## PS Topologie und Funktionalanalysis

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Note: References refer to the lecture notes.

1. Suppose  $\sum_{n=1}^{\infty} |c_n| < \infty$ . Show that

$$u(t,x) := \sum_{n=1}^{\infty} c_n e^{-(\pi n)^2 t} \sin(n\pi x),$$

is continuous for  $(t,x) \in [0,\infty) \times [0,1]$  and solves the heat equation for  $(t,x) \in (0,\infty) \times [0,1]$ . (Hint: Weierstrass M-test. When can you interchange the order of summation and differentiation?)

2. Show that for  $n, m \in \mathbb{N}$  we have

$$2\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 1, & n=m, \\ 0, & n\neq m. \end{cases}$$

Conclude that the Fourier sine coefficients are given by

$$\hat{u}_{0,n} = 2\int_0^1 \sin(n\pi x) u_0(x) dx$$

provided the sum in (\*) converges uniformly. Conclude that in this case the solution can be expressed as

$$u(t,x) = \int_0^1 K(t,x,y)u_0(y)dy, \qquad t > 0,$$

where

$$K(t, x, y) := 2 \sum_{n=1}^{\infty} e^{-(\pi n)^2 t} \sin(n\pi x) \sin(n\pi y)$$
$$= \frac{1}{2} \left( \vartheta(\frac{x-y}{2}, i\pi t) - \vartheta(\frac{x+y}{2}, i\pi t) \right).$$

Here

$$\vartheta(z,\tau) := \sum_{n \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}\pi n^2 \tau + 2\pi \mathrm{i} n z} = 1 + 2 \sum_{n \in \mathbb{N}} \mathrm{e}^{\mathrm{i}\pi n^2 \tau} \cos(2\pi n z), \quad \mathrm{Im}(\tau) > 0,$$

is the Jacobi theta function.

- 3. Show that  $|||f|| ||g||| \le ||f g||$ .
- 4. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if  $f_n \to f$ ,  $g_n \to g$ , and  $\alpha_n \to \alpha$ , then  $||f_n|| \to ||f||$ ,  $f_n + g_n \to f + g$ , and  $\alpha_n g_n \to \alpha g$ .

5. While  $\ell^1(\mathbb{N})$  is separable, it still has room for an uncountable set of linearly independent vectors. Show this by considering vectors of the form

$$a^{\alpha} = (1, \alpha, \alpha^2, \dots), \qquad \alpha \in (0, 1).$$

(Hint: Recall the Vandermonde determinant.)

6. Prove Young's inequality

$$\alpha^{1/p}\beta^{1/q} \leq \frac{1}{p}\alpha + \frac{1}{q}\beta, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \geq 0.$$

Show that equality occurs precisely if  $\alpha = \beta$ . (Hint: Take logarithms on both sides.)

- 7. Show that  $\ell^{\infty}(\mathbb{N})$  is a Banach space.
- 8. Is  $\ell^1(\mathbb{N})$  a closed subspace of  $\ell^{\infty}(\mathbb{N})$  (with respect to the  $\|.\|_{\infty}$  norm)? If not, what is its closure?
- 9. Consider  $\ell^1(\mathbb{N})$ . Show that  $||a|| := \sup_{k \in \mathbb{N}} |\sum_{j=1}^k a_j|$  is a norm. Is  $\ell^1(\mathbb{N})$  complete with this norm?
- 10. Show that  $\ell^{\infty}(\mathbb{N})$  is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)
- 11. Let X be a normed space. Show that the following conditions are equivalent.
  - (i) If ||x + y|| = ||x|| + ||y|| then  $y = \alpha x$  for some  $\alpha \ge 0$  or x = 0.
  - (ii) If ||x|| = ||y|| = 1 and  $x \neq y$  then  $||\lambda x + (1 \lambda)y|| < 1$  for all  $0 < \lambda < 1$ .
  - (iii) If ||x|| = ||y|| = 1 and  $x \neq y$  then  $\frac{1}{2}||x + y|| < 1$ .
  - (iv) The function  $x \mapsto ||x||^2$  is strictly convex.

A norm satisfying one of them is called strictly convex.

