1 Fermat Numbers

Following P.Giordano by Fermat number we understand number of the form

$$x = r + \sum_{i=1}^{n} \alpha_i \tau^{a_i} \tag{1}$$

where $r, \alpha_i, a_i, i = 1, 2, ..., n$ are standard numbers (given e.g. by their decimal representations)such that $a_1 < a_2 < ... < a_n \leq 1$ and τ is nilpotent number,i.e. $\tau^a = 0, a > 1$.

2 Fokker-Planck equations

Let x be a state variable for some system (with one degree of freedom). By a *process* we understand the map $t \mapsto x(t)$. The process $t \mapsto x(t)$ is said to be *forward diffusion* if it obeys the first Îto equation

$$x(t + \tau) = x(t) + b(t, x(t))\tau + \lambda\tau^{1/2}, \tag{2}$$

where $b = b(t, x)$ is called the *forward drift velocity*, λ - *diffusion coefficient* of the process and τ is nilpotent variable such that $\tau^{3/2} = 0$.

The *observables* of the system are the smooth functions $f(t, x)$. Since $(b\tau + \lambda\tau^{1/2})^2 = \lambda^2\tau$ and $(b\tau + \lambda\tau^{1/2})^3 = 0$ we have

$$\begin{aligned}\delta f := f(t + \tau, x(t + \tau) - f(t, x(t)) = [\partial_t f(t, x(t)) + b(t, x(t))\partial_x f(t, x(t)) \\ +\lambda^2/2\partial_{xx} f(t, x(t))]\tau + \lambda\partial_x f(t, x(t))\tau^{1/2}.\end{aligned}$$

The linear part of this expression determines the forward operator D of full time derivative given by

$$D = \partial_t + b(t, x)\partial_x + (\lambda^2/2)\partial_{xx}. \tag{3}$$

The *statistical state* of the system is determined by the probability density (*pd*) $\rho(t, x)$ and the mean value of some observable non-depending on time $f(x)$ in such state is given by

$$\langle f \rangle = \int f(x)\rho(t, x)dx. \tag{4}$$

Assuming that taking the mean value and full time derivative are independent (and therefore commute)we have

$$D\langle f\rangle = \langle Df\rangle \tag{5}$$

.Integrating by parts one can easily obtain the forward Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (b\rho)}{\partial x} - \frac{\lambda^2}{2}\frac{\partial^2 \rho}{\partial x^2} = 0. \tag{6}$$

Analogically the *backward diffusion* is defined by the second Îto equation

$$x(t) = x(t-\tau) + b_*(t, x(t))\tau + \lambda\tau^{1/2}, \tag{7}$$

where b_* is the backward drift velocity. The change of the observable f is equal to

$$\delta_* f := f(t, x(t)) - f(t-\tau, x(t-\tau)). \tag{8}$$

Again the linear part of this expression determines the backward operator of full time derivative

$$D_* = \partial_t + b_*(t, x)\partial_x - (\lambda^2/2)\partial_{xx} \tag{9}$$

which leads to the backward Fokker-Planck equation

$$\frac{\partial \rho_*}{\partial t} + \frac{\partial (b_*\rho_*)}{\partial x} + \frac{\lambda^2}{2}\frac{\partial^2 \rho_*}{\partial x^2} = 0. \tag{10}$$

3 Quantum Systems

In general the solutions of the equations (6) and (10) are different but if we want to consider a particle with coordinate $x(t)$ at time t then we say that this is a *quantum particle* if $\lambda^2 = \hbar/m$,m being the mass of the particle, and that the particle is in *quantum equilibrium* when $\rho = \rho_*$. Summing the two equations we come to the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{11}$$

where $v = (b+b_*)/2$ is the *mean velocity.* The difference of the two equations yields the so called *osmotic velocity*

$$u := (b - b_*)/2 = \frac{\hbar}{2m}\frac{\rho_x}{\rho}, \tag{12}$$

where $\rho_x = \partial\rho/\partial x$.

