DIRAC DELTA AS A GENERALIZED HOLOMORPHIC FUNCTION

SEKAR NUGRAHENI AND PAOLO GIORDANO

Abstract. The definition of a non-trivial space of generalized functions of a complex variable allowing to consider derivatives of continuous functions is a non-obvious task, e.g. because of Morera theorem, because distributional Cauchy-Riemann equations implies holomorphicity and of course because including Dirac delta seems incompatible with the identity theorem. Surprisingly, these results can be achieved if we consider a suitable non-Archimedean extension of the complex field, i.e. a ring where infinitesimal and infinite numbers return to be available. In this first paper, we set the definition of generalized holomorphic function and prove the extension of several classical theorems, such as Cauchy-Riemann equations, Goursat, Looman-Menchoff and Montel theorems, generalized complex differentiability implies smoothness, embedding of distributions, closure with respect to composition and hence non-linear operations on these generalized functions. The theory hence addresses several limitations of Colombeau theory of generalized holomorphic functions. The final aim of this series of papers is to prove the Cauchy-Kowalevski theorem including also distributional PDE or singular boundary conditions and nonlinear operations.

1. Introduction

The extension of at least some Schwartz distributions from the real field to the complex plane has been (sometimes informally) attempted by several authors, because of its uses in quantum physics, [14, 3, 8, 40], in the study of random processes, [52], for physical wavelets, [33], and in general relativity, [20]. However, these efforts sometimes present mathematical drawbacks, as already pointed out, e.g., by [58].

From the purely mathematical point of view, the definition of a non-trivial space of generalized functions of a complex variable that allows one to consider derivatives of continuous functions is a non-obvious task, because its solution must sidestep several impossibility theorems. For example, if we want that these generalized functions embed ordinary continuous maps defined on a domain $D \subseteq \mathbb{C}$ and, at the same time, satisfy the Cauchy integral theorem, then these continuous functions would also be path-independent and, from Morera's theorem, they would actually be holomorphic functions, see e.g. [59]. Likewise, if we want that these generalized functions satisfy the Cauchy-Riemann equations (CRE), even with respect to distributional derivatives, then necessarily the embedded continuous ones will actually be, once again, ordinary holomorphic functions, see [30]. Paraphrasing the

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language of [32] (see also [35]), we could say that the simplest solution of the problem to have derivatives of continuous functions of a complex variable satisfying the Cauchy theorem or the distributional Cauchy-Riemann equations is the sheaf of holomorphic functions itself, and a larger space seems not possible. Once again: it seems impossible to extend the Dirac delta distribution from $\mathbb{R}$ to $\mathbb{C}$ and, at the same time, to prove a general identity theorem for these *generalized holomorphic functions* (GHF). The problem is also related to nonlinear operations on spaces of distributions: in fact, one of the peculiar property of holomorphic functions is their expandability in Taylor series, and hence the converge of the Cauchy product of these series seems a natural consequence, if we think to generalize this expandability to GHF; however, nonlinear operations, even involving only Heaviside function and Dirac delta (assuming that this expandability can also be proved at least for these GHF), collide with well-known impossibility theorems about nonlinear operations on distributions, see e.g. [11, 31] and references therein.

It is rather surprising that all these impossibility results can be avoided if we consider *non-Archimedean extensions* $^\rho\widetilde{\mathbb{C}} \supseteq \mathbb{C}$ and $^\rho\widetilde{\mathbb{R}} \supseteq \mathbb{R}$ of the complex and real fields, i.e. rings $^\rho\widetilde{\mathbb{C}}$ and $^\rho\widetilde{\mathbb{R}}$ where infinitesimal and infinite numbers return to be available. In fact, for our GHF (defined on an open subset of $^\rho\widetilde{\mathbb{C}}$ and valued in this same ring) we can prove the Cauchy integral theorem, the CRE, Morera's theorem, expandability in Taylor hyperseries (i.e. series extended over natural infinite numbers, see Def. 6 below), closure with respect to composition, and hence nonlinear operations, and the generalization of several other results of ordinary holomorphic functions. On the other hand, in essentially every non-Archimedean theory, see e.g. [18] and references therein, the topology of interest always see the set of all the infinitesimals as a clopen set, and hence the corresponding non-Archimedean ring is disconnected (and all its intervals as well). This property allows one to prove only a weak form of the identity theorem for GHF and hence make it possible to extend the Dirac delta from $^\rho\widetilde{\mathbb{R}}^2$ to $^\rho\widetilde{\mathbb{C}}$. Any complex primitive of this extended Dirac delta can also be regarded as an extension of the Heaviside function and, using the closure of GHF with respect to composition and nonlinear operations, among GHF we can include examples such as, e.g., $\delta^a \cdot H^b \circ \delta^c \cdot H^d$ for any a, b, c, $d \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Moreover, in our solution, every GHF is a set-theoretical map defined on a subset of $^\rho\widetilde{\mathbb{C}}$ and valued in the same ring, and this solves the informal use of generalized functions as one can find in physics and engineer, [9]. Using a classical smoothing method, every continuous map f defined on a domain $\Omega \subseteq \mathbb{C}$ and satisfying the distributional CRE can be embedded as a GHF, but if we assume to lose all the infinitesimal information, formally if the regularized values are ordinary complex numbers for all $z \in \Omega$, then the previous result of [30] about distributional CRE implies that necessarily f is a standard holomorphic function, see Thm. 39. Informally, we can hence state that without a suitable language of infinitesimal and infinite numbers, a theory of GHF with meaningful examples and results is impossible, but in an appropriate non-Archimedean setting, this theory is possible and full of rich examples and results.

Schema of this series of papers. This is the first of a series of three papers devoted to GHF. In the present first article, we introduce the rings $^\rho\widetilde{\mathbb{R}}$ and $^\rho\widetilde{\mathbb{C}}$, the definition of GHF, and prove its first properties, we prove that generalized

${}^{\rho}\widetilde{\mathbb{C}}$-differentiability implies smoothness, the CRE, versions of Goursat, Looman-Menchoff, and Montel theorems, and close with a list of meaningful examples, including all distributions, hence the aforementioned δ and its relation with the one dimensional Dirac delta. We also summarize why our approach solves several technical problems of Colombeau setting: intrinsic embedding of distributions preserving all δ derivatives, closure with respect to composition, possibility to have generalized functions defined in infinitesimal or infinite sets (hence solutions of differential equations which are impossible for Colombeau generalized functions (CGF)), and better Fourier transform applicable also to non-tempered distributions.

In the subsequent paper [47], which is already almost finished, we define generalized paths (including distributional ones) and path integral. We prove Morera's theorem, Cauchy-Goursat's theorem, Cauchy's integral formula, Cauchy's inequality, the mean value property, a generalized version of Jordan curve theorem, generalized solutions of Dirichlet boundary value problem, a version of the Riemann mapping theorem for biholomorphic generalized functions, and the Jordan-Schoenflies theorem for generalized functions as homotopies. In [48], we consider the expansion of GHF using hyper power series, i.e. power series where the summation is extended over infinite natural numbers of the ring ${}^{\rho}\widetilde{\mathbb{R}}$. We also include Goursat's theorem, Liouville's theorem, Laurent hyperseries, a suitable version of the identity theorem, and, hopefully, a simple proof of the Paley-Wiener-Schwartz theorem.

A full theory of GHF would open the possibility to generalize the Cauchy-Kowalevski theorem including also singular (e.g. using suitable Schwartz distributions) PDE or singular boundary conditions, and nonlinear (generalized holomorphic) operations and their compositions. Since functions in Sobolev spaces can be approximated using Taylor series, see [7, 10, 53], and GHF solutions of PDE correspond to their Taylor hyperseries, we also hope to find a way to understand when a GHF solution is actually the embedding of a function in a Sobolev space.

This first paper is self-contained in the sense that it contains all the statements required for the proofs we are going to present. If proofs of preliminaries are omitted, we clearly give references to where they can be found. Also the other articles in this series will be written with the same style. Therefore, to understand this series of papers, only a basic knowledge of distribution theory is needed.

1.1. **The Ring of Robinson-Colombeau Numbers.** A sufficiently general rigorous mathematical theory of generalized functions of a complex variable has already been developed within Colombeau theory of generalized functions (see e.g. [12, 2, 49, 57, 1, 51] and references therein). Since the beginning, in that framework it was natural to define a holomorphic generalized function using CRE, see e.g. [12, 57, 2, 1]. Indeed, the notion of partial derivatives (of any order) of a CGF was already clear and fully compatible with distributional derivative; therefore, using pointwise evaluation of CGF, one can define a complex CGF of a complex variable starting from its real and imaginary parts and asking that they satisfy the CRE. Even if an important results such as the CRE and the existence of *all* the derivatives are directly taken in the definition, this initial approach probably appeared more natural than considering the limit of the incremental ratio in a *ring* of scalars (see Sec. 1.1 below) with a topology usually managed through a non-Archimedean ultrametric. On the other hand, already in [2], thanks to the density of invertible elements in the ring of scalars, a more general notion of differentiability in the Colombeau setting, both for the complex and the real case, is considered using a classical Newton quotient. This allows [2] to show that all CGF are also differentiable in this more classical way, but missed to extend the important classical result that from a suitable notion of *generalized* complex differentiability the existence of all *generalized* derivatives of greater order follows, see also [57].

In this article we want to define a GHF by using some kind of limit of the incremental ratio, like for ordinary holomorphic functions, without already starting from the CRE and also showing that first order generalized complex differentiability implies the existence of all greater derivatives. After our proof of the CRE for GHF, see Thm. 34, we will be able to show that our approach to GHF is more general than the classical Colombeau definition, and this allows us to include examples such as the Dirac delta and the closure with respect to composition.

In this section, we introduce the non-Archimedean ring of (real and complex) scalars. For more details and proofs about the basic notions introduced here, the reader can refer e.g. to [28, 11, 27]. As we will see better below, in order to accomplish our definition of GHF, we introduce Colombeau generalized numbers by considering an arbitrary asymptotic scale instead of the usual net (ε) used in Colombeau theory (see also [43] for a more general notion of scale, and [24] and references therein for a comparison).

Definition 1. Let $\rho = (\rho_\varepsilon) : (0, 1] \longrightarrow (0, 1] =: I$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a *gauge* and all the asymptotic relation will be for $\varepsilon \to 0^+$), then

(i) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ is *ρ-moderate*, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if $\exists N \in \mathbb{N}: \ x_\varepsilon = \mathcal{O}(\rho_\varepsilon^{-N})$.

(ii) Let $(x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$, and we read it saying that $(x_\varepsilon - y_\varepsilon)$ is *ρ-negligible*, if $\forall N \in \mathbb{N}: \ |x_\varepsilon - y_\varepsilon| = \mathcal{O}(\rho_\varepsilon^N)$. This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$$ {}^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho, $$

which we call *Robinson-Colombeau ring of generalized numbers*. The corresponding equivalence classes are simply denoted by $x = [x_\varepsilon]$ or $x = [x_\varepsilon]_\rho$ in case we have to underscore the dependence from the gauge ρ. In particular, $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

(iii) If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0\varepsilon : \ \mathcal{P}(\varepsilon)$ to denote $\exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \ \mathcal{P}(\varepsilon)$. We can read $\forall^0\varepsilon$ as: *"for ε small"*.

(iv) Let $x, y \in {}^\rho\widetilde{\mathbb{R}}$. We write $x \le y$ if for all representative $[x_\varepsilon] = x$, there exists $[y_\varepsilon] = y$ such that $\forall^0\varepsilon : \ x_\varepsilon \le y_\varepsilon$.

(v) We denote by ${}^\rho\widetilde{\mathbb{R}}_{>0}$ the set of positive invertible generalized numbers. In general, we write $x < y$ to say that $x \le y$ and $x - y$ is invertible.

(vi) A generalized complex number can be written as $z = x + iy \in {}^\rho\widetilde{\mathbb{C}}$, where $x, y \in {}^\rho\widetilde{\mathbb{R}}$ and i is the imaginary unit. Note that $\mathbb{R} \subseteq {}^\rho\widetilde{\mathbb{R}}$ and $\mathbb{C} \subseteq {}^\rho\widetilde{\mathbb{C}}$ are embedded using constant nets.

Like in any non-Archimedean setting, also in ${}^\rho\widetilde{\mathbb{R}}$ we can define infinitesimal and infinite numbers in the usual customary way (see Def. 2 below). Differently to other frameworks, in ${}^\rho\widetilde{\mathbb{R}}$ every representative of an infinitesimal is also an infinitesimal function as $\varepsilon \to 0$. A similar property holds for infinite numbers. For example, this result does not hold in nonstandard analysis, see e.g. [13], where we necessarily need a P-point ultrafilter to prove that every representative of an infinitesimal number converges to 0 along a subsequence, but there are always representatives that do not converge to 0.

Definition 2. Let $z \in {}^{\rho}\widetilde{\mathbb{C}}$ be a generalized number. Then

- (i) z is *infinitesimal* if for all $r \in \mathbb{R}_{>0}$, we have $|z| \le r$. If $z = [z_\varepsilon]$, this is equivalent to $\lim_{\varepsilon\to 0^+} z_\varepsilon = 0$. We write $z \approx w$ if $z - w$ is infinitesimal, and $D_\infty := \left\{h \in {}^{\rho}\widetilde{\mathbb{C}} \mid h \approx 0\right\}$ for the set of all infinitesimals in ${}^{\rho}\widetilde{\mathbb{C}}$.
- (ii) z is *infinite* if for all $r \in \mathbb{R}_{>0}$, we have $|z| \ge r$. If $z = [z_\varepsilon]$, this is equivalent to $\lim_{\varepsilon\to 0^+} |z_\varepsilon| = +\infty$.
- (iii) z is *finite* if for some $r \in \mathbb{R}_{>0}$, we have $|z| \le r$.
- (iv) z is *near-standard* if for some representative (and hence for all) $[z_\varepsilon] = z$, we have $\exists \lim_{\varepsilon\to 0^+} z_\varepsilon =: z^\circ \in \mathbb{C}$, which is called the *standard part* of z. If $U \subseteq {}^{\rho}\widetilde{\mathbb{C}}$, the set of near-standard points in U is $U^\bullet := \{z \in U \mid \exists z^\circ \in U\}$.