- 12. Consider X = C([-1,1]). Which of the following subsets are subspaces of X? Which of them are closed?
  - (i) monotone functions
  - (ii) even functions
  - (iii) polynomials
  - (iv) polynomials of degree at most k for some fixed  $k \in \mathbb{N}_0$
  - (v) continuous piecewise linear functions
  - (vi)  $C^1([-1,1])$
  - (vii)  $\{f \in C([-1,1]) \mid f(c) = f_0\}$  for some fixed  $c \in [-1,1]$  and  $f_0 \in \mathbb{R}$
- 13. Let I be a compact interval. Show that the set  $Y := \{ f \in C(I, \mathbb{R}) \mid f(x) > 0 \}$  is open in  $X := C(I, \mathbb{R})$ . Compute its closure.
- 14. Compute the closure of the following subsets of  $\ell^2(\mathbb{N})$ : (i)  $B_1 := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j \in \mathbb{N}} |a_j| \leq 1\}$ . (ii)  $B_{\infty} := \{a \in \ell^2(\mathbb{N}) \mid \sum_{j \in \mathbb{N}} |a_j| < \infty\}$ .

15. Show that, in a Hilbert space,

$$\sum_{1 \le j < k \le n} \|f_j - f_k\|^2 + \|\sum_{1 \le j \le n} f_j\|^2 = n \sum_{1 \le j \le n} \|f_j\|^2$$

for every  $n \in \mathbb{N}$ . The case n = 2 is the parallelogram law.

- 16. Show that the maximum norm on C[0,1] does not satisfy the parallelogram law
- 17. Let X and Y be real normed spaces. Show that an additive (i.e. T(x+y) = T(x) + T(y)) continuous map  $T: X \to Y$  is linear. What about complex spaces?
- 18. Suppose  $\mathfrak{Q}$  is a complex vector space. Let s(f,g) be a sesquilinear form on  $\mathfrak{Q}$  and q(f) := s(f,f) the associated quadratic form. Prove the **parallelogram law**

$$q(f+g) + q(f-g) = 2q(f) + 2q(g)$$

and the polarization identity

$$s(f,g) = \frac{1}{4} (q(f+g) - q(f-g) + i q(f-ig) - i q(f+ig)).$$

Show that s(f,g) is symmetric if and only if q(f) is real-valued.

Note, that if  $\mathfrak Q$  is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form  $s(f,g)=\frac{1}{4}(q(f+g)-q(f-g))$  will only hold if s(f,g) is symmetric.

19. Show that the integral defined in Example 1.15 satisfies

$$\int_{c}^{e} f(x)dx = \int_{c}^{d} f(x)dx + \int_{d}^{e} f(x)dx, \qquad \left| \int_{c}^{d} f(x)dx \right| \le \int_{c}^{d} |f(x)|dx.$$

How should |f| be defined here?

- 20. Is it possible to define  $\sin(f)$  for  $f \in L^1(I)$ ? If yes, how should this be done? What about  $\exp(f)$ ?
- 21. Show that in a metric space X a totally bounded set U is bounded.
- 22. Find a compact subset of  $\ell^{\infty}(\mathbb{N})$  which does not satisfy (ii) from Theorem 1.12.
- 23. Which of the following families are relatively compact in C[0,1]?
  - (i)  $F := \{ f \in C^1[0,1] \mid ||f||_{\infty} < 1 \}$
  - (ii)  $F := \{ f \in C^1[0,1] \mid ||f'||_{\infty} < 1 \}$
  - (iii)  $F := \{ f \in C^1[0,1] \mid ||f||_{\infty} \le 1, ||f'||_2 \le 1 \}$

24. Consider  $X = \mathbb{C}^n$  and let  $A \in \mathfrak{L}(X)$  be a matrix. Equip X with the norm (show that this is a norm)

$$||x||_{\infty} := \max_{1 \le j \le n} |x_j|$$

and compute the operator norm ||A|| with respect to this norm in terms of the matrix entries. Do the same with respect to the norm

$$||x||_1 := \sum_{1 \le j \le n} |x_j|.$$

- 25. Let X := C[0,1]. Investigate the operator  $A: X \to X$ ,  $f(x) \mapsto x f(x)$ . Show that this is a bounded linear operator and compute its norm. What is the closure of Ran(A)?
- 26. Show that the Fredholm integral operator

$$(Kf)(x) := \int_0^1 K(x, y) f(y) dy,$$

where  $K(x,y) \in C([0,1] \times [0,1])$ , defined on  $\mathfrak{D}(K) := C[0,1]$ , is a bounded operator both in X := C[0,1] (max norm) and  $X := L^2(0,1)$ . Show that the norm in the X = C[0,1] case is given by

