Proseminar Advanced Functional Analysis

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Please see the lecture notes for further details.

1. Let X be a topological vector space. Show that $U + V$ is open if one of the sets is open.
2. Show that Corollary 5.4 fails even in $\mathbb{R}^2$ unless one set is compact.
3. Show that the nonempty intersection of extremal sets is extremal. Show that if $L \subseteq M$ is extremal and $M \subseteq K$ is extremal, then $L \subseteq K$ is extremal as well.
4. Show that the closed unit ball in $L^1(0, 1)$ has no extremal points.

5. Let X be a topological vector space. Show that the closure and the interior of a convex set is convex. (Hint: One way of showing the first claim is to consider the continuous map $f : X\times X \to X$ given by $(x, y) \mapsto \lambda x+(1-\lambda)y$ and use Problem B.14.)

6. Show that (5.11) generates the weak topology on $B_1(0) \subset X$. Show that (5.13) generates the weak topology on $B_1^*(0) \subset X^*$.

7. Let p, q be two seminorms. Then $p(x) \le Cq(x)$ if and only if $q(x) < 1$ implies $p(x) < C$.

8. Instead of (5.17) one frequently uses
$$\tilde{d}(x, y) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{q_n(x-y)}{1+q_n(x-y)}.$$
Show that this metric generates the same topology.

   Consider the Fréchet space $C(\mathbb{R})$ with $q_n(f) = \sup_{[-n,n]} |f|$. Show that the metric balls with respect to $\tilde{d}$ are *not* convex.

9. Find an equivalent norm for $\ell^1(\mathbb{N})$ such that it becomes strictly convex (cf. Problems 1.13 and 1.17).

10. Show that a Hilbert space is uniformly convex. (Hint: Use the parallelogram law.)

11. Consider a linear operator $A : \mathfrak{D}(A) \subseteq X \to Y$, where X and Y are Banach spaces. Show that $A : \mathfrak{D}(A) \to Y$ is bounded if we equip $\mathfrak{D}(A)$ with the graph norm

$$\|x\|_A := \|x\|_X + \|Ax\|_Y, \quad x \in \mathfrak{D}(A).$$

Show that the completion X_A of $(\mathfrak{D}(A), \|.\|_A)$ can be regarded as a subset of X if and only if A is closable. Show that in this case the completion can be identified with $\mathfrak{D}(\overline{A})$ and that the closure of A in X coincides with the extension from Theorem 1.16 of A in X_A. In particular, A is closed if and only if $(\mathfrak{D}(A), \|.\|_A)$ is complete.

12. Let $X := \ell^2(\mathbb{N})$ and $(Aa)_j := j\, a_j$ with $\mathfrak{D}(A) := \{a \in \ell^2(\mathbb{N}) | (ja_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N})\}$ and $Ba := (\sum_{j\in\mathbb{N}} a_j)\delta^1$. Then we have seen that A is closed while B is not closable. Show that $A + B$, $\mathfrak{D}(A + B) = \mathfrak{D}(A) \cap \mathfrak{D}(B) = \mathfrak{D}(A)$ is closed.

13. Discuss the spectrum of the right shift R on $\ell^1(\mathbb{N})$. Show $\sigma(R) = \sigma_r(R) = \bar{B}_1(0)$ and $\sigma_p(R) = \sigma_c(R) = \emptyset$.

14. Suppose $A \in \mathscr{L}(X)$. Show that generalized eigenvectors corresponding to different eigenvalues or with different order are linearly independent.

15. Let X_j be finite dimensional vector spaces and suppose

$$0 \longrightarrow X_1 \xrightarrow{A_1} X_2 \xrightarrow{A_2} X_3 \cdots X_{n-1} \xrightarrow{A_{n-1}} X_n \longrightarrow 0$$

is exact. Show that

$$\sum_{j=1}^{n} (-1)^j \dim(X_j) = 0.$$

(Hint: Rank-nullity theorem.)

16. Suppose $A \in \Phi(X)$. If the kernel chain stabilizes then $\operatorname{ind}(A) \leq 0$. If the range chain stabilizes then $\operatorname{ind}(A) \geq 0$. Moreover, if $A \in \Phi_0(X)$, then the kernel chain stabilizes if and only if the range chain stabilizes.

17. Let a and b be some real-valued sequences in $\ell^\infty(\mathbb{Z})$. Consider the operator

$$Jf_n = a_n f_{n+1} + a_{n-1} f_{n-1} + b_n f_n, \qquad f \in \ell^2(\mathbb{Z}).$$

Show that J is a bounded self-adjoint operator.