For example, we have that $\mathrm{d}\rho^n \in {}^{\rho}\widetilde{\mathbb{R}}$, $n \in \mathbb{N}_{>0}$, is an invertible infinitesimal, whose reciprocal is $\mathrm{d}\rho^{-n} = [\rho_\varepsilon^{-n}]$, which is necessarily a positive infinite number. Of course, in the ring ${}^{\rho}\widetilde{\mathbb{R}}$ there exist generalized numbers which are not in any of the three classes of Def. 2, like e.g. $x_\varepsilon = \frac{1}{\varepsilon}\sin\left(\frac{1}{\varepsilon}\right)$. It is possible to prove that ${}^{\rho}\widetilde{\mathbb{R}}$ is the simplest (co-universal) quotient ring having a prescribed element $\mathrm{d}\rho$ and where every representative of $0 = [z_\varepsilon]$ is an infinitesimal net: $\lim_{\varepsilon\to 0^+} z_\varepsilon = 0$, see [35].

On ${}^{\rho}\widetilde{\mathbb{R}}^n$, we consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^{\rho}\widetilde{\mathbb{R}}$. Even if this generalized norm takes value in ${}^{\rho}\widetilde{\mathbb{R}}$, it shares essential properties with classical norms, like the triangle inequality and absolute homogeneity. It is therefore natural to consider on ${}^{\rho}\widetilde{\mathbb{R}}^n$ the topology generated by balls $B_r(x) := \left\{y \in {}^{\rho}\widetilde{\mathbb{R}} \mid |x-y| < r\right\}$, $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, which is called *sharp topology*. Note that $\mathrm{d}\rho \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, and therefore we can also have balls $B_{\mathrm{d}\rho^n}(x)$ of infinitesimal radius. Similarly, the absolute value (modulus) of a generalized complex number $z = [z_\varepsilon] = [x_\varepsilon + iy_\varepsilon] = x + iy \in {}^{\rho}\widetilde{\mathbb{C}}$ is defined by $|z| := |[z_\varepsilon]| = \left[\left(x_\varepsilon^2 + y_\varepsilon^2\right)^{\frac{1}{2}}\right]$ and takes value in ${}^{\rho}\widetilde{\mathbb{R}}$. Therefore, the topology of the set of generalized complex numbers is the same as the sharp topology of ${}^{\rho}\widetilde{\mathbb{R}}^2$.

The following result is useful in dealing with positive and invertible generalized numbers, see [31, 28].

Lemma 3. *Let $x \in {}^{\rho}\widetilde{\mathbb{R}}$. Then the following are equivalent:*

- *(i)* *x is invertible and $x \ge 0$, i.e. $x > 0$.*
- *(ii)* *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x we have $\forall^0\varepsilon : \ x_\varepsilon > 0$.*
- *(iii)* *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x we have $\exists m \in \mathbb{N}\,\forall^0\varepsilon : \ x_\varepsilon > \rho_\varepsilon^m$, i.e. $x > \mathrm{d}\rho^m$.*
- *(iv)* *There exists a representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x such that $\exists m \in \mathbb{N}\,\forall^0\varepsilon : \ x_\varepsilon > \rho_\varepsilon^m$.*

A consequence of this lemma is that the sharp topology is generated by all the infinitesimal balls of the type $B_{\mathrm{d}\rho^q}(z)$ for any $q \in \mathbb{N}$ and $z \in {}^{\rho}\widetilde{\mathbb{C}}$, and that $x < y$ is equivalent to $x_\varepsilon < y_\varepsilon$ for ε small and for all representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$.

Lemma 4. *For all $z \in {}^{\rho}\widetilde{\mathbb{C}}$, z is invertible if and only if $|z|$ is invertible. Moreover, invertible generalized complex numbers are dense in ${}^{\rho}\widetilde{\mathbb{C}}$.*

Proof. If $z\cdot w-1=[z_\varepsilon\cdot w_\varepsilon-1]=0$, then $z_\varepsilon\cdot w_\varepsilon-1\to0$ for $\varepsilon\to0^+$. Therefore, $z_\varepsilon\ne0$ for ε small, and hence $|z_\varepsilon|>0$. By Lem. 3, the absolute value $|z|$ is invertible. Conversely, assume that $|z|$ is invertible. Since $|z|\ge0$, by Lem. 3, for every representative $(z_\varepsilon)\in\mathbb{C}_\rho$ of z we have $\forall^0\varepsilon:\ |z_\varepsilon|>0$ and hence z is invertible with inverse $[z_\varepsilon^{-1}]$. For the second part of the statement, we have to prove that for all $r\in{}^\rho\widetilde{\mathbb{R}}_{>0}$ and $z\in{}^\rho\widetilde{\mathbb{C}}$ there exists an invertible $z^*\in B_r(z)$. Let $z=[z_\varepsilon]$ and $r=[r_\varepsilon]$ be any representatives, we set

$$z_\varepsilon^*:=\begin{cases}\frac{r_\varepsilon}{2} & \text{if } z_\varepsilon=0\\ z_\varepsilon & \text{otherwise}\end{cases},$$

Then, once again by Lem. 3, we have that z^* is invertible and $z^*\in B_r(z)$. □

A natural way to obtain particular subsets of ${}^\rho\widetilde{\mathbb{K}}^n$, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, is by using a net (A_ε) of subset $A_\varepsilon\subseteq\mathbb{K}$.

Definition 5. Let (A_ε) be a net of subsets of $\mathbb{K}$, then

(i) A set of the type

$$[A_\varepsilon]:=\left\{[x_\varepsilon]\in{}^\rho\widetilde{\mathbb{K}}^n\mid\forall^0\varepsilon:\ x_\varepsilon\in A_\varepsilon\right\}$$

is called *internal set*. It is possible to show that $[A_\varepsilon]$ is sharply closed and $[A_\varepsilon]=[\mathrm{cl}(A_\varepsilon)]$, where $\mathrm{cl}(S)$ is the closure of $S\subseteq\mathbb{K}^n$, see e.g. [50, 28].

(ii) A set of the type

$$\langle A_\varepsilon\rangle:=\left\{x\in{}^\rho\widetilde{\mathbb{K}}^n\mid\forall[x_\varepsilon]=x\,\forall^0\varepsilon:\ x_\varepsilon\in A_\varepsilon\right\}$$

is called *strongly internal set*. It is possible to show that $\langle A_\varepsilon\rangle$ is sharply open and $\langle A_\varepsilon\rangle=\langle\mathrm{int}(A_\varepsilon)\rangle$, where $\mathrm{int}(S)$ is the interior of $S\subseteq\mathbb{K}^n$, see e.g. [27].

It is quite useful to intuitively think at $\varepsilon\to0^+$ as a time variable, and hence $x=[x_\varepsilon]$ as a *dynamical number* (static numbers are ordinary reals) *with a fuzzy halo of ρ-negligible amplitude around* (x_ε). A similar intuitive interpretation can be considered for (strongly) internal sets. A thorough investigation of internal sets can be found in [50, 1]; see [27] for strongly internal sets.

For example, it is not hard to show that the open and closed balls of center $c=[c_\varepsilon]\in{}^\rho\widetilde{\mathbb{C}}$ and radius $r=[r_\varepsilon]>0$ can be written as

$$B_r(c)=\left\langle B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)\right\rangle,\quad\overline{B_r(c)}=\left[B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)\right],\tag{1.1}$$

where $B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon):=\{z\in\mathbb{C}\mid|z-c_\varepsilon|<r_\varepsilon\}$ denotes an ordinary Euclidean ball in $\mathbb{C}$. Moreover, by contradiction, it is easy to prove that

$$\overline{B_r(c)}\subseteq\langle A_\varepsilon\rangle\ \Rightarrow\ \forall^0\varepsilon:\ B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)\subseteq A_\varepsilon,\tag{1.2}$$

see e.g. [28, Lem. 11].

Before continuing with the formal mathematics, we want to explain why considering an arbitrary gauge ρ is important. In fact, any good theory of GHF has to be well linked with a good notion of power series. However, in the ring $\widetilde{\mathbb{R}}:={}^{(\varepsilon)}\widetilde{\mathbb{R}}$ of Colombeau generalized numbers we have that $(x_n)_{n\in\mathbb{N}}\in\widetilde{\mathbb{R}}^{\mathbb{N}}$ is a Cauchy sequence if and only if $\lim_{n\to+\infty}|x_{n+1}-x_n|=0$ (in the sharp topology; see [45, 54]). As a consequence, a series of Colombeau generalized numbers $a_n\in\widetilde{\mathbb{R}}$

$$\sum_{n=0}^{\infty}a_n\text{ converges}\iff a_n\to0\text{ (in the sharp topology)}.\tag{1.3}$$

Actually, this is a well-known property of every ultrametric space, cf., e.g., [36]. For example, $\sum_{n=1}^{\infty}\frac{1}{n^2}\in\widetilde{\mathbb{R}}$ converges in the sharp topology if and only if $\frac{1}{n^2}\to 0$ in the same topology, for $n\to+\infty$, $n\in\mathbb{N}_{>0}$. But the sharp topology on $\widetilde{\mathbb{R}}$ necessarily has to deal with balls having infinitesimal radius $r\in\widetilde{\mathbb{R}}$ (because generalized functions can have infinite derivatives and are continuous in this topology, see also [28]), and thus $\frac{1}{n^2}\not\to 0$ if $n\to+\infty$, $n\in\mathbb{N}_{>0}$, because we never have $\frac{1}{n^2}<r$ if r is infinitesimal. Similarly, one can easily prove that if the exponential series $\sum_{n=0}^{+\infty}\frac{x^n}{n!}$ converges, then necessarily $x\approx 0$ must be an infinitesimal number because if $\frac{x^n}{n!}\to 0$ in the sharp topology of $\widetilde{\mathbb{R}}$, then $|x_\varepsilon|\le n!\varepsilon^{1/n}$ for all $n\in\mathbb{N}$ sufficiently large.

Intuitively, the only way to have $\frac{1}{n^2}<r\approx 0$ is to take for $n\in\widetilde{\mathbb{R}}$ an infinite number, in this case an infinite natural number. On the other hand, we intuitively would like to have $\frac{1}{\log n}\to 0$, but we have $\frac{1}{\log n_\varepsilon}<\varepsilon^q$ if and only if $n_\varepsilon>e^{\varepsilon^{-q}}$ and the net $\left(e^{\varepsilon^{-q}}\right)_\varepsilon$ is not (ε)-moderate. In order to settle this problem, it is hence important to generalize the role of the net (ε) as used in Colombeau theory, into a more general gauge $\rho=(\rho_\varepsilon)\to 0$, and hence to generalize $\widetilde{\mathbb{R}}$ into ${}^\rho\widetilde{\mathbb{R}}$.

We can consider the aforementioned set of infinite natural numbers, called *hypernatural numbers,* and the set of ρ-moderate nets of natural numbers in the following:

Definition 6. We set

(i) ${}^\rho\widetilde{\mathbb{N}}:=\left\{[n_\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}\mid n_\varepsilon\in\mathbb{N}\ \forall\varepsilon\right\}$.

(ii) $\mathbb{N}_\rho:=\left\{(n_\varepsilon)\in\mathbb{R}_\rho\mid n_\varepsilon\in\mathbb{N}\ \forall\varepsilon\right\}$.

Therefore, $n\in{}^\rho\widetilde{\mathbb{N}}$ if and only if there exists $(x_\varepsilon)\in\mathbb{R}_\rho$ such that $n=[\text{int}\left(|x_\varepsilon|\right)]$, where $\text{int}\left(-\right)$ is the integer part function. Note that $\text{int}\left(-\right)$ is not well-defined on ${}^\rho\widetilde{\mathbb{N}}$: In fact, if $x=1=\left[1-\rho_\varepsilon^{1/\varepsilon}\right]=\left[1+\rho_\varepsilon^{1/\varepsilon}\right]$, then $\text{int}\left(1-\rho_\varepsilon^{1/\varepsilon}\right)=0$ whereas $\text{int}\left(1+\rho_\varepsilon^{1/\varepsilon}\right)=1$, for ε sufficiently small. However, the nearest integer function can be correctly defined as follows (see e.g. [45]):

Definition 7. The *nearest integer function* $\text{ni}\left(-\right)$ is defined by the following two properties:

(i) $\text{ni}:{}^\rho\widetilde{\mathbb{N}}\longrightarrow\mathbb{N}_\rho$.

(ii) If $[x_\varepsilon]\in{}^\rho\widetilde{\mathbb{N}}$ and $\text{ni}\left([x_\varepsilon]\right)=(n_\varepsilon)$ then $\forall^0\varepsilon: n_\varepsilon=\left\lfloor x_\varepsilon+\frac{1}{2}\right\rfloor$, where $\lfloor-\rfloor$ is the floor function.

In other words, if $x\in{}^\rho\widetilde{\mathbb{N}}$, then $x=\left[\text{ni}\left(x\right)_\varepsilon\right]$ and $\text{ni}\left(x\right)_\varepsilon\in\mathbb{N}$ for all ε.

