( A \) is self-adjoint, \( \lambda \in \mathbb{R} \), and \( R \) is bounded. Show that \( R = R_A(\lambda) \) if and only if \( \langle (A - \lambda)\varphi, R\psi \rangle = \langle \varphi, \psi \rangle \) for all \( \varphi \in \mathcal{D}(A) \), \( \psi \in \mathcal{H} \).

Corollary 6.32: Then this holds for all \( z \) in the interior of \( \Gamma \).

Proof of Theorem 6.38, penultimate line: \( \ldots, \) which implies \( \| R_A(\lambda) - i \| < 1 \) for \( n \) sufficiently large, \( \ldots \)

Discussion after Lemma 7.20: \( |\psi(x, t)|^2d^n x = |\hat{\psi}(\frac{x}{2t})|^2 \frac{d^n x}{(2t)^n} \)

Last line of the proof of Theorem 8.2: \( 0 = (z^* z^*) \| \hat{A}\psi \|^2 \)

(8.13) \[ \psi(x) = \left( \frac{\lambda}{\pi} \right)^{n/4} e^{-\frac{1}{2} |x - x_0|^2} \psi_{x_0} \]

First equation in the proof of Lemma 8.3:

\[ \frac{1}{\sqrt{2\pi}} \int \frac{\varphi(x)e^{-\frac{1}{2} |x|^2}}{2\pi} \sum_{j=0}^{k} \frac{(itx)^j}{j!} \frac{d^n x}{dx} = 0 \]

Theorem 8.4: There exists an orthonormal basis of simultaneous eigenvectors for the operators \( L^2 \) and \( L^3 \).

Line before (8.41): If \( N\psi_0 = 0 \), then we must have \( A_+ \psi_0 = 0 \), and the normalized solution of this last equation is given by

Proof of Theorem 8.6:

\[ R_{H_1}(z) = R_{H_1}(z)(1 - P_1) \frac{1}{z} P_1 = U R_{H_0}(z) U^* \frac{1}{z} P_1 \]

\[ \geq \frac{1}{z} \left( U (|H_0|^{1/2} R_{H_0}(z)|H_0|^{1/2} - 1) U^* P_1 \right) \]

\[ = \frac{1}{z} \left( A R_{H_0}(z) A^* (1 - P_1) P_1 \right) = \frac{1}{z} \left( A R_{H_0}(z) A^* P_1 \right) \]

Proof of Lemma 9.5: Choosing \( f_1 = v, f_2 = f, f_3 = v^*, f_4 = f^* \), we infer (9.15).
Problem 9.1: and $f(d) = \gamma$, $(pf')(d) = \delta$.
Problem 9.3: Let $\phi \in L^1_{loc}(I)$ be real-valued.
Problem 9.4: Add the assumption that $a$ is regular. Otherwise one can also start the integration at an arbitrary point in $(a,b)$.

Problem 9.4: where

$$Q = \frac{q}{r} - \frac{(pr)^{1/4}}{r} (p((pr)^{-1/4}))'.$$

Problem 9.4: Replace the last sentence by: Moreover, the following set is a core for $A$

$$(9.21) \quad \mathcal{D}_1 = \{ f \in \mathcal{D}(\tau) \mid \exists x_0 \in I : \forall x \in (a,x_0), V_x(f) = 0, \exists x_1 \in I : \forall x \in (x_1,b), W_x(f) = 0 \},$$

where we set $V_x(f) = W_x(v,f)$, $W_x(f) = W_x(w,f)$ if $\tau$ is l.c. at $a$, $b$ and $V_x(f) = f(x)$, $W_x(f) = f(x)$ if $\tau$ is l.p. at $a$, $b$, respectively.

Problem 224.

$$(9.23) \quad W_a(v,f) = 0 \iff \cos(\alpha)BC_2^a(f) + \sin(\alpha)BC_1^a(f) = 0,$$

where $\tan(\alpha) = \frac{BC_2^a(v)}{BC_1^a(v)}$.

Page 228. Theorem 9.10: Delete ”(which are simple)”. And the following claim about simplicity of eigenvalues only applies to separated boundary conditions as in Theorem 9.6.

Problem 231.

$$(9.37) \quad (Uf)(\lambda) = \frac{1}{\sqrt{2\pi}} \left( \int_\mathbb{R} e^{i\sqrt{\lambda}x} f(x) \, dx \int_\mathbb{R} e^{-i\sqrt{\lambda}x} f(x) \, dx \right), \quad \lambda \in \sigma(H_0) = [0, \infty).$$

Problem 233. Proof of Lemma 9.13:

$$\sum_j \int_\mathbb{R} F_j(\lambda)^* \int_a^b u_j(\lambda,x)g(x)r(x) \, dx d\mu_j(\lambda) = \int_a^b (U^{-1} F)(x)^* g(x)r(x) \, dx.$$ 

Interchanging integrals on the left-hand side

Page 233. Delete the last sentence: Note that since we can replace $u_j(\lambda, x)$ by $\gamma_j(\lambda)u_j(\lambda, x)$ where $|\gamma_j(\lambda)| = 1$, it is no restriction to assume that $u_j(\lambda, x)$ is real-valued.