18. Show that $(\alpha A)^* = \alpha^* A^*$ for $\alpha \in \mathbb{C} \setminus \{0\}$ and $(A+B)^* \supseteq A^* + B^*$ (where $\mathfrak{D}(A^* + B^*) = \mathfrak{D}(A^*) \cap \mathfrak{D}(B^*)$) with equality if one operator is bounded. Give an example where equality does not hold.

19. Suppose AB is densely defined. Show that $(AB)^* \supseteq B^* A^*$. Moreover, if A is bounded or if B has a bounded inverse (defined on all of $\mathfrak{H}$), then $(AB)^* = B^* A^*$.

20. Show that normal operators are closed. (Hint: A^* is closed.)

21. Suppose that A is closable and $B \in \mathscr{L}(\mathfrak{H})$. Show that $\overline{\alpha A} = \alpha \overline{A}$ for $\alpha \in \mathbb{C} \setminus \{0\}$ and $\overline{A + B} = \overline{A} + B$.

22. Let $A = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A) = \{f \in H^2(0, \pi) \,|\, f(0) = f(\pi) = 0\}$ and let $\psi(x) = \frac{1}{2\sqrt{\pi}} x(\pi - x)$. Find the error in the following argument: Since A is symmetric, we have $1 = \langle A\psi, A\psi\rangle = \langle \psi, A^2\psi\rangle = 0$.

23. Suppose A is a densely defined closed operator. Show that A^*A (with $\mathfrak{D}(A^*A) = \{\psi \in \mathfrak{D}(A) | A\psi \in \mathfrak{D}(A^*)\}$) is self-adjoint. Show $\mathfrak{Q}(A^*A) = \mathfrak{D}(A)$. (Hint: $A^*A \geq 0$.)

24. Suppose a densely defined operator A_0 can be written as $A_0 = S^*S$, where S is a closable operator with $\mathfrak{D}(S) = \mathfrak{D}(A_0)$. Show that the Friedrichs extension is given by $A = S^*\overline{S}$.

    Use this to compute the Friedrichs extension of $A_0 = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A_0) = \{f \in C^2(0, \pi) | f(0) = f(\pi) = 0\}$. Compute also the self-adjoint operator $\overline{S}S^*$ and its form domain.

25. Find a Weyl sequence for the self-adjoint operator $A = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A) = H^2(\mathbb{R})$ for $z \in (0, \infty)$. What is $\sigma(A)$? (Hint: Cut off the solutions of $-u''(x) = z\,u(x)$ outside a finite ball.)

26. Show that for a normal operator eigenvectors corresponding to different eigenvalues are orthogonal.

27. Show that for $A = \bigoplus_j A_j$ as defined in the lecture, we have $(\bigoplus_j A_j)^* = \bigoplus_j A_j^*$.

28. Show that for $A = \bigoplus_j A_j$ as defined in the lecture, we have $\|A\| = \sup_j \|A_j\|$.

29. Let $\mathfrak{H} = L^2(\mathbb{R})$ and let f be a real-valued measurable function. Show that

$$P(\Omega) = \chi_{f^{-1}(\Omega)}$$

is a projection-valued measure. What is the corresponding operator?

30. Show that a resolution of the identity $P(\lambda) = P((-\infty, \lambda])$ satisfies properties (i)–(iv) stated in the book.

31. Show that for a self-adjoint operator A we have $\|R_A(z)\| = \operatorname{dist}(z, \sigma(A))^{-1}$.

32. Suppose A is self-adjoint. Let λ_0 be an eigenvalue and ψ a corresponding normalized eigenvector. Compute μ_ψ.

33. Construct a multiplication operator A on $L^2(\mathbb{R})$ which has dense point spectrum, $\overline{\sigma_p(A)} = \mathbb{R}$.

34. Let $d\mu(\lambda) = \chi_{[0,1]}(\lambda)d\lambda$ and $f(\lambda) = \chi_{(-\infty,t]}(\lambda)$, $t \in \mathbb{R}$. Compute $f_\star\mu$.

35. Show the missing direction in the proof of Lemma 3.12

36. Compute $\sigma(A)$, $\sigma_{ac}(A)$, $\sigma_{sc}(A)$, and $\sigma_{pp}(A)$ for the multiplication operator $A(x) = \lfloor x \rfloor$ in $L^2(\mathbb{R})$. What is its spectral multiplicity?

37. Let $\mathfrak{H} = L^2(0, 2\pi)$ and consider the one-parameter unitary group given by $U(t)f(x) = f(x - t \mod 2\pi)$. Show that it is strongly continuous. What is the generator of U?