To glimpse the necessity of studying ${}^\rho\widetilde{\mathbb{N}}$, it suffices to note that we have $\frac{1}{\log n}\to 0$ in ${}^\sigma\widetilde{\mathbb{R}}$ as $n\to+\infty$ for $n\in{}^\sigma\widetilde{\mathbb{N}}$, but only for a suitable gauge σ (depending on ρ, e.g. $\sigma_\varepsilon:=\exp(-\rho_\varepsilon^{-1/\rho_\varepsilon})$), whereas this limit does not exist if $\sigma=\rho$ (cf. [45, Example 27]). This is also related to the former problem of series of generalized numbers $a_n\in{}^\rho\widetilde{\mathbb{R}}$ because instead of ordinary series, we have better results summing over all $n\in{}^\sigma\widetilde{\mathbb{N}}$: e.g. we have ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}}\frac{z^n}{n!}=e^z$ for all $z\in{}^\rho\widetilde{\mathbb{C}}$ where the exponential is moderate, i.e. if $|z_\varepsilon|\le\log\left(\rho_\varepsilon^{-R}\right)$ for some $R\in\mathbb{N}$ (clearly, this includes all finite numbers z), see [48, 55]. Intuitively, the smaller is the second gauge σ, the greater are the infinite numbers we can consider with it (i.e. represented by σ-moderate nets), and hence the greater is the number of summands in summations of the form

${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n \in {}^{\rho}\widetilde{\mathbb{R}}$. This kind of summations are called *hyperseries*, their theory has been developed in [54, 55], and we will use them in [48] to show that any GHF is expandable in Taylor hyperseries.

We now recall the notion of *generalized smooth function*, because the real and imaginary part of every GHF always belong to this class.

1.2. **Generalized Smooth Functions.** Generalized smooth functions (GSF) are the simplest way (in the technical sense that they satisfy a suitable universal property, see [35]) to deal with a very large class of generalized functions and singular problems, by working directly with all their ρ-moderate smooth regularizations. In fact, GSF are a universal way to have set-theoretical functions defined on generalized numbers and having arbitrary derivatives (see [35]). GSF are close to the historically original conception of generalized function, [17, 37, 34]: in essence, the idea of authors such as Dirac, Cauchy, Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside (who informally worked with "numbers" which also comprise infinitesimals and infinite scalars) was to view generalized functions as certain types of smooth set-theoretical maps obtained from ordinary smooth maps by introducing a dependence on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, [34]. More generally, in the GSF approach, generalized functions are seen as suitable maps defined on, and attaining values in, the non-Archimedean ring of scalars ${}^{\rho}\widetilde{\mathbb{R}}$. The calculus of GSF is closely related to classical analysis sharing several properties of ordinary smooth functions. On the other hand, GSF include all Colombeau generalized functions and hence also all Schwartz distributions [27, 31, 28]. They allow nonlinear operations on generalized functions and unrestricted composition [27, 28]. For GSF, we can prove a number of analogues of theorems of classical analysis for generalized functions: e.g., mean value theorem, intermediate value theorem, extreme value theorem, Taylor's theorems, local and global inverse function theorems, integrals via primitives, and multidimensional integrals [26, 25, 28]. We can develop calculus of variations and optimal control for generalized functions, with applications e.g. in collision mechanics, singular optics, quantum mechanics and general relativity, see [38, 22]. We have new existence results for nonlinear singular ODE and PDE (e.g. a Picard-Lindelöf theorem for PDE), [42, 29], and with the notion of *hyperfinite Fourier transform* we can consider the Fourier transform of any GSF, without restriction to tempered type, [46]. GSF with their particular sheaf property define a Grothendieck topos, [28].

Definition 8. Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$. We say that $f : X \longrightarrow Y$ is a GSF ($f \in {}^{\rho}\mathcal{GC}^{\infty}(X, Y)$), if

(i) $f : X \longrightarrow Y$ is a map.

(ii) There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ such that $X \subseteq \langle\Omega_\varepsilon\rangle$ and for all $[x_\varepsilon] \in X$ (i.e. for all representatives (x_ε)):
  - a) $f(x) = [f_\varepsilon(x_\varepsilon)]$.
  - b) $\forall\alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is ρ-moderate.

In Thm. 22, we will prove that real and imaginary part of any GHF are GSF.

1.3. **The language of subpoints.** The following simple language allow us to simplify some proofs using steps recalling the classical real field $\mathbb{R}$, particularly in dealing with a negation of an equality or of an inequality, or even of an arbitrary property in $^{\rho}\widetilde{\mathbb{R}}$. This notion is studied in [45].

Definition 9.

(i) For subsets J, $K \subseteq I$ we write $K \subseteq_0 J$ if 0 is an accumulation point of K and $K \subseteq J$ (we read it as: *K is co-final in J*). Note that for any $J \subseteq_0 I$, the constructions introduced so far in Def. 1 can be repeated using nets $(x_\varepsilon)_{\varepsilon \in J}$. We indicate the resulting ring with the symbol $^{\rho}\widetilde{\mathbb{R}}|_J$. More generally, no peculiar property of $I = (0, 1]$ will ever be used in the following, and hence all the presented results can be easily generalized considering any other directed set.

(ii) If $K \subseteq_0 J$, $x \in {^{\rho}\widetilde{\mathbb{R}}}|_J$ and $x' \in {^{\rho}\widetilde{\mathbb{R}}}|_K$, then x' is called a *subpoint* of x, denoted as $x' \subseteq x$, if there exist representatives $(x_\varepsilon)_{\varepsilon \in J}$, $(x'_\varepsilon)_{\varepsilon \in K}$ of x, x' such that $x'_\varepsilon = x_\varepsilon$ for all $\varepsilon \in K$. In this case, we write $x' = x|_K$, $\mathrm{dom}(x') := K$, and the restriction $(-)|_K : {^{\rho}\widetilde{\mathbb{R}}} \longrightarrow {^{\rho}\widetilde{\mathbb{R}}}|_K$ is a well defined operation. In general, for $X \subseteq {^{\rho}\widetilde{\mathbb{R}}}$ we set $X|_J := \left\{ x|_J \in {^{\rho}\widetilde{\mathbb{R}}}|_J : x \in X \right\}$.

For example, Lem. 3 implies that $x \in {^{\rho}\widetilde{\mathbb{R}}}$ is not invertible if and only if 0 is a subpoint of x, and this also gives a better intuition about Lem. 4.

In the next definition, we now consider binary relations that hold only *on subpoints*.

Definition 10. Let x, $y \in {^{\rho}\widetilde{\mathbb{R}}}$, $L \subseteq_0 I$, then we say:

(i) $x <_L y$ if $x|_L < y|_L$, the latter inequality has to be meant in the ordered ring $^{\rho}\widetilde{\mathbb{R}}|_L$. We read $x <_L y$ as "x is less than y on L".

(ii) $x <_{\mathrm{s}} y$ if $\exists L \subseteq_0 I : x <_L y$. We read $x <_{\mathrm{s}} y$ as "x is less than y on subpoints".

Analogously, we can define other relations holding only on subpoints such as e.g.: $\in_{\mathrm{s}}$, $\le_{\mathrm{s}}$, $=_{\mathrm{s}}$, $\subseteq_{\mathrm{s}}$, etc.

For example, we have

$$\begin{aligned} & x < y \iff \forall L \subseteq_0 I : x <_L y \text{ and} \\ & x \ne y \iff x >_{\mathrm{s}} y \text{ or } x <_{\mathrm{s}} y. \end{aligned} \tag{1.4}$$

Moreover, if $\mathcal{P}\{x_\varepsilon\}$ is an arbitrary property of x_ε, then

$$\neg \left(\forall^0 \varepsilon : \mathcal{P}\{x_\varepsilon\} \right) \iff \exists L \subseteq_0 I \, \forall \varepsilon \in L : \neg \mathcal{P}\{x_\varepsilon\}.$$

As one can guess from (1.4) (recall that $x >_{\mathrm{s}} 0$ implies that x is invertible on subpoints), this language allows one to deal with good substitute of the field and of the total order properties, see [45] for a deeper study.

2. Generalized Holomorphic Functions

2.1. **Basic (locally Lipschitz) functions, limits and weak little-o.** Before defining the notion of GHF, we need to reformulate in our non-Archimedean setting some more basic notions: what is the class of functions we want to consider, their limits and continuity in the sharp topology, and Landau little-o relation. These are the concepts that allow us to get a good theory of GHF.

Considering the theory of GSF, we still want to have generalized holomorphic function of the form $f(z) = [f_\varepsilon(z_\varepsilon)] \in {^{\rho}\widetilde{\mathbb{C}}}$ but without assuming the CRE for f_ε directly in the definition like in Colombeau theory; we can think e.g. (f_ε) as obtained by a suitable holomorphic regularization of a compactly supported distribution, see Sec. 3.

We therefore start by considering the notion of basic function as follows:

Definition 11. Let $U \subseteq {}^{\rho}\widetilde{\mathbb{C}}$. We say that $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ is a *basic function* (or a ρ*-basic function* in case we have to underscore the dependence from the gauge ρ), if

(i) $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ is a map.

(ii) There exists a net (f_ε) such that $f_\varepsilon : U_\varepsilon \longrightarrow \mathbb{C}$, $U \subseteq \langle U_\varepsilon \rangle$, $U_\varepsilon \subseteq \mathbb{C}$, and for all $z = [z_\varepsilon] \in U$
$$f(z) = [f_\varepsilon(z_\varepsilon)], \tag{2.1}$$
and in this case, we say that f *is defined by* the net (f_ε).

(iii) If $U = \langle \Omega \rangle$, the *standard part of* f is $f^\circ : \{z \in \Omega \mid \exists f(z)^\circ\} \mapsto f(z)^\circ \in \mathbb{C}$, so that $f^\circ(z) = \lim_{\varepsilon \to 0^+} f_\varepsilon(z) \in \mathbb{C}$ for all $z \in \Omega$ where such a limit exists.

Moreover, we say that the basic function f is (ρ-)*locally Lipschitz* if for all $z \in U$, there exist constants $L \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that
$$|f(x) - f(y)| \le L \cdot |x - y| \quad \forall x, y \in B_r(z). \tag{2.2}$$
We also refer to f as L*-Lipschitz function* on the sharply open ball $B_r(z)$ if (2.2) holds, and we finally say that f is *Lipschitz at* z if (2.2) holds for some $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and some $L \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$.

A similar concept was first introduced in [21] to define basic linear maps, whereas the notion of *internal function* as defined by [50] seems to be a particular case of basic function, see [50, Prop. 3.3]. Note also that equality (2.1) implies that if $(z_\varepsilon) \sim_\rho (z'_\varepsilon)$, then $(f_\varepsilon(z_\varepsilon)) \sim_\rho (f_\varepsilon(z'_\varepsilon))$, which is a strong limitation on the maps f_ε: e.g. $f_\varepsilon(z) = |z|$ defines a basic function, but $f_\varepsilon(z) = 1$ if $z \ne 0$ and $f_\varepsilon(0) = 0$ does not. Therefore, basic functions are particular cases of set theoretical maps of the type $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ because they need to have suitable defining nets (f_ε) satisfying (ii) above.

Clearly, we expect that our GHF are locally Lipschitz. We therefore start studying locally Lipschitz maps and proving that being locally Lipschitz can also be formulated ε-wise.

Theorem 12. *Let $U_0 \subseteq {}^{\rho}\widetilde{\mathbb{C}}$ be a sharp neighbourhood of 0. If $R = [R_\varepsilon(-)] : U_0 \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ is a basic function, then R is Lipschitz at 0 if and only if there exists $Q \in \mathbb{R}_{>0}$ and $(L_\varepsilon) \in \mathbb{R}_\rho$ such that R_ε is L_ε-Lipschitz (hence continuous) over $B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0) \subseteq \mathbb{C}$ for ε small.*

Proof. $\Rightarrow$: Property (2.2) yields the existence of some $L = [L_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $r = [r_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that R is L-Lipschitz on the sharply open ball $B_r(0)$. From (1.1) and Lem. 3, there exists $Q \in \mathbb{N}$ such that $\forall^0 \varepsilon : \ r_\varepsilon > \rho_\varepsilon^Q$ and hence R is L-Lipschitz on $\left\langle B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0) \right\rangle$. By contradiction, assume that there exist a strictly decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}} \to 0^+$ and $h_{\varepsilon_n}, k_{\varepsilon_n} \in B^{\mathrm{E}}_{\rho_{\varepsilon_n}^Q}(0)$ such that
$$|R_{\varepsilon_n}(h_{\varepsilon_n}) - R_{\varepsilon_n}(k_{\varepsilon_n})| > L_{\varepsilon_n} |h_{\varepsilon_n} - k_{\varepsilon_n}| \quad \forall n \in \mathbb{N}. \tag{2.3}$$
Defining $h_\varepsilon := h_{\varepsilon_n}$ and $k_\varepsilon := k_{\varepsilon_n}$ for $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]$, and taking any h_ε, $k_\varepsilon \in B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0)$ otherwise, we have that $h := [h_\varepsilon]$, $k := [k_\varepsilon] \in B_r(0)$. Setting $K := \{\varepsilon_n \mid n \in \mathbb{N}\} \subseteq_0$

I, we have $|h-k|>_K 0$ from (2.3), and

$$L|h-k|<_K |R(h)-R(k)|\le L\cdot|h-k| \Longrightarrow L\cdot|h-k|<_K L\cdot|h-k|.$$

Therefore, $L<_K L$ for some $K\subseteq_0 I$, which is a contradiction.
$\Leftarrow$: For ε small and h, $k\in B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0)$, we have

$$|R_\varepsilon(h)-R_\varepsilon(k)|\le L_\varepsilon|h-k|.$$

For all $x=[x_\varepsilon]$, $y=[y_\varepsilon]\in\left\langle B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0)\right\rangle=B_{\mathrm{d}\rho^Q}(0)$, we have x_ε, $y_\varepsilon\in B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0)$ and

$$|R_\varepsilon(x_\varepsilon)-R_\varepsilon(y_\varepsilon)|\le L_\varepsilon\cdot|x_\varepsilon-y_\varepsilon|,$$

for small ε. Hence, R is locally Lipschitz for $L:=[L_\varepsilon]\in{}^\rho\widetilde{\mathbb{R}}$. □

It is clear that any basic locally Lipschitz function $f:U\longrightarrow{}^\rho\widetilde{\mathbb{C}}$ is *sharply continuous on* U, i.e. for all $z_0\in U$,

$$\forall q\in\mathbb{N}\,\exists H\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall h\in U: |h-z_0|<H \implies |f(h)-f(z_0)|<\mathrm{d}\rho^q. \tag{2.4}$$

In the next definition, we introduce the notion of *limit* of a basic functions.