$$\|K\| = \max_{x \in [0,1]} \int_0^1 |K(x,y)| dy.$$

27. Let  $A \in \mathfrak{L}(X)$  be a bijection. Show

$$||A^{-1}||^{-1} = \inf_{x \in X, ||x|| = 1} ||Af||.$$

- 28. Prove Lemma 1.19. (Hint: Hölder's inequality in  $\mathbb{C}^n$  and note that equality is attained.)
- 29. Let X, Y be normed spaces and suppose  $A \in \mathfrak{L}(X, Y)$ . Show that  $\operatorname{Ker}(A)$  is closed. Suppose  $M \subseteq \operatorname{Ker}(A)$  is a closed subspace. Show that the induced map  $\tilde{A}: X/M \to Y, [x] \mapsto Ax$  is a well-defined operator satisfying  $\|\tilde{A}\| = \|A\|$  and  $\operatorname{Ker}(\tilde{A}) = \operatorname{Ker}(A)/M$ ,  $\operatorname{Ran}(\tilde{A}) = \operatorname{Ran}(A)$ . In particular,  $\tilde{A}$  is injective for  $M = \operatorname{Ker}(A)$ .
- 30. Given some vectors  $f_1, \ldots, f_n$  we define their **Gram determinant** as

$$\Gamma(f_1,\ldots,f_n) := \det \left( \langle f_j, f_k \rangle \right)_{1 \leq j,k \leq n}$$

Show that the Gram determinant is nonzero if and only if the vectors are linearly independent. Moreover, show that in this case

$$\operatorname{dist}(g,\operatorname{span}\{f_1,\ldots,f_n\})^2 = \frac{\Gamma(f_1,\ldots,f_n,g)}{\Gamma(f_1,\ldots,f_n)}$$

and

$$\Gamma(f_1,\ldots,f_n) \leq \prod_{j=1}^n ||f_j||^2.$$

with equality if the vectors are orthogonal. (Hint: First establish  $\Gamma(f_1, \ldots, f_j + \alpha f_k, \ldots, f_n) = \Gamma(f_1, \ldots, f_n)$  for  $j \neq k$  and use it to investigate how  $\Gamma$  changes when you apply the Gram–Schmidt procedure?)

- 31. Let  $\{u_j\}$  be some orthonormal basis. Show that a bounded linear operator A is uniquely determined by its matrix elements  $A_{jk} := \langle u_j, Au_k \rangle$  with respect to this basis.
- 32. Let I = (a, b) be some interval and consider the scalar product

$$\langle f, g \rangle := \int_a^b f(x)^* g(x) w(x) dx$$

associated with some positive weight function w(x). Let  $P_j(x) = x^j + \beta_j x^{j-1} + \gamma_j x^{j-2} + \ldots$  be the corresponding monic orthogonal polynomials obtained by applying the Gram–Schmidt procedure (without normalization) to the monomials:

$$\int_{a}^{b} P_{i}(x)P_{j}(x)w(x)dx = \begin{cases} \alpha_{j}^{2}, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $\bar{P}_j(x) := \alpha_j^{-1} P_j(x)$  be the corresponding orthonormal polynomials and show that they satisfy the three term recurrence relation

$$a_j \bar{P}_{j+1}(x) + b_j \bar{P}_j(x) + a_{j-1} \bar{P}_{j-1}(x) = x \bar{P}_j(x),$$

or equivalently

$$P_{j+1}(x) = (x - b_j)P_j(x) - a_{j-1}^2 P_{j-1}(x),$$

where

$$a_j := \int_a^b x \bar{P}_{j+1}(x) \bar{P}_j(x) w(x) dx, \qquad b_j := \int_a^b x \bar{P}_j(x)^2 w(x) dx.$$

Here we set  $P_{-1}(x) = \bar{P}_{-1}(x) \equiv 0$  for notational convenience (in particular we also have  $\beta_0 = \gamma_0 = \gamma_1 = 0$ ).