Page 250. Second line in Section 9.7: on $(a,b) = \mathbb{R}$. 

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Proof of Lemma 9.35: where $M_n = \sup_{|m| \geq n} \int_m^{m+1} |q(x)|dx$.

First line: the zeros of $\psi_n$ interlace the zeros of $\psi_{n+1}$.

(Hint: Let $\varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2)$ and investigate $\langle \varphi_\varepsilon, H \varphi_\varepsilon \rangle$.)

Problem 9.18: Change the hint according to:

$$\text{(Hint: Let } \varphi_\varepsilon(x) = \exp(-\varepsilon^2 x^2) \text{ and investigate } \langle \varphi_\varepsilon, H \varphi_\varepsilon \rangle.)$$

$$F(z) = \int \frac{f(z, y) d\mu(y)}{y}$$

$$\pi^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2} d^n x = nV_n \int_0^\infty e^{-r^2} r^{n-1} dr = \frac{nV_n}{2} \int_0^\infty e^{-s} s^{n/2-1} ds = \frac{nV_n}{2} \Gamma\left(\frac{n}{2}\right) = V_n \Gamma\left(\frac{n}{2} + 1\right)$$

Proof of Lemma A.36: Now set $\varphi(x) = 1$ for $|x| \leq R$, $\varphi(x) = 1 - |x| - R$ for $R \leq |x| \leq R + 1$, and $\varphi(x) = 0$ for $|x| \geq R + 1$.

Problem A.32 can be deleted as the claim is part of Lemma A.36.

Problem A.34. This claim is clearly wrong (take a function which is constant on an interval). It should be deleted.

Example:

$$(D\mu)(x) = \lim_{\varepsilon \downarrow 0} \frac{\mu((x - \varepsilon, x + \varepsilon))}{2\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\mu(x + \varepsilon) - \mu(x - \varepsilon)}{2\varepsilon} = \mu'(x).$$

Proof of Lemma A.47.

$$|M_\varepsilon| = \int_{M_\varepsilon} d^n x \leq \frac{1}{\varepsilon} \int_{M_\varepsilon} (D\mu)(x) d^n x = \frac{1}{\varepsilon} \mu_{ac}(M_\varepsilon) \leq \frac{1}{\varepsilon} \mu_{ac}(M_0) = 0$$

Addendum

Proof of Theorem 2.14: Since the rest is not so straightforward, here is a complete proof:
Proof. Since $\mathcal{H}_q$ is dense, $\tilde{\psi}$ and hence $A$ is a well-defined operator. Moreover, replacing $q$ by $q(\cdot) - \gamma\|\cdot\|^2$ and $A$ by $A - \gamma$, it is no restriction to assume $\gamma = 0$. Next it will be convenient to look at the definition from a somewhat more abstract point of view: We have a conjugate linear continuous embedding $j : \mathcal{H} \to \mathcal{H}_q^*$, $\psi \mapsto \langle \psi, \cdot \rangle$ (here $\mathcal{H}_q$ is equipped with $\|\cdot\|_q$ with $\text{Ran}(j)$ dense. Indeed, if $\text{Ran}(j)$ were not dense, there would be some nonzero $\varphi \in \mathcal{H}_q^{**} \cong \mathcal{H}_q$ (the identification given by the Riesz lemma via evaluation) such that $j(\psi)(\varphi) = \langle \psi, \varphi \rangle = 0$ for all $\psi \in \mathcal{H}$ implying the contradiction $\varphi = 0$.

Next, there is a conjugate linear isometric isomorphism $\hat{A} : \mathcal{H}_q \to \mathcal{H}_q^*$, $\psi \mapsto s(\psi,.) + \langle \psi, \cdot \rangle$ (Riesz lemma) and our operator $A$ is given by $j^{-1}\hat{A} - I$. Moreover, $\mathcal{D}(A) = \hat{A}^{-1}\text{Ran}(j)$ is dense in $\mathcal{H}_q$ and hence also in $\mathcal{H}$. By construction, $q_A(\psi) = g(\psi)$ for $\psi \in \mathcal{D}(A)$, which shows that $A$ is nonnegative and as in the proof of Lemma 2.11, it follows that $\text{Ran}(A + 1) = \mathcal{H}$. Thus $A$ is self-adjoint. Finally, note that the fact that $\mathcal{D}(A)$ is dense in $\mathcal{H}_q$ implies $\mathcal{D}_A = \mathcal{D}_q$.