Definition 13. Let $U_0\subseteq{}^\rho\widetilde{\mathbb{C}}$ be a sharp neighbourhood of z_0. Let $R:U_0\longrightarrow{}^\rho\widetilde{\mathbb{C}}$ be a basic function. We say that the ρ-*limit* of R, as h tends to z_0 in the sharp topology is $\lambda\in{}^\rho\widetilde{\mathbb{C}}$, if the following property holds:

$$\forall q\in\mathbb{N}\,\exists H\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall h\in U_0:\ 0<|h-z_0|<H\Rightarrow|R(h)-\lambda|<\mathrm{d}\rho^q. \tag{2.5}$$

In that case, we write $\lim_{h\to z_0}R(h)=\lambda$, or ${}^\rho\lim_{h\to z_0}R(h)=\lambda$ in case we have to underscore the dependence from the gauge ρ.

Remark 14. In the assumptions of Def. 13, let $k\in{}^\rho\widetilde{\mathbb{R}}_{>0}$ and $N\in\mathbb{N}$, then the following are equivalent:

(i) $\lambda\in{}^\rho\widetilde{\mathbb{C}}$ is the ρ-limit of R as h tends to z_0.
(ii) $\forall\eta\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\exists H\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall h\in U_0:\ 0<|h-z_0|<H\Rightarrow|R(h)-\lambda|<\eta$.
(iii) For all ρ-sharply open sets $U\subseteq{}^\rho\widetilde{\mathbb{C}}$ then there exists a ρ-sharply open set $V\subseteq{}^\rho\widetilde{\mathbb{C}}$ containing z_0 such that $\forall h\in V\cap U_0: R(h)\in U$.
(iv) $\forall q\in\mathbb{N}\,\exists H\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall h\in U_0:\ 0<|h-z_0|<H\Rightarrow|R(h)-\lambda|<k\cdot\mathrm{d}\rho^q$.
(v) $\forall q\in\mathbb{N}\,\exists H\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall h\in U_0:\ 0<|h-z_0|<H\Rightarrow|R(h)-\lambda|<\mathrm{d}\rho^{q-N}$.

Directly by the inequality $|\lambda_1-\lambda_2|\le|\lambda_1-R(h)|+|\lambda_2-R(h)|\le 2\mathrm{d}\rho^{q+1}<\mathrm{d}\rho^q$ and by Lem. 4 (or by using that the sharp topology on ${}^\rho\widetilde{\mathbb{C}}$ is Hausdorff) it follows that there exists at most one limit. Moreover, since any basic locally Lipschitz function $R:U_0\longrightarrow{}^\rho\widetilde{\mathbb{C}}$ is sharply continuous on U_0, in this case we have

$$\lim_{h\to z_0}R(h)=R(z_0). \tag{2.6}$$

Lemma 15. *Let $U_0\subseteq{}^\rho\widetilde{\mathbb{C}}$ be a sharp neighbourhood of 0. Let $R:U_0\longrightarrow{}^\rho\widetilde{\mathbb{C}}$ be a basic locally Lipschitz function. Then $R(0)=0$ if and only if there exists a net of functions $(R_\varepsilon(-))$ that defines R such that $R_\varepsilon(0)=0$ for small ε.*

Proof. If $R(0)=0$ and the net $(\bar{R}_\varepsilon)$ defines R, then $R_\varepsilon(h):=\bar{R}_\varepsilon(h)-\bar{R}_\varepsilon(0)$ satisfies the claim. The opposite implication is trivial. □

In the following corollary, we use Thm. 12 to get an ε-wise characterization of limits.

Corollary 16. *Let $U_0 \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharp neighbourhood of 0. Let $R : U_0 \longrightarrow {}^\rho\widetilde{\mathbb{C}}$ be a ρ-basic function which is Lipschitz at 0. Then the following are equivalent:*

(i) ${}^\rho\lim_{h\to 0} R(h) = 0$.

(ii) *There exist $Q \in \mathbb{R}_{>0}$ and a net of functions $(R_\varepsilon(-))$ that defines R, it is L_ε-Lipschitz over $B^{\mathrm{E}}_{\rho_\varepsilon^Q}(0) \subseteq \mathbb{C}$, and $\lim_{h\to 0} R_\varepsilon(h) = 0$ for small ε.*

A natural definition of ${}^\rho\widetilde{\mathbb{C}}$-differentiable function $f : U \longrightarrow {}^\rho\widetilde{\mathbb{C}}$ at $z_0 \in U$ has to be linked to a linearization property of the form

$$f(z_0 + h) = f(z_0) + h \cdot m + o(h) \text{ as } h \to 0, \tag{2.7}$$

for some $m \in {}^\rho\widetilde{\mathbb{C}}$. In pursuing this idea, we have to first define the notion of Landau little-o in a ring such as ${}^\rho\widetilde{\mathbb{C}}$.

Definition 17. Let $U_0 \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharp neighbourhood of 0 and $f_1, f_2 : U_0 \longrightarrow {}^\rho\widetilde{\mathbb{C}}$ be maps. Then we say that

$$f_1(h) = \bar{o}(f_2(h)) \text{ as } h \to 0,$$

if there exists a map $r : U_0 \longrightarrow {}^\rho\widetilde{\mathbb{C}}$ such that

(i) ${}^\rho\lim_{h\to 0} r(h) = 0$.

(ii) $\forall h \in U_0 : \ f_1(h) = f_2(h) r(h)$.

We call this relation *weak little-o*.

Remark 18.

a) If $f_2(h)$ is invertible for all sufficiently small invertible h, and both r, $\bar{r}$ satisfy Def. 17, then $r(h) = \bar{r}(h)$ for these h.

b) If $f_2(h) = h$, then Def. 17 implies ${}^\rho\lim_{h\to 0} \frac{f_1(h)}{h} = 0$, because in the definition of limit (2.5) we always take $0 < |h|$, i.e. h invertible. This and (2.7) yield a comparison with the approach followed by [2].

Here are some general rules that are easily satisfied by the definition of weak little-o:

Lemma 19. *In the assumptions of Def. 17, we have*

(i) *If $f(h) = \bar{o}(h)$ and $g(h) = \bar{o}(h)$ then $f(h)+g(h) = \bar{o}(h)$ and $f(h)g(h) = \bar{o}(h^2)$.*

(ii) *If $f(h) = \bar{o}(h)$ and $c \in {}^\rho\widetilde{\mathbb{C}}$, then $cf(h) = \bar{o}(h)$.*

2.2. **Complex generalized differentiable function.** Having the previous notion of weak little-o relation, it would now be natural to use the linearization property (2.7) to define a pointwise complex differentiable function. However, considering the following motivations, we prefer to introduce a stronger notion of little-o:

1) Def. 17 of weak little-o do not use the notion of basic function at all, and indeed it includes the classical example given by $i(z) := 1$, if z is infinitesimal and $i(z) := 0$ otherwise. This function i cannot be a GHF on any neighbourhood $B_r(0)$ defined by a standard radius $r \in \mathbb{R}_{>0}$, because, otherwise, its real and imaginary parts would be GSF (see (v) of Thm. 22 below), which is not the case because the intermediate value theorem does not hold for them, see [28, Cor. 48] and Sec. 1.2 above). In particular, $i(z)$ cannot be globally defined $i(z) = [i_\varepsilon(z_\varepsilon)]$ by a net of ordinary holomorphic functions (i_ε). On the other hand, the function i seems well-behaving, at least if we restrict i to infinitesimal neighbourhoods and standard points, i.e. on balls of the type $B_r(c)$, where $c \in \mathbb{C}$ and $r \approx 0$, e.g. $r = \mathrm{d}\rho^Q$.

2) We would then have the drawback that the function i does not coincide with its Taylor series in its entire set of convergence, even if the latter is the trivial (constant) series. Therefore, in general, if we define GHF using (2.7) and the weak little-o, we cannot prove that if f is a GHF, its Taylor series converges to f in the entire convergence set. All this is related to the total disconnectedness property of ${}^{\rho}\widetilde{\mathbb{C}}$, due to the fact that the set of infinitesimals is a clopen set.
3) We therefore follow the idea to ask a relation of the type (2.7), but remaining in the class of basic functions for all the maps appearing in (2.7), $f(z_0+h)-f(z_0)-h\cdot m$ and $o(h)=o(z_0,h)$, and for all the variables z_0 and h. In other words, the previous function i satisfies a property of the form "for any standard point $z_0\in\mathbb{C}$ there exists a net of functions (i_ε) defining i in an infinitesimal neighbourhood of z_0 and such that the linearization property (2.7) holds", whereas we want to restrict to a stronger definition of the form "there exists a net of functions (f_ε) globally defining f, and at the point $z_0\in{}^{\rho}\widetilde{\mathbb{C}}$ a linearization property of the form (2.7) holds".
4) Finally, we also want to prove that $f'(z_0):=m$ is still a basic function defined by a net (f'_ε) of ordinary complex differentiable functions because this would ensure that our class of GHF is not too large but remains strictly tied to regularizations of GF using ordinary holomorphic ones, see Sec. 3. Starting only from (2.7), this desired property does not hold: in fact, since $[\rho_\varepsilon^{1/\varepsilon}]=0\in{}^{\rho}\widetilde{\mathbb{C}}$, we could have $f_\varepsilon(z_{0\varepsilon}+h)=f_\varepsilon(z_{0\varepsilon})+h\cdot m_\varepsilon+h\cdot o_\varepsilon(z_{0\varepsilon},h)+\rho_\varepsilon^{1/\varepsilon}\cdot n_\varepsilon(z_{0\varepsilon},h)$ for ε and h small, even if $n_\varepsilon(z_{0\varepsilon},h)$ is not holomorphic in its variables. For this reason, we ask that (2.7) holds also at the ε-level and without any other non-regular additive remainder.
5) As we already mentioned in Sec. 1.1, one of the aim of this work is to prove, in this setting of GF, that a condition of first order differentiability implies both the CRE and the existence of all greater derivatives.

We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω, where Ω is an open subset of $\mathbb{C}$.

Definition 20. Let $\emptyset\neq V\subseteq U\subseteq{}^{\rho}\widetilde{\mathbb{C}}$ and $f:U\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$ be a basic function. Then f is said to be *${}^{\rho}\widetilde{\mathbb{C}}$-differentiable on V* if there exist $U_0\subseteq{}^{\rho}\widetilde{\mathbb{C}}$ and nets $f_\varepsilon:U_\varepsilon\longrightarrow\mathbb{C}$, $r_\varepsilon:V_\varepsilon\times U_{0\varepsilon}\longrightarrow\mathbb{C}$ such that:

(i) $\forall z\in V\,\exists s\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}:\ z+B_s(0)\subseteq z+U_0\subseteq U$.
(ii) (f_ε) defines f.
(iii) (r_ε) defines a basic function $r:V\times U_0\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$.

Moreover, the following properties hold for all $z\in V$:

(iv) ${}^{\rho}\lim_{h\to0}r(z,h)=0$.
(v) There exists $[m_\varepsilon]=m\in{}^{\rho}\widetilde{\mathbb{C}}$ such that for all representatives $[z_\varepsilon]=z\in V$ and all $[h_\varepsilon]\in U_0$, we have

$$\forall^0\varepsilon:\ f_\varepsilon(z_\varepsilon+h_\varepsilon)=f_\varepsilon(z_\varepsilon)+h_\varepsilon\cdot m_\varepsilon+h_\varepsilon\cdot r_\varepsilon(z_\varepsilon,h_\varepsilon). \tag{2.8}$$

$$r_\varepsilon(z_\varepsilon,0)=0=\lim_{h\to0}r_\varepsilon(z_\varepsilon,h). \tag{2.9}$$

If conditions (i) - (v) hold, we simply write

$$f(z+h)=f(z)+h\cdot m+o(h)\text{ as }h\to0\text{ in }z\in V. \tag{2.10}$$

The relation $o(h)$ is called *strong little-o*, or simply *little-o*.

Finally, we say that:

1) f is ${}^{\rho}\widetilde{\mathbb{C}}$*-differentiable at* z_0 if it is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable on $V = \{z_0\}$.
2) f is a *generalized holomorphic function* (GHF) at z_0 if there exists a neighbourhood $V \supseteq B_r(z_0)$ of z_0 such that it is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable on V and the functions f_ε of Def. 20 are all holomorphic: $f_\varepsilon \in \mathcal{O}(B^{\mathrm{E}}_{r_\varepsilon}(z_{0\varepsilon}))$ $\forall\varepsilon$, where $r = [r_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $z_0 = [z_{0\varepsilon}]$.
3) f is a *GHF on a set* $V \subseteq U$, if V is sharply open and f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable on V and the functions f_ε of Def. 20 are all holomorphic: $f_\varepsilon \in \mathcal{O}(U_\varepsilon)$ $\forall\varepsilon$.
4) We write $f \in {}^{\rho}\mathcal{GH}(W, Y)$ if $f : W \longrightarrow Y$ and f is a GHF on W, or simply $f \in {}^{\rho}\mathcal{GH}(W)$ if $Y = {}^{\rho}\widetilde{\mathbb{C}}$.