- 33. Consider  $M := \{ f \in L^2(0,1) | \int_0^1 f(x) dx = 0 \} \subset L^2(0,1)$ . Compute  $M^{\perp}$ .
- 34. Consider the subspace  $M_e:=\{f\in C[-1,1]|f(x)=f(-x)\}$  of all even continuous functions in  $L^2(-1,1)$ . Is  $M_e$  closed? What is  $P_{\overline{M_e}}$  and  $M_e^{\perp}$ ? Compute  $\operatorname{dist}(\exp(x),M_e)$ .
- 35. Show that  $\ell(a) = \sum_{j=1}^{\infty} \frac{a_j + a_{j+2}}{2^j}$  defines a bounded linear functional on  $X := \ell^2(\mathbb{N})$ . Compute its norm.
- 36. Let  $\mathfrak{H}_1$ ,  $\mathfrak{H}_2$  be Hilbert spaces and let  $u \in \mathfrak{H}_1$ ,  $v \in \mathfrak{H}_2$ . Show that the operator

$$Af := \langle u, f \rangle v$$

is bounded and compute its norm. Compute the adjoint of A.

- 37. Let  $\mathfrak{H}$  be a Hilbert space and  $\{f_j\}_{j=1}^n\subset \mathfrak{H}$  some vectors. Show that  $A:\mathbb{C}^n\to \mathfrak{H},\ \alpha\mapsto \sum_{j=1}^n\alpha_jf_j$  is bounded and compute  $A^*$ .
- 38. Let  $\mathfrak{H}_1$ ,  $\mathfrak{H}_2$  be Hilbert spaces and suppose  $A \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$  has a bounded inverse  $A^{-1} \in \mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ . Show  $(A^{-1})^* = (A^*)^{-1}$ .

39. Let  $\mathfrak H$  be a Hilbert space and  $s:\mathfrak H^2\to\mathbb C$  a bounded coercive sesquilinear form. Show that the solution of

$$s(h, f) = \langle h, g \rangle, \quad \forall h \in \mathfrak{H}.$$
 (\*)

is also the unique minimizer of

$$h \mapsto \operatorname{Re}\left(\frac{1}{2}s(h,h) - \langle h, g \rangle\right)$$

if s is nonnegative with  $s(w, w) \ge \varepsilon ||w||^2$  for all  $w \in \mathfrak{H}$ .

40. Consider  $A \in \mathfrak{L}(\mathfrak{H})$  and denote the set of fixed points by  $\operatorname{Fix}(A) := \operatorname{Ker}(A-\mathbb{I}) = \{f \in \mathfrak{H} | Af = f\}$ . Show  $\operatorname{Fix}(A) = \operatorname{Fix}(A^*)$  provided  $||A|| \leq 1$ . Use this to establish the mean ergodic theorem for  $A \in \mathfrak{L}(\mathfrak{H})$  with  $||A|| \leq 1$ :

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} A^{j} f = P_{\text{Fix}} f,$$

where  $P_{\text{Fix}}$  denotes the orthogonal projection onto Fix(A). (Hint: To see Fix(A) = Fix(A\*) think about when you have equality in the Cauchy–Schwarz inequality. To compute the limit consider the cases  $f \in \text{Fix}(A)$  and  $f \in \text{Ran}(A - \mathbb{I})$ . Also  $\text{Ker}(A^*) = \text{Ran}(A)^{\perp}$  might be useful.)

- 41. Is the left shift  $(a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots)$  compact in  $\ell^2(\mathbb{N})$ ?
- 42. Is the multiplication operator  $M_t: C^k[0,1] \to C[0,1]$  with  $M_t f(t) = t f(t)$  compact for k = 0, 1? (Hint: Problem 23 and Example 3.2.)
- 43. Show that the adjoint of the Fredholm integral operator K on  $L^2(a,b)$  from Lemma 3.4 is the integral operator with kernel  $K(y,x)^*$ :

$$(K^*f)(x) = \int_a^b K(y,x)^*f(y)dy.$$

(Hint: Fubini.)

44. Find the eigenvalues and eigenfunctions of the integral operator  $K \in \mathfrak{L}(L^2(0,1))$  given by

$$(Kf)(x) := \int_0^1 u(x)v(y)f(y)dy,$$

where  $u, v \in C([0, 1])$  are some given continuous functions.

45. Find the eigenvalues and eigenfunctions of the integral operator  $K \in \mathfrak{L}(L^2(0,1))$  given by

$$(Kf)(x) := 2 \int_0^1 (2xy - x - y + 1)f(y)dy.$$

46. Show that the inverse  $(A-z)^{-1}$  (provided it exists and is densely defined) of a symmetric operator A is again symmetric for  $z \in \mathbb{R}$ .