38. Suppose $\psi(t)$ is differentiable on $\mathbb{R}$. Show that

$$\|\psi(t) - \psi(s)\| \le M|t - s|, \qquad M = \sup_{\tau \in [s,t]} \|\frac{d\psi}{dt}(\tau)\|.$$

(Hint: Consider $f(\tau) = \|\psi(\tau) - \psi(s)\| - \tilde{M}(\tau - s)$ for $\tau \in [s, t]$. Suppose τ_0 is the largest τ for which the claim holds with $\tilde{M} > M$ and find a contradiction if $\tau_0 < t$.)

39. (**Mean ergodic theorem**) Let A be self-adjoint and $\lambda_0 \in \mathbb{R}$. Show

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \varphi, \mathrm{e}^{\mathrm{i}t(A-\lambda_0)}\psi\rangle dt = \langle \varphi, P_A(\{\lambda_0\})\psi\rangle$$

and conclude

$$\operatorname*{s-lim}_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{\mathrm{i}t(A-\lambda_0)} dt = P_A(\{\lambda_0\}).$$

40. Prove Corollary 5.10 from the notes.

41. Let A be the self-adjoint operator $A = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A) = \{f \in H^2[0,1] | f(0) = f(1) = 0\}$ in the Hilbert space $L^2(0,1)$ and $q \in L^2(0,1)$.

    Show that for every $f \in \mathfrak{D}(A)$ we have

    $$\|f\|_\infty^2 \le \frac{\varepsilon}{2}\|f''\|^2 + \frac{1}{2\varepsilon}\|f\|^2$$

    for every $\varepsilon > 0$. Conclude that the relative bound of q with respect to A is zero. (Hint: $|f(x)|^2 \le \int_0^1 |f'(t)|^2 dt = -\int_0^1 f(t)^* f''(t) dt$.)

42. Suppose A is closed and B relatively bounded with A-bound less than one. Show that $A+B$ is closed. Show that this fails without the restriction on the A-bound of B.

43. Show that the singular values $s_j(K)$ of a compact operator K satisfy

    $$\|K\| = \max_j s_j(K).$$

44. Show that every bounded operator can be written as a linear combination of two self-adjoint operators. Furthermore, show that every bounded self-adjoint operator can be written as a linear combination of two unitary operators. (Hint: $x \pm \mathrm{i}\sqrt{1-x^2}$ has absolute value one for $x \in [-1,1]$.)

45. Suppose $f \in L^2(\mathbb{R}^n)$. Then the set $\{f(x+a)|a \in \mathbb{R}^n\}$ is total in $L^2(\mathbb{R}^n)$ if and only if $\hat{f}(p) \neq 0$ a.e. (Hint: Use Lemma 7.2 and the fact that a subspace is total if and only if its orthogonal complement is zero.)

46. The free relativistic Hamiltonian is given by $H_0 = \sqrt{-\Delta + m^2}$, $\mathfrak{D}(H_0) = H^1(\mathbb{R}^n)$. Show that H_0 is self-adjoint, find its spectrum and compute the spectral measure of ψ.

47. Let $f : \mathbb{R}^n \to \mathbb{R}$ be polynomially bounded. Show that $\mathcal{S}(\mathbb{R}^n)$ is a core for $f(p)$, $\mathfrak{D}(f) = \{\psi \in L^2(\mathbb{R}^n)|f(p)\hat{\psi}(p) \in L^2(\mathbb{R}^n)\}$. (Hint: Have a look at the examples on page 73).

48. Show that $\mathfrak{D}_0 = \{\psi \in \mathcal{S}(\mathbb{R})|\psi(0) = 0\}$ is dense but *not* a core for $H_0 = -\frac{d^2}{dx^2}$. Can you give another self-adjoint extension? (Hint: Have a look at the examples on page 73).

49. The **Bessel function** of order $\nu \in \mathbb{C}$ can be defined as

$$J_\nu(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(\nu+j+1)} \left(\frac{z}{2}\right)^{2j+\nu}.$$

Show that $J_\nu(z)$ is a solution of the Bessel differential equation

$$z^2 u'' + z u' + (z^2 - \nu^2) u = 0.$$

Prove the following properties of the Bessel functions.

(a) $(z^{\pm\nu} J_\nu(z))' = \pm z^{\pm\nu} J_{\nu\mp 1}(z)$.

50. Given $\alpha, \beta, \gamma, \delta$, show that there is a function f in $\mathfrak{D}(\tau)$ restricted to $[c,d] \subseteq (a,b)$ such that $f(c) = \alpha$, $(pf')(c) = \beta$ and $f(d) = \gamma$, $(pf')(d) = \delta$. (Hint: Lemma 9.2 from the notes.)