Remark 21.

a) Note explicitly that Def. 20 is of the form $\exists(f_\varepsilon)\,\exists(r_\varepsilon)\,\forall z \in V$ and not $\forall z \in V\,\exists(f_\varepsilon)\,\exists(r_\varepsilon)$, as we already mentioned in 3) at the beginning of the present section.
b) Property (i) of Def. 20 allows us to consider both $f(z+h)$ and $r(z,h)$ for all $z \in V$ and $h \in B_s(0)$. If we set $U_0(z) := \{h \in U_0 \mid z + h \in U\}$ for each $z \in V$, we have $U_0(z) \supseteq B_s(0)$ and $U_0 = \bigcup_{z\in V} U_0(z)$. Moreover, (i) also implies that V is contained in the sharply interior of U.
c) By contradiction, it is not hard to prove that (2.8) implies
$$\forall^0\varepsilon\,\forall h \in B^{\mathrm{E}}_{s_\varepsilon}(0):\ f_\varepsilon(z_\varepsilon + h) = f_\varepsilon(z_\varepsilon) + h\cdot m_\varepsilon + h\cdot r_\varepsilon(z_\varepsilon, h), \tag{2.11}$$
where $z + \overline{B}_s(0) \subseteq z + U_0 \subseteq U$. In fact, assume that for some $J \subseteq_0 I$, we can find $h_\varepsilon \in B^{\mathrm{E}}_{s_\varepsilon}(0)$ for $\varepsilon \in J$ and such that
$$f_\varepsilon(z_\varepsilon + h_\varepsilon) \neq f_\varepsilon(z_\varepsilon) + h_\varepsilon\cdot m_\varepsilon + h_\varepsilon\cdot r_\varepsilon(z_\varepsilon, h_\varepsilon). \tag{2.12}$$
Defining $h_\varepsilon := 0$ for $\varepsilon \in I \setminus J$, we have $[h_\varepsilon] \in [B^{\mathrm{E}}_{s_\varepsilon}(0)] = \overline{B}_s(0) \subseteq U_0$, and hence (2.12) contradicts (2.8). From (2.11) and (2.9) of Def. 20, it follows that for ε small, each function f_ε is differentiable at $z_\varepsilon \in U_\varepsilon$.
d) Properties (iv) and (2.8) of Def. 20 imply
$$f(z+h) = f(z) + h\cdot m + h\cdot r(z,h), \tag{2.13}$$
for all $z \in V$ and all $h \in U_0(z)$, i.e. a weak little-o relation of the form (2.7). Clearly, all the present construction of GHF is based on the classical theory of holomorphic functions, and indeed we are using a net $(f_\varepsilon) \in \mathcal{O}(U_\varepsilon)$ in Def. 20.3) of GHF; however, the notion of differentiability (2.13) or (2.10) for the *generalized* function f is stated only as a first order condition. Indeed, instead of being part of the definition like in Colombeau theory, CRE and higher order differentiability will be explicitly proved later.
e) It is important to note that conditions (2.8) and (2.13), express a condition only on generalized points $z = [z_\varepsilon] \in V$, and not on standard points $z \in U_\varepsilon$, which instead are considered by the properties of the type $f_\varepsilon \in \mathcal{O}(B^{\mathrm{E}}_{r_\varepsilon}(z_{0\varepsilon}))$ or $f_\varepsilon \in \mathcal{O}(U_\varepsilon)$. For example, we can even have that $V = \{[z_\varepsilon]\}$ is given by a single infinite point.
f) We recall that in the case $V = \{z_0\}$, a basic function $r : \{z_0\} \times U_0 \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ is defined by $r_\varepsilon : V_\varepsilon \times U_{0\varepsilon} \longrightarrow \mathbb{C}$ if $\{z_0\} \times U_0 \subseteq \langle V_\varepsilon \times U_{0\varepsilon}\rangle$, and $\left[r_\varepsilon(\bar{z}_{0\varepsilon}, \bar{h}_\varepsilon)\right] = r(z_0, h) = [r_\varepsilon(z_{0\varepsilon}, h_\varepsilon)]$ for all representatives $[\bar{z}_{0\varepsilon}] = z_0 = [z_{0\varepsilon}]$ and all $h = [h_\varepsilon] =$

$[\bar{h}_\varepsilon] \in U_0$, see Def. 11. Note that from $0 \in U_0$ it also follows $(z_{0\varepsilon}, 0) \in V_\varepsilon \times U_{0\varepsilon}$ for ε small.

g) Let $q \in \mathbb{N}$. Since $\lim_{h\to 0} r(z,h) = 0$, then there exists $H \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $\forall h \in U_0(z) : 0 < |h| < H$, we have $|r(z,h)| < \mathrm{d}\rho^q$. Hence,

$$\left|\frac{f(z+h)-f(z)}{h} - m\right| = |r(z,h)| < \mathrm{d}\rho^q, \quad \forall h \in U_0(z) : 0 < |h| < H.$$

Recall Lem. 4 about the density of invertible elements, and that $|h| > 0$ means that $|h|$ is invertible. In the other words, we have

$$m = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}.$$

Therefore, m is unique and we can define

$$f'(z) = m = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

which is called *the derivative of f at $z \in V$*. Here, we can only conclude that $f'(z)$ is a number dependent on z. To prove that $f'(-)$ is a basic function, we need the following:

Theorem 22. *Let f be a* GHF *on U and $[z_\varepsilon] = z \in U$. Then, we have:*

(i) $\exists Q \in \mathbb{R}_{>0}\, \forall^0\varepsilon :\ f_\varepsilon|_{B_{\rho_\varepsilon^Q}(z_\varepsilon)}$ *is a holomorphic function and $f'_\varepsilon(z_\varepsilon) = m_\varepsilon$ for ε small, so that $f'(z) = [f'_\varepsilon(z_\varepsilon)]$.*

(ii) $\exists Q, R \in \mathbb{R}_{>0}\, \forall^0\varepsilon\, \forall n \in \mathbb{N} :\ \left|\frac{f_\varepsilon^{(n)}(z_\varepsilon)}{n!}\right| \le \rho_\varepsilon^{-nQ-R}$. *In particular, all ε-wise derivatives are ρ-moderate: $\forall n \in \mathbb{N} :\ \left(f_\varepsilon^{(n)}(z_\varepsilon)\right) \in \mathbb{C}_\rho$.*

(iii) The function r_ε in Def. 20 satisfies

$$\exists [a_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall^0\varepsilon\, \forall h \in B^{\mathrm{E}}_{a_\varepsilon}(z_\varepsilon) :\ r_\varepsilon(z_\varepsilon, h) = \sum_{n=2}^{\infty} \frac{f^{(n)}(z_\varepsilon)}{n!} h^{n-1}.$$

Therefore, it is locally Lipschitz in the variable h, and hence the ρ-limit Def. 20.(iv) always follows by the ε-wise limit Def. 20.(v).

(iv) The function r_ε in Def. 20 can also be written as

$$\exists [s_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall^0\varepsilon\, \forall h \in B^{\mathrm{E}}_{s_\varepsilon}(z_\varepsilon) :\ r_\varepsilon(z_\varepsilon, h) = \int_0^1 f'_\varepsilon(z_\varepsilon + ht)\,\mathrm{d}t - f'_\varepsilon(z_\varepsilon). \tag{2.14}$$

(v) Real and imaginary parts of f are GSF defined on $\left\{(x,y) \in {}^\rho\widetilde{\mathbb{R}}^2 \mid x + iy \in U\right\}$ and hence f is locally Lipschitz.

(vi) The derivative of f, $f' \in {}^\rho\mathcal{GH}(U)$. Therefore, recursively defining the derivatives of f, we have that $f^{(n)} \in {}^\rho\mathcal{GH}(U)$ for all $n \in \mathbb{N}$.

(vii) The function $r := [r_\varepsilon(-,-)]$ is locally Lipschitz in both variables z, h.

Proof. (i): In fact $z = [z_\varepsilon] \in U$ and $U \subseteq \langle U_\varepsilon \rangle$ because (f_ε) defines f. Therefore, there exists $Q \in \mathbb{N}$ such that $\overline{B_{\mathrm{d}\rho^Q}(z)} \subseteq \langle U_\varepsilon \rangle$ because $\langle U_\varepsilon \rangle$ is a sharply open set (see Def. 5), and hence $B^{\mathrm{E}}_{\rho_\varepsilon^Q}(z_\varepsilon) \subseteq U_\varepsilon$ for ε small by (1.2). This yields the conclusion because $f_\varepsilon \in \mathcal{O}(U_\varepsilon)$. Finally, (2.11) and (2.9) imply $f'_\varepsilon(z_\varepsilon) = m_\varepsilon$ for small ε, so that $f'(z) = m = [m_\varepsilon] = [f'_\varepsilon(z_\varepsilon)]$.

(ii): From Def. 20.(i) we can find $a=[a_\varepsilon]$, $b=[b_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ with $a<b$ and such that $B_a(z)\subseteq\overline{B_b(z)}\subseteq U$. As we proved in (i), we have $f_\varepsilon\in\mathcal{O}(B^{\text{E}}_{b_\varepsilon}(z_\varepsilon))$ for ε small, and hence for these ε Cauchy's formula yields

$$f_\varepsilon^{(n)}(z_\varepsilon)=\frac{n!}{2\pi i}\int_\gamma\frac{f_\varepsilon(z)}{(z-z_\varepsilon)^{n+1}}\mathrm{d}z, \tag{2.15}$$

where γ is the counterclockwise oriented circle forming the boundary of $B^{\text{E}}_{a_\varepsilon}(z_\varepsilon)$. Therefore, $|f_\varepsilon^{(n)}(z_\varepsilon)|\le n!a_\varepsilon^{-n}\sup_{z\in\gamma}|f_\varepsilon(z)|$. Since f_ε is continuous and $\gamma\subseteq\mathbb{C}$ is compact, we have $\sup_{z\in\gamma}|f_\varepsilon(z)|=|f_\varepsilon(\bar{z}_\varepsilon)|$ for some $\bar{z}_\varepsilon\in\gamma$, and hence $\bar{z}:=[\bar{z}_\varepsilon]\in U$. Thereby $(|f_\varepsilon(\bar{z}_\varepsilon)|)\in\mathbb{R}_\rho$ because $f:U\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$ is a basic function, and this proves claim (ii), where we can take $Q\in\mathbb{R}_{>0}$ such that $a\ge\mathrm{d}\rho^Q$, hence depending only on a, and $R\in\mathbb{R}_{>0}$ such that $|f_\varepsilon(\bar{z}_\varepsilon)|\le\rho_\varepsilon^{-R}$. Moreover, if $\varepsilon_0\in I$ satisfies both $a_\varepsilon\ge\rho_\varepsilon^Q$ and $|f_\varepsilon(\bar{z}_\varepsilon)|\le\rho_\varepsilon^{-R}$ for all $\varepsilon\le\varepsilon_0$, then for all $n\in\mathbb{N}$ we also have $\left|f_\varepsilon^{(n)}(z_\varepsilon)\right|\le n!\rho_\varepsilon^{-nQ-R}$.

(iii): Taking a, $b\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ as above, we have that the Taylor formula

$$f_\varepsilon(z)=\sum_{n=0}^{\infty}\frac{f_\varepsilon^{(n)}(z_\varepsilon)}{n!}(z-z_\varepsilon)^n \tag{2.16}$$

pointwise converges for $z\in B^{\text{E}}_{a_\varepsilon}(z_\varepsilon)$. The conclusion now follows from (2.11).

(iv): From (2.11), for $h\in B^{\text{E}}_{s_\varepsilon}(0)\setminus\{0\}$ it suffices to multiply by h both sides of (2.14) and integrate. For $h=0$, the equality follows from $r_\varepsilon(z_\varepsilon,0)=0$.

(v): This follows from (i), (ii) and Def. 8.

(vi): Set $U_0':=\bigcup\left\{B_a(0)\mid a>0,\ \exists b>a\,\exists z\in U:\ \overline{B_b(z)}\subseteq U\right\}$. For each $z\in U$, from Def. 20.(i) (for $V=U$), it follows the existence of a, $b\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $a<b$ and $z+B_a(0)\subseteq\overline{B_b(z)}\subseteq z+U_0\subseteq U$. Thereby, $B_a(0)\subseteq U_0'$ and hence $z+B_a(0)\subseteq z+U_0'\subseteq U$. This shows property (i) for U_0'. The map $z=[z_\varepsilon]\in U\mapsto[f_\varepsilon'(z_\varepsilon)]\in{}^{\rho}\widetilde{\mathbb{C}}$ is basic because of property (ii) and because if $(z_\varepsilon)\simeq_\rho(\bar{z}_\varepsilon)$, then $|f_\varepsilon'(z_\varepsilon)-f_\varepsilon'(\bar{z}_\varepsilon)|\le|z_\varepsilon-\bar{z}_\varepsilon|\,|f_\varepsilon''(w_\varepsilon)|\simeq_\rho 0$ for some w_ε in the segment $[z_\varepsilon,\bar{z}_\varepsilon]$. Using (iv), we can define $r_\varepsilon'(z,h):=\int_0^1 f_\varepsilon''(z+ht)\,\mathrm{d}t-f_\varepsilon''(z)$ for all $z\in U_\varepsilon$ and all h such that $[z,z+h]\subseteq U_\varepsilon$. Once again from (ii) it follows that r_ε defines a basic function of the type $U\times U_0'\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$. Proceeding as in (iv), we can also prove that $f_\varepsilon'(z_\varepsilon+h_\varepsilon)=f_\varepsilon'(z_\varepsilon)+h_\varepsilon\cdot f_\varepsilon''(z_\varepsilon)+h_\varepsilon\cdot r_\varepsilon'(z_\varepsilon,h_\varepsilon)$.