Show that a densely defined bounded operator is symmetric if and only if its extension, according to Theorem 1.16, is.

(Hint: 
$$g \in \mathfrak{D}((A-z)^{-1})$$
 if and only if  $g = (A-z)f$  for some  $f \in \mathfrak{D}(A)$ .)

- 47. The finite union of nowhere dense sets is nowhere dense and the finite intersections of sets with dense interior has dense interior.
- 48. Let X be a topological space. Prove that the meager sets form a  $\sigma$ -ideal. Moreover, prove that every superset of a fat set is fat.
- 49. Consider X:=C[-1,1]. Show that  $M:=\{x\in X|x(-t)=x(t)\}$  is meager.
- 50. Consider  $f = \chi_{\mathbb{Q}}$  on [0,1]. Show that there cannot be a sequence of continuous functions  $f_n \in C[0,1]$  which converges to f pointwise. (Hint: Apply Baire's theorem with  $F_n := \{t \in [0,1] | |f_n(t) f_m(t)| \leq \frac{1}{3} \forall m \geq n\}$ .)
- 51. Let X be a Banach space and Y, Z normed spaces. Show that a bilinear map  $B: X \times Y \to Z$  is bounded,  $\|B(x,y)\| \leq C\|x\|\|y\|$ , if and only if it is separately continuous with respect to both arguments. (Hint: Uniform boundedness principle.)
- 52. Consider a Schauder basis as in (1.31). Show that the coordinate functionals  $\alpha_n$  are continuous. (Hint: Denote the set of all possible sequences of Schauder coefficients  $\alpha = (\alpha_n)_{n=1}^N$  by  $\mathcal{A}$  and equip it with the norm  $\|\alpha\| := \sup_n \|\sum_{k=1}^n \alpha_k u_k\|$ . By construction the operator  $A: \mathcal{A} \to X$ ,  $\alpha \mapsto \sum_k \alpha_k u_k$  has norm one. Now show that  $\mathcal{A}$  is complete and apply the inverse mapping theorem.)
- 53. Show that the operator

$$\mathfrak{D}(A) := \{ a \in \ell^p(\mathbb{N}) | j \, a_j \in \ell^p(\mathbb{N}) \}, \qquad (Aa)_j := j \, a_j,$$

is a closed operator in  $X := \ell^p(\mathbb{N})$ .

- 54. Suppose  $A: X \to Y$  and  $B: Y \to Z$  are linear operators between Banach spaces X, Y, Z. Show that A is bounded if BA and B are bounded with B injective.
- 55. Let  $X := \mathbb{C}^3$  equipped with the norm  $|(x,y,z)|_1 := |x| + |y| + |z|$  and  $Y := \{(x,y,z) \in X | x+y=0, z=0\}$ . Find at least two extensions of  $\ell(x,y,z) := x$  from Y to X which preserve the norm. What if we take  $Y := \{(x,y,z) \in X | x+y=0\}$ ?
- 56. Consider X := C[0,1] and let  $f_0(x) := 1 2x$ . Find at least two linear functionals with norm one such that  $\ell(f_0) = 1$ .
- 57. Show that the extension from Corollary 4.13 is unique if  $X^*$  is strictly convex. (Hint: Problem 11.)
- 58. Let X be some normed space. Show that

$$||x|| = \sup_{\ell \in V, ||\ell|| = 1} |\ell(x)|,$$

where  $V \subset X^*$  is some dense subspace. Show that equality is attained if  $V = X^*$ .

59. Show that every  $l \in \ell^p(\mathbb{N})^*, 1 \leq p < \infty$ , can be written as

$$l(a) = l_b(a) := \sum_{j \in \mathbb{N}} b_j a_j$$

with some unique  $b \in \ell^q(\mathbb{N})$ . (Hint: To see  $b \in \ell^q(\mathbb{N})$  consider  $a^N$  defined such that  $a_j^n := |b_j|^q/b_j$  for  $j \le n$  with  $b_j \ne 0$  and  $b_j^n := 0$  else. Now look at  $|l(a^n)| \le |l| ||a^n||_p$ .)

60. Let X be a Banach space. Show that a subset  $U\subseteq X$  is bounded if and only if  $\ell(U)\subseteq\mathbb{C}$  is bounded for every  $\ell\in X^*$ . (Hint: Uniform boundedness principle.)