51. Let $A_0 = -\frac{d^2}{dx^2}$, $\mathfrak{D}(A_0) = \{f \in H^2[0,1] | f(0) = f(1) = 0\}$ and $B = q$, $\mathfrak{D}(B) = \{f \in L^2(0,1) | qf \in L^2(0,1)\}$. Find a $q \in L^1(0,1)$ such that $\mathfrak{D}(A_0) \cap \mathfrak{D}(B) = \{0\}$. (Hint: Problem 0.41 in the notes.)

52. Show that every Sturm–Liouville equation can be transformed into one with $r = p = 1$ as follows: Show that the transformation $U : L^2((a,b), r\,dx) \to L^2(d,e)$, $d = -\int_a^c \sqrt{\frac{r(t)}{p(t)}} dt$, $e = \int_c^b \sqrt{\frac{r(t)}{p(t)}} dt$, defined via $u(x) \mapsto v(y)$, where

$$y(x) = \int_c^x \sqrt{\frac{r(t)}{p(t)}} dt, \qquad v(y) = \sqrt[4]{r(x(y))p(x(y))}\, u(x(y)),$$

is unitary. Moreover, if $p, r, p', r' \in AC(a,b)$, then

$$-(pu')' + qu = r\lambda u$$

transforms into

$$-v'' + Qv = \lambda v,$$

where

$$Q = \frac{q}{r} - \frac{(pr)^{1/4}}{r} \big(p((pr)^{-1/4})'\big)'.$$

53. Compute the spectrum and the resolvent of $\tau = -\frac{d^2}{dx^2}$, $I = (0, \infty)$ defined on $\mathfrak{D}(A) = \{f \in \mathfrak{D}(\tau) | f(0) = 0\}$.

54. Suppose a is regular and $\lim_{x \to b} q(x)/r(x) = \infty$. Show that $\sigma_{ess}(A) = \emptyset$ for every self-adjoint extension. (Hint: Fix some positive constant n, choose $c \in (a, b)$ such that $q(x)/r(x) \geq n$ in (c, b), and use Theorem 9.11.)

55. Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and $c \in (a, b)$. Introduce

$$[u]_x = \frac{W_x(u, u^*)}{z - z^*} \in \mathbb{R}$$

and use (9.4) to show that

$$[u]_x = [u]_c + \int_c^x |u(y)|^2 \, r(y) dy, \qquad (\tau - z)u = 0.$$

Hence $[u]_x$ is increasing and $[u]_b = \lim_{x \uparrow b} [u]_x$ exists if and only if $u \in L^2((c, b), r\, dx)$.

Let $u_{1,2}$ be two solutions of $(\tau - z)u = 0$ which satisfy $[u_1]_c = [u_2]_c = 0$ and $W(u_1, u_2) = 1$. Then, all (nonzero) solutions u of $(\tau - z)u = 0$ that satisfy $[u]_b = 0$ can be written as

$$u = u_2 + m\, u_1, \qquad m \in \mathbb{C},$$

up to a complex multiple (note $[u_1]_x > 0$ for $x > c$).

Show that

$$[u_2 + m\, u_1]_x = [u_1]_x \Big(|m - M(x)|^2 - R(x)^2 \Big),$$

where

$$M(x) = -\frac{W_x(u_2, u_1^*)}{W_x(u_1, u_1^*)}$$

and

$$\begin{aligned} R(x)^2 &= \Big(|W_x(u_2, u_1^*)|^2 + W_x(u_2, u_2^*) W_x(u_1, u_1^*) \Big) \Big(|z - z^*| [u_1]_x \Big)^{-2} \\ &= \Big(|z - z^*| [u_1]_x \Big)^{-2}. \end{aligned}$$

Hence the numbers m for which $[u]_x = 0$ lie on a circle which either converges to a circle (if $\lim_{x \to b} R(x) > 0$) or to a point (if $\lim_{x \to b} R(x) = 0$) as $x \to b$. Show that τ is l.c. at b in the first case and l.p. in the second case.

56. Show that the dependence of the Weyl function on the boundary condition is given by

$$m_{b,\alpha}(z) = \frac{\cos(\alpha - \beta) m_{b,\beta}(z) + \sin(\alpha - \beta)}{\cos(\alpha - \beta) - \sin(\alpha - \beta) m_{b,\beta}(z)}.$$

(Hint: The case $\beta = 0$ is (9.52).)