(vii): It follows from (vi) and (v) that f' is locally Lipschitz. The claim then follows from (iv). $\square$

Using the natural definition of differentiability and Lem. 19, it is possible to give intrinsic proof (i.e. without using nets of functions that define given basic functions) of several classical theorems of differential calculus, such as algebraic properties and chain rule.

Theorem 23. *Let $U\subseteq{}^{\rho}\widetilde{\mathbb{C}}$ be a sharply open set, f, $g:U\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$ be basic functions, $z_0\in U$, and $c\in{}^{\rho}\widetilde{\mathbb{C}}$. If f, g are ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at z_0, then*

(i) $f+g$ *is* ${}^{\rho}\widetilde{\mathbb{C}}$*-differentiable at* z_0 *and* $(f+g)'(z_0)=f'(z_0)+g'(z_0)$.

(ii) cf *is* ${}^{\rho}\widetilde{\mathbb{C}}$*-differentiable at* z_0 *and* $(cf)'(z_0)=cf'(z_0)$.

(iii) fg *is* ${}^{\rho}\widetilde{\mathbb{C}}$*-differentiable at* z_0 *and* $(fg)'(z_0)=f'(z_0)g(z_0)+f(z_0)g'(z_0)$.

(iv) *If* $g(z_0)$ *is invertible, then* $\left(\frac{f}{g}\right)'(z_0)=\frac{f'(z_0)g(z_0)-g'(z_0)f(z_0)}{g(z_0)^2}$.

To show that GHF allow to easily adapt classical proofs of certain fundamental theorems, we prove here the generalized version of the chain rule. This also shows that GHF are closed with respect to composition and hence to all holomorphic nonlinear operations:

Theorem 24. *Let $U, V \subseteq {}^{\rho}\widetilde{\mathbb{C}}$ be sharply open sets, $f : V \longrightarrow U$, $g : U \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$, and $z_0 \in V$. If f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at z_0 and g is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at $f(z_0)$, then $(g \circ f)$ is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. Since f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at z_0 and g is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at $f(z_0)$ then there exists $U_0 \subseteq {}^{\rho}\widetilde{\mathbb{C}}$ a sharply neigborhood of 0, such that $z_0 + h \in V$ and $f(z_0) + h \in U$ for all $h \in U_0$, and

$$\begin{aligned} f(z_0 + h) &= f(z_0) + f'(z_0)h + o(h), \\ g(f(z_0) + h) &= g(f(z_0)) + g'(f(z_0))h + o(h) \end{aligned}$$

as $h \to 0$. Therefore, there exist ρ-basic locally Lipschitz functions $r_1(z_0, -)$, $r_2(z_0, -) : U_0 \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ such that for all $h \in U_0$, we have

$$\begin{aligned} f(z_0 + h) &= f(z_0) + f'(z_0)h + r_1(z_0, h)h \\ g(f(z_0) + h) &= g(f(z_0)) + g'(f(z_0))h + r_2(z_0, h)h. \end{aligned}$$

Then, for all $h \in U_0$ sufficiently small such that $f(z_0 + h) \in U$, we have

$$\begin{aligned} g(f(z_0 + h)) &= g\left(f(z_0) + f'(z_0)h + r_1(z_0, h)h\right) \\ &= g(f(z_0)) + g'(f(z_0))\left(f'(z_0)h + r_1(z_0, h)h\right) + r_2(z_0, h)h \\ &= g(f(z_0)) + g'(f(z_0))f'(z_0)h + \left[g'(f(z_0))r_1(z_0, h) + r_2(z_0, h)\right]h. \end{aligned}$$

The function $r(z_0, h) := g'(f(z_0))r_1(z_0, h) + r_2(z_0, h)$, then it satisfies (2.10) of Def. 20, from which the conclusion follows. ☐

We close this section by characterizing sharply continuous functions using the little-o notations. We will use this result in the definition of path integral in [47]. Using a little abuse of language, in the following result we use the notion of strong little-o $o(1)$, which can be easily defined reformulating Def. 20 with $m_\varepsilon = 0$ and replacing $h_\varepsilon \cdot r_\varepsilon(z_\varepsilon, h_\varepsilon)$ with $r_\varepsilon(z_\varepsilon, h_\varepsilon)$, see also Def. 29 below with $N = 0$.

Lemma 25. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{C}}$ be a sharply open set, $f : U \longrightarrow {}^{\rho}\widetilde{\mathbb{C}}$ be a map and $z_0 \in U$. Then f is sharply continuous at z_0 if and only if*

$$f(z_0 + h) - f(z_0) = \bar{o}(1) \quad \text{as} \quad h \to 0.$$

The same relation holds for the strong little-o $o(1)$ if f is a basic function defined by a net $f_\varepsilon \in \mathcal{C}^0(U_\varepsilon)$ of continuous functions.

Proof. We only have to define $r_\varepsilon(z, h) := f_\varepsilon(z + h) - f_\varepsilon(z)$ for all $z \in U_\varepsilon$ and all h sufficiently small to show the necessary condition for the strong little-o, because the remaining part follows directly from Def. 17 and the relations between strong and weak little-o. ☐

2.3. **A criterion of generalized holomorphicity.** The next problem that we want to solve is an ε-wise criterion of generalized holomorphicity: If we start from an arbitrary net $f_\varepsilon : U_\varepsilon \longrightarrow \mathbb{C}$ of holomorphic function (e.g. a ρ-moderate net coming from regularizing singularities in a given PDE, see Sec. 3), when does this net define a GHF?

Theorem 26. *Let $U_\varepsilon \subseteq \mathbb{C}$ be a net of open sets, and $z_\varepsilon \in U_\varepsilon$ for all ε. Let (f_ε) be a net of functions $f_\varepsilon : U_\varepsilon \longrightarrow \mathbb{C}$ and $q \in \mathbb{R}_{>0}$. Set $z := [z_\varepsilon] \in {}^\rho\widetilde{\mathbb{C}}$, and suppose that the following properties hold:*

(i) $B_{\mathrm{d}\rho^q}(z) \subseteq \langle U_\varepsilon \rangle$.

(ii) $f_\varepsilon \in \mathcal{O}(B^{\mathrm{E}}_{\rho^q_\varepsilon}(z_\varepsilon))$ *and (f_ε) defines a basic function of the type* $B_{\mathrm{d}\rho^q}(z) \longrightarrow {}^\rho\widetilde{\mathbb{C}}$.

(iii) $\forall [c_\varepsilon] \in B_{\mathrm{d}\rho^q}(z) : \ (f'_\varepsilon(c_\varepsilon)) \in \mathbb{C}_\rho$.

Then $f := [f_\varepsilon(-)]\,|_{B_{\mathrm{d}\rho^q}(z)} \in {}^\rho\mathcal{GH}(B_{\mathrm{d}\rho^q}(z))$.

In particular, this theorem also shows that ordinary holomorphic functions on the open set $\Omega \subseteq \mathbb{C}$ are embedded as GHF with

$$f \in \mathcal{O}(\Omega) \mapsto [f(-)]_\rho \in {}^\rho\mathcal{GH}(d_f(\Omega)),$$

where $z \in d_f(\Omega)$ if and only if for some $q \in \mathbb{R}_{>0}$, conditions (i), (ii) and (iii) hold with $U_\varepsilon = \Omega$. Note that if $f \in \mathcal{O}(\Omega)$, then $d_f(\Omega) \subseteq {}^\rho\widetilde{\mathbb{C}}$ is sharply open, and it always contains finite generalized points strictly contained in Ω (usually called *compactly supported points*), i.e.:

$$\langle \Omega \rangle_{\mathrm{fin}} := \{ z \in \langle \Omega \rangle \mid z \text{ is finite, } d(z, \partial\Omega) \in \mathbb{R}_{>0} \} \subseteq d_f(\Omega).$$

Proof. We proceed by proving that f is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at z. To show that it is also ${}^\rho\widetilde{\mathbb{C}}$-differentiable at any other $w \in B_{\mathrm{d}\rho^q}(z)$ it suffices to consider a suitable ball $B_{\mathrm{d}\rho^p}(w) \subseteq B_{\mathrm{d}\rho^q}(z)$ and argue similarly. By assumption, (f_ε) defines a basic function of type $B_{\mathrm{d}\rho^q}(z) \longrightarrow {}^\rho\widetilde{\mathbb{C}}$, so that we have to focus only on the remainder functions r_ε that we define for all ε as

$$r_\varepsilon(z,h) := \int_0^1 f'_\varepsilon(z+ht)\,\mathrm{d}t - f'_\varepsilon(z) \quad \forall z \in U_\varepsilon := B^{\mathrm{E}}_{\rho^q_\varepsilon}(z_\varepsilon)\,\forall h \in U_{0\varepsilon} := B^{\mathrm{E}}_{\rho^q_\varepsilon}(0). \tag{2.17}$$

Recall that $f_\varepsilon \in \mathcal{O}(B^{\mathrm{E}}_{\rho^q_\varepsilon}(z_\varepsilon))$ by assumption. As we did above, considering the product $h_\varepsilon r_\varepsilon(z_\varepsilon, h_\varepsilon)$ and integrating, we get (2.8) and (2.9) for all $[z_\varepsilon] = z$ and all $[h_\varepsilon] = h \in B_{\mathrm{d}\rho^q}(0)$. Therefore, it remains only to prove that the net (r_ε) defines a basic function of the type $\{z\} \times B_{\mathrm{d}\rho^q}(0) \longrightarrow {}^\rho\widetilde{\mathbb{C}}$. From (iii) and (2.17), it follows that $(r_\varepsilon(z_\varepsilon, h_\varepsilon)) \in \mathbb{C}_\rho$. We want to prove that $r(w,h) := [r_\varepsilon(w_\varepsilon, h_\varepsilon)]$ is well-defined for all $w = [w_\varepsilon] \in B_{\mathrm{d}\rho^q}(z)$ and all $h \in B_{\mathrm{d}\rho^q}(0)$. Take representatives $[\bar{w}_\varepsilon] = [w_\varepsilon]$, $[\bar{h}_\varepsilon] = [h_\varepsilon]$ and let γ_ε be any counterclockwise oriented circle centered at w_ε and with radius $[s_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ contained in $B^{\mathrm{E}}_{\rho^q_\varepsilon}(z_\varepsilon)$. By Cauchy formula and the integral mean value theorem we have

$$\begin{aligned} |f'_\varepsilon(w_\varepsilon) - f'_\varepsilon(\bar{w}_\varepsilon)| &\le \frac{s_\varepsilon^{-2}}{2\pi} \int_0^{2\pi} \left| f_\varepsilon(w_\varepsilon + e^{it}) - f_\varepsilon(\bar{w}_\varepsilon + e^{it}) \right| \mathrm{d}t \\ &= s_\varepsilon^{-2} \left| f_\varepsilon(w_\varepsilon + e^{it_\varepsilon}) - f_\varepsilon(\bar{w}_\varepsilon + e^{it_\varepsilon}) \right| \\ &\le s_\varepsilon^{-2} |f'_\varepsilon(c_\varepsilon)| \cdot |w_\varepsilon - \bar{w}_\varepsilon| \end{aligned} \tag{2.18}$$

for some $t_\varepsilon \in [0, 2\pi]$ and some $c_\varepsilon \in [w_\varepsilon + e^{it_\varepsilon}, \bar{w}_\varepsilon + e^{it_\varepsilon}]$. This proves that $(f'_\varepsilon(w_\varepsilon)) \simeq_\rho (f'_\varepsilon(\bar{w}_\varepsilon))$ (and the Lipschitz constant in (2.18) depends only on f'_ε and not on higher derivatives, so we can use assumption (iii)). Similarly, we can prove independence on representatives in the first integral summand in (2.17). □

Comparing the previous Thm. 26 with [49, Thm. 2], we can note several differences:

1) [49, Thm. 2] considers only the case of holomorphicity at ordinary points $z_0 \in \Omega \subseteq \mathbb{C}$. On the contrary, in Thm. 26, in general, $\rho \neq (\varepsilon)$ and $[z_\varepsilon]_\rho$ can also be, e.g., an infinite generalized number.
2) In assumption (iii), we require that only the first derivative is moderate and not estimates of the type $\left|f_\varepsilon^{(n)}(z_0)\right| \le n!\eta^{n+1}\varepsilon^{-Q}$ (ε small) with $\eta \in \mathbb{R}_{>0}$ for all derivatives. On the one hand, this is due to our effort to start from a first order differentiability condition (2.8), that allows us to prove (ii) and (vi) of Thm. 22. On the other hand, as we already showed in the proof of Thm. 22.(ii), we also have an estimate of the form $|f_\varepsilon^{(n)}(z_\varepsilon)| \le n!a_\varepsilon^{-n}\sup_{z\in\gamma}|f_\varepsilon(z)| \le n!(\rho_\varepsilon^{-Q})^n\rho_\varepsilon^{-R}$, but the term $\eta = \rho_\varepsilon^{-Q}$ cannot usually be replaced by a finite number.
3) This allows us to include a larger class of GHF with respect to the classical Colombeau theory (in particular, the Dirac delta, see Sec. 3 below). One could also conceptually describe the present approach as follows: depending on a given problem, let (f_ε) be a ρ-moderate net of holomorphic functions obtained by regularizing the singularities of a given problem; find a gauge ρ that satisfies the assumption of Thm. 26, then $\overline{f} := [f_\varepsilon(-)]_\rho$ is a GHF that can be studied in the present theory.

In general, it could happen that z_ε approaches ∂U_ε too quickly with respect to ρ, so that (z_ε) does not yield an interior point in the ρ-sharp topology. This problem is solved by changing the gauge into a smaller one $\sigma_\varepsilon \le \min\{\rho_\varepsilon, d(z_\varepsilon, \partial U_\varepsilon)\}$, which would hence allow us to measure smaller but still σ-invertible infinitesimal numbers and thus greater infinite numbers.