This is not all however. While the probability density $\rho(t,x)$ reflects our ignorance of the true position, the velocity at some point x is intrinsically not well defined - due to the vacuum fluctuations it is contained with predominating probability into interval $[b, b_*]$, when $b < b_*$, or $[b_*, b]$, when $b_* < b$. (Let us note that according to the Îto equations b_* can be considered as incoming velocity into x and b as outgoing velocity from x.) This implies that we can consider the momentum of the particle as a random variable with mean value

$$\langle p|t,x\rangle = mv = m(b+b_*)/2 = S_x := \frac{\partial S}{\partial x}, \tag{13}$$

where we have introduced the momentum potential

$$S(t,x) = \int_{x_0}^{x} mv(t,y)dy, \tag{14}$$

x_0 being arbitrary standard number.(Let us note that S is determined up to additive constant.) and uncertainty

$$\Delta p(t,x) = m|u| = m|b - b_*|/2 = \frac{\hbar}{2}\frac{|\rho_x|}{\rho}. \tag{15}$$

In other words there should exists a conditional *pd* $w(p|t,x)$ such that the *local* mean value of some observable $f(x,p)$ at space -time point (t,x) is given by

$$\langle f|t,x\rangle = \int f(x,p)w(p|t,x)dp. \tag{16}$$

In particular the local dispersion is determined by

$$\Delta p(t,x)^2 = \int (p - \langle p|t,x\rangle)^2 w(p|t,x)dp = \langle p^2|t,x\rangle - \langle p|t,x\rangle^2. \tag{17}$$

4 Schrödinger equation

It is natural to consider $w(t, x, p) = w(p|t, x)\rho(t, x)$ as the joint probability density of the coordinate and momentum of the particle with *pd* of coordinate $\rho(t, x)$ and local *pd* of momentum at (t, x) equal to $w(p|t, x)$. Then the mean energy of the particle is

$$\begin{aligned}\langle H \rangle &= \int \{ \frac{p^2}{2m} + V(t, x) \} w(t, x, p) dp dx \\ &= \int \{ \frac{\langle p^2 | t, x \rangle}{2m} + V(t, x) \} \rho(t, x) dx. \end{aligned} \tag{18}$$

Using Eqs.(13),(15),(17) we come to the following expression

$$\langle H \rangle = \int H dx, \tag{19}$$

where

$$H = (\frac{S_x^2}{2m} + V(t, x)) \rho + \frac{\hbar^2}{8m} \frac{\rho_x^2}{\rho} \tag{20}$$

is the local density of energy at (t, x). We consider the pair (ρ, S) as a statistical state of the particle describing its time evolution. Making change of variables

$$\psi = \sqrt{\rho} e^{\frac{i}{\hbar} S}, \psi^* = \sqrt{\rho} e^{-\frac{i}{\hbar} S} \tag{21}$$

we obtain

$$H = \frac{\hbar^2}{2m} \psi_x^* \psi_x + V \psi^* \psi. \tag{22}$$

The Hamilton equations of the system are

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\delta \langle H \rangle}{\delta \psi^*} := \frac{\partial H}{\partial \psi^*} - \partial_x \frac{\partial H}{\partial \psi_x^*}, \tag{23}$$

and the complex conjugate. Equation (23)is just the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V \psi. \tag{24}$$

5 Conclusion

Every successful derivation of Schrödinger equation allows to make some conclusions about the motion of a quantum particle. Given external conditions (potential $V(t,x)$) the quantum state of the particle is given by the pair of drift velocities (b, b_*) (or equivalently the *pd* of coordinate ρ and the momentum potential S). First of them gives the outgoing velocity $b(t,x)$ from x at time t, and the second gives the incoming velocity $b_*(t,x)$ to x at the *same* time t. Since the two velocities are (in general) different this means $infinite$ force acting on the particle. This leads in effect that the velocity (as well the corresponding momentum) becomes a random quantity with local *pd* (for momentum)$w(p|t,x)$. The product $W(t,x,p) = w(p|t,x)\rho(t,x)$ can be considered as joint *pd* and the mean value of the energy with respect this *pd* is functional depending on the field variables (ρ, S). Let us note that $w(p|t,x)$ is not known - it can be every *pd* with the same mean value $\langle p|t,x\rangle$ and same dispersion $\Delta p(t,x)^2$ of momentum. Any two such *pd* are therefore equivalent with respect quantum state under consideration. In other words in a given quantum state not only momentum at some point is not specified but a whole class of possible local *pds* of momentum at that point are possible. Quantum state specifies only the local mean value and the local dispersion (uncertainty) at the point. As a result, choosing e.g. the Gaussian distribution one can suggest the following expression for the mean value of some observable $f(x,p)$

$$\langle f\rangle = \int dx \int dp f(x,p) \frac{\rho(t,x)}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(p-S_x)^2}{2\sigma^2}\}, \tag{25}$$

where $\sigma = \hbar|\rho_x/2\rho$.