Concerning uniqueness let $\hat{A}$ be another self-adjoint operator with the same properties. Then equality of the associated quadratic forms (and hence of the sesquilinear forms) on $\mathcal{O}$ implies $\langle A\psi, \varphi \rangle = \langle \hat{A}\psi, \varphi \rangle$ for $\psi \in \mathcal{D}(A)$, $\varphi \in \mathcal{D}(\hat{A})$. But this shows $\psi \in \mathcal{D}(\hat{A}^*) = \mathcal{D}(\hat{A})$ and $\hat{A}\psi = A^*\psi = A\psi$ and vice versa. \qed

Page 118. Here is an amplification of Theorem 3.16:

**Theorem 3.16.** For every self-adjoint operator $A$ there is an ordered spectral basis $\{\psi_j\}_{j=1}^N$. Moreover, it can be chosen such that $d\mu_{\psi_j} = \chi_{\Omega_j} d\mu$, where $\mu$ is a maximal spectral measure and $\Omega_{j+1} \subseteq \Omega_j$. The dimension $N$ is the spectral multiplicity of $A$.

**Proof.** First of all observe that for every $\varphi$ there is a maximal spectral vector $\psi$ such that $\varphi \in \mathcal{H}_\psi$. To see this start with a maximal spectral vector $\tilde{\psi}$. Then $d\mu_{\tilde{\psi}} = \gamma d\mu_{\tilde{\psi}}$ and we set $\Omega = \{\lambda : f(\lambda) > 0\}$. Then $P_A(\Omega)\varphi = \varphi$ since $|P_A(\Omega)\varphi|^2 = \int_{\Omega} d\mu_{\tilde{\psi}} = \int_{\Omega} f d\mu_{\tilde{\psi}} = \|\varphi\|^2$. Now set $\psi = \varphi + P(\mathbb{R}\setminus\Omega)\tilde{\psi}$ and observe $d\mu_{\psi} = d\mu_{\tilde{\psi}} + \chi_{\mathbb{R}\setminus\Omega} d\mu_{\tilde{\psi}} = (f + \chi_{\mathbb{R}\setminus\Omega}) d\mu_{\tilde{\psi}}$. Since $f + \chi_{\mathbb{R}\setminus\Omega} > 0$ we see that $d\mu_{\tilde{\psi}}$ is absolutely continuous with respect to $d\mu_{\tilde{\psi}}$ and hence $\psi$ is a maximal spectral vector with $\varphi = P_A(\Omega)\psi \in \mathcal{H}_\psi$ as required.

Now start with some total set $\{\tilde{\psi}_j\}$ and proceed as in Lemma 3.4 to obtain an ordered spectral basis $\{\psi_j\}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to $\mu_{\psi_j}$ all spectral measures are absolutely continuous with respect to $\mu = \mu_{\psi_1}$, that is, $d\mu_{\psi_j} = f_j d\mu$. Choosing $\Omega_j = \{\lambda : f_j(\lambda) > 0\}$ we can replace $\psi_j \to \chi_{\Omega_j}(A)^{-1/2}\psi_j$ such that $f_j \to \chi_{\Omega_j}$. Since $\mu_{\psi_{j+1}}$ is absolutely continuous with respect to $\mu_{\psi_j}$ we can even assume $\Omega_{j+1} \subseteq \Omega_j$.

Finally, we show that the spectral multiplicity of $A$ is $N$. By the first part we can assume that $A$ is multiplication by $\lambda$ in $\bigoplus_{j=1}^N L^2(\Omega_j, d\mu)$. Let $\{\psi_j\}_{j=1}^n$ be a spectral basis with $n < N$. We will show that there is some vector in the orthogonal complement of $\bigoplus_{j=1}^n \mathcal{H}_{\psi_j}$. Of course such a vector exists pointwise for every $\lambda$ but it is not clear that the components can be chosen measurable. To see this we use a Gauss-type elimination: For this note that we can multiply every vector $\psi_j$ with a non-vanishing function or add multiples of the other.
vectors to a given one without changing $\bigoplus_j \delta_j$. Hence we can first normalize the first component of every $\psi_j$ to be a characteristic function. Moreover, by adding all other vectors to $\psi_1$ we can assume that its first component is positive on a maximal set $\tilde{\Omega}_1$. In fact, after another normalization we can assume that $\psi_{1,1} = \chi_{\tilde{\Omega}_1}$ and after subtracting multiples of $\psi_1$ from the remaining vectors we can assume $\psi_{j,1} = 0$ for $j \geq 2$. If $\mu_1(R \setminus \tilde{\Omega}_1) > 0$ then $\varphi = (\chi_{R \setminus \tilde{\Omega}_1}, 0, \ldots)$ would be in the orthogonal complement and we are done. So assume $\chi_{\Omega_1} = 1$ and continue with the other components until they satisfy $\psi_{j,k} = \delta_{j,k}$ for $1 \leq j, k \leq n$. Then $\varphi = (-\psi_{1,n+1}, \ldots, -\psi_{n,n+1}, 1, 0, \ldots)$ is in the orthogonal complement contradicting our assumption that $\{\psi_j\}_{j=1}^n$ is a spectral basis. $\square$