Definition 27. Let σ, ρ be two gauges, then we say $\sigma \le \rho$ if $\forall^0\varepsilon: \ \sigma_\varepsilon \le \rho_\varepsilon$.

The relation $\le$ is reflexive, transitive, and antisymmetric in the sense that $\sigma \le \rho$ and $\rho \le \sigma$ imply $\sigma_\varepsilon = \rho_\varepsilon$ for ε small. Clearly, $\sigma \le \rho$ implies the inclusion of ρ-moderate nets $\mathbb{R}_\rho \subseteq \mathbb{R}_\sigma$. We now need to link ρ-moderate numbers in ${}^\sigma\widetilde{\mathbb{C}}$ with those in ${}^\rho\widetilde{\mathbb{C}}$: If $\sigma \le \rho$ and $U \subseteq {}^\rho\widetilde{\mathbb{C}}$, we define ${}^\sigma_\rho U := \left\{[x_\varepsilon]_\sigma \in {}^\sigma\widetilde{\mathbb{C}} \mid (x_\varepsilon) \in \mathbb{C}_\rho \text{ and } [x_\varepsilon]_\rho \in U\right\}$ as the set of ${}^\sigma\widetilde{\mathbb{C}}$ numbers which are ρ-moderate (recall that $[x_\varepsilon]_\sigma$ is the equivalent class with respect to the relation $\sim_\sigma$ as in Def. 1). We can also well-define a natural map $\iota : [x_\varepsilon]_\sigma \in {}^\sigma_\rho\widetilde{\mathbb{C}} \mapsto [x_\varepsilon]_\rho \in {}^\rho\widetilde{\mathbb{C}}$ because $\sigma \le \rho$. This map is surjective but generally not injective, even if $\iota(x) = \iota(y)$ implies $|x - y| \le [\rho_\varepsilon]_\sigma^q$ for all $q \in \mathbb{R}_{\ge 0}$.

For instance, if $\rho = (\varepsilon)$ and $\sigma = \left(e^{-\frac{1}{\varepsilon}}\right)$, then a generalized number of the form $\left[2^{-\frac{1}{\varepsilon}}\right]_\rho = 0$ but $\left[2^{-\frac{1}{\varepsilon}}\right]_\sigma > 0$. However, if $[x_\varepsilon]_\sigma \in {}^\sigma_\rho\widetilde{\mathbb{C}}$ and $[x_\varepsilon]_\rho > 0$ then $[x_\varepsilon]_\sigma > 0$.

Furthermore, we denote by ρ and σ two arbitrary gauges; only when it will be needed, we will assume a relation between them, such as $\sigma \le \rho$.

In the following result, we show that a representative of a ρ-basic locally Lipschitz function also defines a σ-basic locally Lipschitz function,

Lemma 28. *Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$. Let (U_ε) be a net of open subsets of $\mathbb{C}$ such that $U \subseteq \langle U_\varepsilon\rangle$ and (f_ε) be a net of functions $f_\varepsilon : U_\varepsilon \longrightarrow \mathbb{C}$. If (f_ε) defines a ρ-basic locally*

Lipschitz function $f : U \longrightarrow {}^\rho\widetilde{\mathbb{C}}$*, then for all* $\sigma \le \rho^Q$*, for some* $Q \in \mathbb{R}_{>0}$*, the net* (f_ε) *defines a* σ*-basic locally Lipschitz map* $\overline{f} := [f_\varepsilon(-)] : {}^\sigma_\rho U \longrightarrow {}^\sigma\widetilde{\mathbb{C}}$*.*

Proof. Let $[z_\varepsilon]_\sigma = \overline{z} \in {}^\sigma_\rho U$. Then there exists $(x_\varepsilon) \in \mathbb{C}_\rho$ such that $[x_\varepsilon]_\sigma = \overline{z} = [z_\varepsilon]_\sigma$ and $[x_\varepsilon]_\rho \in U$. Therefore, we have $[f_\varepsilon(x_\varepsilon)]_\rho \in {}^\rho\widetilde{\mathbb{C}}$ and $(f_\varepsilon(x_\varepsilon)) \in \mathbb{C}_\rho \subseteq \mathbb{C}_\sigma$, hence $[f_\varepsilon(x_\varepsilon)]_\sigma \in {}^\sigma\widetilde{\mathbb{C}}$. By (2.2), there exist $L = [L_\varepsilon]_\rho$, $r = [r_\varepsilon]_\rho \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $|f(x) - f(y)| \le L \cdot |x - y|$ whenever $x, y \in B_r(z)$. By Lem. 3, there exist $N, \overline{q} \in \mathbb{N}$ such that $L_\varepsilon \le \rho_\varepsilon^{-N}$ and $r_\varepsilon \ge \rho_\varepsilon^{\overline{q}}$, for small ε. Since $(x_\varepsilon) \sim_\sigma (z_\varepsilon)$ and $\sigma \le \rho$, then for sufficiently small ε, we have

$$|f_\varepsilon(x_\varepsilon) - f_\varepsilon(z_\varepsilon)| \le L_\varepsilon |x_\varepsilon - z_\varepsilon| \le \rho_\varepsilon^{-N} \cdot \sigma_\varepsilon^{\overline{q}} \le \sigma_\varepsilon^q \quad \forall q \ge \overline{q} - N.$$

This implies $[f_\varepsilon(z_\varepsilon)]_\sigma = \overline{f}(\overline{z}) = [f_\varepsilon(x_\varepsilon)]_\sigma$, and hence conditions (i) and (ii) follow. To prove (2.2), by setting $\overline{L} := [L_\varepsilon]_\sigma$ and $\overline{r} := [r_\varepsilon]_\sigma$, we have $\overline{L}, \overline{r} \in {}^\sigma\widetilde{\mathbb{R}}_{>0}$, and $|\overline{f}(x) - \overline{f}(y)| \le \overline{L} \cdot |x - y|$ for every $x, y \in {}^\sigma_\rho U \cap B_{\overline{r}}(\overline{z})$. □

A simple case of this construction is the exponential map

$$e^{(-)} : [z_\varepsilon] \in \left\{ z \in {}^\rho\widetilde{\mathbb{C}} \mid \exists c \in {}^\rho\widetilde{\mathbb{R}}_{>0} : |z| \le \log c \right\} \mapsto [e^{z_\varepsilon}] \in {}^\rho\widetilde{\mathbb{C}}.$$

The domain of this map depends on the infinitesimal net ρ. For instance, if $\rho = (\varepsilon)$ then all its points are bounded by generalized numbers of the form $[-N \log \varepsilon]$, $N \in \mathbb{N}$; whereas if $\rho = \left(e^{-\frac{1}{\varepsilon}}\right)$, all points are bounded by $\left[N\varepsilon^{-1}\right]$, $N \in \mathbb{N}$. However, if we consider a gauge $\sigma \le \rho$ such that $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{\frac{1}{\varepsilon}}\right)$, then $\overline{e}^{(-)} : [z_\varepsilon] \in {}^\sigma_\rho\widetilde{\mathbb{C}} \mapsto [e^{z_\varepsilon}] \in {}^\sigma\widetilde{\mathbb{C}}$ is well defined.

2.4. **The Cauchy-Riemann equations.** In this section, we want to prove that these GHF satisfy the Cauchy-Riemann equations. We start from the definitions of ${}^\rho\widetilde{\mathbb{R}}$-differentiability and ${}^\rho\widetilde{\mathbb{R}}$-partial differentiability. Even if in this series of papers we will apply these definitions mainly to the real and imaginary parts of a GHF (which are GSF, and hence all partial derivatives are well-defined), these notions can be introduced independently from the theory of GSF using the language of strong little-o. Moreover, notions of differentiability which are more general and independent from GSF theory are used in the present work to extend Goursat, Looman-Menchoff and Montel theorems to GHF.

We already pointed out, see e) of Rem. 21 and Thm. 22, the importance of using the sheaf $\mathcal{O}(-)$ of holomorphic functions in Def. 20. Also in Lem. 25, in the definition of strong $o(1)$, it is clear that we are using the same idea, but with the sheaf $\mathcal{C}^0(-)$ of continuous functions and, instead of a first order Taylor difference $f_\varepsilon(z_\varepsilon + h_\varepsilon) - f_\varepsilon(z_\varepsilon) - h_\varepsilon \cdot m_\varepsilon$, a 0-order Taylor difference. The following Def. 30 of ${}^\rho\widetilde{\mathbb{R}}$-(partially) differentiable maps follows the same idea, but using the sheaf of (partially) differentiable maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and suitable Taylor (partial) differences.

In general, $T_N(u, z, h, (m_j)_j)$ denotes a (possibly partial) Taylor difference (i.e. the function u evaluated at an incremented argument minus the corresponding Taylor polynomial) of order $N \in \mathbb{N}$ of the function u at the point z, with increment h (in some direction) and coefficients $(m_j)_{j=1}^N$, and we have:

Definition 29. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $U \subseteq {}^\rho\widetilde{\mathbb{K}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{K}}^d$, T_N be a Taylor difference of order N. Let $S(X) \subseteq (\mathbb{R}^d)^X$ be a family of maps for all open subsets $X \subseteq \mathbb{K}^n$ (e.g. a sheaf). Then, we write $u \in {}^\rho\mathcal{G}(U, Y; S(-), T_N)$ if $u : U \longrightarrow Y$ is a basic function and there exist $U_0 \subseteq {}^\rho\widetilde{\mathbb{K}}^n$ and nets $u_\varepsilon \in S(U_\varepsilon)$, $r_\varepsilon : U_\varepsilon \times U_{0\varepsilon} \longrightarrow \mathbb{K}^d$ such that:

(i) U_0 is a sharp neighborhood of 0.

(ii) (u_ε) defines u.

(iii) (r_ε) defines a basic function $r : U \times U_0 \longrightarrow {}^\rho\widetilde{\mathbb{K}}^d$.

Moreover, the following properties hold for all sharply interior point $z \in U$:

(iv) ${}^\rho\lim_{h\to 0} r(z,h) = 0$ and there exists $[m_{1\varepsilon}] = m_1, \dots, [m_{N\varepsilon}] = m_N \in {}^\rho\widetilde{\mathbb{K}}^d$ such that for all representatives $[z_\varepsilon] = z \in U$ and all $[h_\varepsilon] \in U_0$, we have

$$\forall^0\varepsilon : \; T_N(u_\varepsilon, z_\varepsilon, h_\varepsilon, (m_{j\varepsilon})_j) = h_\varepsilon^N \cdot r_\varepsilon(z_\varepsilon, h_\varepsilon). \tag{2.19}$$

$$r_\varepsilon(z_\varepsilon, 0) = 0 = \lim_{h\to 0} r_\varepsilon(z_\varepsilon, h). \tag{2.20}$$

(v) If $\partial^\alpha(-)$, with $\alpha \in \mathbb{N}^n$ and $|\alpha| = N$, is the highest derivative considered in the Taylor difference T_N, then the map $[z_\varepsilon] \in U \mapsto [\partial^\alpha u_\varepsilon(z_\varepsilon)] \in {}^\rho\widetilde{\mathbb{K}}^d$ is well-defined and sharply continuous.

If conditions (i) - (iv) hold, we simply write

$$T_N(u, z, h, (m_j)_j) = o(h^N) \text{ as } h \to 0 \text{ in } z \in U,$$

and we say that u is *T_N-regular with representatives $u_\varepsilon \in S(U_\varepsilon)$ (and with respect to ${}^\rho\widetilde{\mathbb{K}}^n$* in case we have to underscore the dependence from ${}^\rho\widetilde{\mathbb{K}}^n$). Usually, specifying the family $S(X)$ it is already implicitly clear what is the Taylor difference T_N we are considering. As already proved in Thm. 22, in case of GHF, the last condition (v) can be proved from the remaining ones.

We have the following examples:

Definition 30. Let $V \subseteq U \subseteq {}^\rho\widetilde{\mathbb{R}}^2$ be a sharply open set, $f : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}^2$ be a basic function, $\mathrm{Diff}(\bar{U})$ be the family of all differentiable functions on the open set $\bar{U} \subseteq \mathbb{R}^2$, and $\mathrm{PDiff}(\bar{U})$ be the family of all partially differentiable on the open set $\bar{U} \subseteq \mathbb{R}^2$. Then,

(i) A function u is said to be *${}^\rho\widetilde{\mathbb{R}}$-differentiable on V* if for all $(x,y) \in V$ there exist a linear ρ-basic function $D_{(x,y)} : {}^\rho\widetilde{\mathbb{R}}^2 \longrightarrow {}^\rho\widetilde{\mathbb{R}}^2$ satisfying the condition $u(x+h_1, y+h_2) - u(x,y) - D_{(x,y)}(h) = o(h)$ as $h = (h_1, h_2) \to 0$ and with representatives in $\mathrm{Diff}(\bar{U}_\varepsilon)$, where $U \subseteq \langle \bar{U}_\varepsilon \rangle$. This unique $D_{(x,y)}$ (see Rem. 32.b) below) is called the differential of f at (x,y) and is denoted by $f'(x,y)$ or $\mathrm{d}f(x,y)$.

(ii) A function f is said to be ${}^\rho\widetilde{\mathbb{R}}$-*partially differentiable* with respect to x (resp. y) on V if for all $(x,y) \in V$ there exist m_x (resp. m_y)$\in {}^\rho\widetilde{\mathbb{R}}$ such that $f(x+h, y) = f(x,y) + h \cdot m_x + o(h)$ as $h \to 0$ (resp. $f(x+h, y) = f(x+y) + h \cdot m_x + o(h)$ as $h \to 0$) and with representatives in $\mathrm{PDiff}(\bar{U}_\varepsilon)$, where $U \subseteq \langle \bar{U}_\varepsilon \rangle$. We denote the ${}^\rho\widetilde{\mathbb{R}}$-partial derivative of f with respect to x (resp. y) as $\frac{\partial f}{\partial x}(x,y) := \partial_1 f(x,y) := \partial_x f(x,y) := m_x$ (or $\frac{\partial f}{\partial y}(x,y) := \partial_2 f(x,y) := \partial_2 f(x,y) := m_y$).

The following lemma is essentially a reformulation of Def. 20.

Lemma 31. *Let U, $Y \subseteq {}^\rho\widetilde{\mathbb{C}}$, $V \subseteq U$, $u : U \longrightarrow Y$ be a basic function. Then, u is* ${}^\rho\widetilde{\mathbb{C}}$-differentiable *on V if and only if there exists a ρ-basic function $m : U \longrightarrow {}^\rho\widetilde{\mathbb{C}}$ satisfying the condition for all $z \in V$, $u(z+h) - u(z) - m(z) \cdot h = o(h)$ as $h \to 0$ and with representatives $u_\varepsilon \in \mathcal{O}(\bar{U})$, where $U \subseteq \langle \bar{U} \rangle$.*

Remark 32.

a) Linear ρ-basic functions $D : {}^{\rho}\widetilde{\mathbb{R}}^2 \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^2$ are precisely the ρ-basic functions that can be written as $D(z) = Az$, where $A \in M_{2\times 2}({}^{\rho}\widetilde{\mathbb{R}})$ is a matrix with entries in the ring ${}^{\rho}\widetilde{\mathbb{R}}$. That is, each linear ρ-basic function D is a multiplication by a unique $A \in M_{2\times 2}({}^{\rho}\widetilde{\mathbb{R}})$ and conversely.

b) The only weak $\bar{o}(h)$-linear function D on ${}^{\rho}\widetilde{\mathbb{R}}^2$, i.e. such that it can be written as $D(h) = h \cdot r(h)$ with $r(h) \to 0$, is the zero function. In fact, assume that $D(h_0) \neq 0$ for some $h_0 \in {}^{\rho}\widetilde{\mathbb{R}}^2$. From (1.4), there exists $L \subseteq_0 I$ and $r \in \mathbb{N}$ such that $|D(h_0)| >_L 0$ is invertible. We cannot have $h_0 =_L 0$ because, otherwise, $D(h_0) =_L 0$ by linearity. Once again from (1.4) (applied to the ring ${}^{\rho}\widetilde{\mathbb{R}}|_L$) there exists $K \subseteq_0 L$ such that both $|h_0| >_K 0$ and $|D(h_0)| >_K 0$ are invertible. For all invertible $\alpha \in {}^{\rho}\widetilde{\mathbb{R}}|_K$, we have
$$D(\alpha h_0) =_K \alpha D(h_0) =_K \alpha h_0 r(\alpha h_0),$$
and we get the contradiction $|D(h_0)|/|h_0| =_K |r(\alpha h_0)| \to 0$ as $\alpha \to 0$ in ${}^{\rho}\widetilde{\mathbb{R}}|_K$.

In the following, we prove the equivalent condition for differentiability on partial derivatives. Thanks to our language of little-os, the proof is formally identical to the classical one, and we hence omit it.

Theorem 33. *Let $V \subseteq {}^{\rho}\widetilde{\mathbb{R}}^2$ be a sharply open set and $z = (z_1, z_2) \in V$*

(i) *If $f := (f_1, f_2) : V \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^2$ be a ${}^{\rho}\widetilde{\mathbb{R}}$-differentiable at z, then for i,$j \in \{1, 2\}$, there exists the ${}^{\rho}\widetilde{\mathbb{R}}$-partial derivative $\partial_j f_i$ and is sharply continuous. Furthermore,*
$$f'(z) = \begin{bmatrix} \partial_1 f_1(z) & \partial_2 f_1(z) \\ \partial_1 f_2(z) & \partial_2 f_2(z) \end{bmatrix}.$$

(ii) *If $f := (f_1, f_2) : V \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^2$ be a basic function. Suppose that for each i, $j \in \{1, 2\}$, the ${}^{\rho}\widetilde{\mathbb{R}}$-partial derivative $\partial_j f_i$ exists on some neighbourhood of z and the ${}^{\rho}\widetilde{\mathbb{R}}$-partial derivative $\partial_j f_i$ is sharply continuous at z. Then f is ${}^{\rho}\widetilde{\mathbb{R}}$-differentiable at z.*

We can finally prove the CRE, and the related Goursat-like theorems. The proof of the CRE is again formally identical to the classical one. For Goursat theorem, we need to take representatives in
$$\mathrm{CRE}(U) := \{u \in \mathrm{PDiff}(U) \mid \partial_j u_k \text{ satisfy the CRE, } j, k = 1, 2\}.$$

Theorem 34 (Cauchy-Riemann equations and Goursat theorem)**.** *Consider a sharply open set $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^2$, set $\overline{U} := \left\{z \in {}^{\rho}\widetilde{\mathbb{C}} : (\mathrm{Re}(z), \mathrm{Im}(z)) \in U\right\}$ and take $(x_0, y_0) \in U$. Let u, $v : U \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}^2$ be maps, and for all $z = x + iy \in \overline{U}$ set $f(z) := u(x, y) + iv(x, y)$. Then, we have:*

(i) *If f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at $z_0 := x_0 + iy_0$, then u, v are sharply continuously ${}^{\rho}\widetilde{\mathbb{R}}$-differentiable at (x_0, y_0) and satisfy the CRE:*
$$\partial_1 u = \partial_2 v \quad \textit{and} \quad \partial_2 u = -\partial_1 v. \tag{2.21}$$

(ii) *Let u, v be ${}^{\rho}\widetilde{\mathbb{R}}$-partially differentiable functions on U with representatives in*
$$\left\{u \in \mathrm{CRE}(U_\varepsilon) \cap \mathcal{C}^0(U_\varepsilon) \mid \partial_j u_k \in \mathcal{C}^0(U_\varepsilon),\ j, k = 1, 2\right\}, \tag{2.22}$$
then $f \in {}^{\rho}\mathcal{GH}(U)$.

(iii) *Let u, v be ${}^{\rho}\widetilde{\mathbb{R}}$-partially differentiable functions on $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^2$ with representatives in* $\mathrm{CRE}(U_\varepsilon) \cap \mathrm{Diff}(U_\varepsilon)$*, then $f \in {}^{\rho}\mathcal{GH}(U)$.*

Proof. (i): Assume that f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at z_0. By Thm. 33, the partial derivatives $\partial_j u$ and $\partial_j v$ ($j = 1,\ 2$) exist and are sharply continuous. Since f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at z_0, then there exists $f'(z_0) \in {}^{\rho}\widetilde{\mathbb{C}}$ satisfying

$$f(z_0 + h) = f(z_0) + m \cdot h + o(h) \quad \text{as} \quad h \to 0.$$

Taking $h = h_1 \in {}^{\rho}\widetilde{\mathbb{R}}$, we have

$$\begin{aligned} f(z_0 + h_1) - f(z_0) &= u(x_0 + h_1, y_0) + iv(x_0 + h_1, y_0) - u(x_0, y_0) - iv(x_0, y_0) \\ &= u(x_0 + h_1, y_0) - u(x_0, y_0) + i\left[v(x_0 + h_1, y_0) - v(x_0, y_0)\right] \\ &= \partial_1 u(x_0, y_0) h_1 + o(h_1) + i\left[v(x_0, y_0) h_1 + o(h_1)\right] \\ &= \left[\partial_1 u(x_0, y_0) + i\partial_1 v(x_0, y_0)\right] h_1 + o(h_1) \quad \text{as} \quad h_1 \to 0. \end{aligned}$$

Since the derivative is unique, then

$$f'(z_0) = \partial_1 u(x_0, y_0) + i\partial_1 v(x_0, y_0). \tag{2.23}$$

Taking $h = ih_2$, $h_2 \in {}^{\rho}\widetilde{\mathbb{R}}$ we have

$$\begin{aligned} f(z_0 + ih_2) - f(z_0) &= u(x_0, y_0 + h_2) + iv(x_0, y_0 + h_2) - u(x_0, y_0) - iv(x_0, y_0) \\ &= u(x_0, y_0 + h_2) - u(x_0, y_0) + i\left[v(x_0, y_0 + h_2) - v(x_0, y_0)\right] \\ &= \partial_2 u(x_0, y_0) h_2 + o(h_2) + i\left[v(x_0, y_0) h_2 + o(h_2)\right] \\ &= \left[\partial_2 u(x_0, y_0) + i\partial_2 v(x_0, y_0)\right] h_2 + o(h_2) \quad \text{as} \quad h_2 \to 0. \end{aligned}$$

On the other hand, we have

$$f(z_0 + ih_2) - f(z_0) = f'(z_0) ih_2 + o(h_2) \quad \text{as} \quad h_2 \to 0.$$

Since the derivative is unique, then

$$f'(z_0) = \frac{1}{i}\left(\partial_2 u(x_0, y_0) + i\partial_2 v(x_0, y_0)\right). \tag{2.24}$$

By (2.23) and (2.24), we hence have

$$\partial_1 u(x_0, y_0) + i\partial_1 v(x_0, y_0) = \partial_2 v(x_0, y_0) - i\partial_2 u(x_0, y_0)$$

from which the conclusion follows.
(ii): In order to prove that $f \in {}^{\rho}\mathcal{GH}(U)$, we have to show that $f_\varepsilon \in \mathcal{O}(U_\varepsilon)$ (and here we clearly have to use the classical Goursat theorem, see e.g. [30, Thm. 1]), and the strong little-o differentiability condition (2.10).

To prove that $f_\varepsilon \in \mathcal{O}(U_\varepsilon)$, take representatives $u = [u_\varepsilon(-)]$ and $v = [v_\varepsilon(-)]$ in (2.22). Then, $\partial_j u_\epsilon$ and $\partial_j v_\varepsilon$ exist and are continuous in U_ε, they satisfy there the CRE, and $f_\varepsilon := u_\varepsilon + iv_\varepsilon$ is continuous because u_ε, $v_\varepsilon \in \mathcal{C}^0(U_\varepsilon)$. Therefore, for the classical Goursat theorem [30, Thm. 1], we have the conclusion.

To show (2.10), by Thm. 33, we have that the ${}^{\rho}\widetilde{\mathbb{R}}$-partial derivatives $\partial_j u$ and $\partial_j v$ exist at (x_0, y_0), $j = 1, 2$, and satisfy the CRE because e.g. $\partial_j u(x, y) = [\partial_j u_\varepsilon(x_\varepsilon, y_\varepsilon)]$ and we assumed that u_ε, $v_\varepsilon \in \mathrm{CRE}(U_\varepsilon)$, where $U \subseteq \langle U_\varepsilon \rangle$. Hence, for all $h = h_1 + ih_2 \in {}^{\rho}\widetilde{\mathbb{C}}$ sufficiently small, by applying Lemma 19 and the CRE, we obtain

$$
\begin{aligned}
f(z_0+h) &= u(x_0+h_1,y_0+h_2)+iv(x_0+h_1,y_0+h_2)\\
&= [u(x_0,y_0)+\partial_1 u(x_0,y_0)h_1+\partial_2 u(x_0,y_0)h_2+o(h)]\\
&\quad +i\,[v(x_0,y_0)+\partial_1 v(x_0,y_0)h_1+\partial_2 v(x_0,y_0)h_2+o(h)]\\
&= [u(x_0,y_0)+iv(x_0,y_0)]+[\partial_1 u(x_0,y_0)+i\partial_1 v(x_0,y_0)]\,h_1\\
&\quad +[\partial_2 u(x_0,y_0)+i\partial_2 v(x_0,y_0)]\,h_2+o(h)\\
&= [u(x_0,y_0)+iv(x_0,y_0)]+[\partial_1 u(x_0,y_0)-i\partial_2 u(x_0,y_0)]\,h_1\\
&\quad +[\partial_1 u(x_0,y_0)-i\partial_2 u(x_0,y_0)]\,ih_2+o(h)\\
&= f(z_0)+[\partial_1 u(x_0,y_0)-i\partial_2 u(x_0,y_0)]\,h+o(h)\text{ as }h\to 0. \quad (2.25)
\end{aligned}
$$

(iii): We can proceed as above using [30, Thm. 2]. □

Proceeding like in these extensions of Goursat theorem and using the classical Looman-Menchof theorem [30, Thm. 3], we obtain the following

Theorem 35 (Looman-Menchof)**.** *In the general assumptions of Thm. 34, let* u*,* v *be* ${}^\rho\widetilde{\mathbb{R}}$*-partially differentiable functions on* U *with representatives in* $\mathrm{CRE}(U_\varepsilon)\cap\mathcal{C}^0(U_\varepsilon)$*, then* $f\in{}^\rho\mathcal{GH}(U)$*.*

If we only assume that u, v are ${}^\rho\widetilde{\mathbb{R}}$-partially differentiable functions on U with representatives in $\mathrm{CRE}(U_\varepsilon)$ and f is sharply bounded on a closed ball, we obtain the following generalized version of Montel theorem:

Theorem 36 (Montel)**.** *In the general assumptions of the previous Thm. 34, let* u*,* v *be* ${}^\rho\widetilde{\mathbb{R}}$*-partially differentiable functions on* U *with representatives in* $\mathrm{CRE}(U_\varepsilon)$*, and assume that* f *is sharply bounded on the closed ball* $\overline{B}_r(z_0)\subseteq U$*, i.e.*

$$\exists M\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall z\in\overline{B}_r(z_0):\ |f(z)|<M. \tag{2.26}$$

Then $f\in{}^\rho\mathcal{GH}(B_r(z_0))$*.*

Proof. In fact, take representatives u_ε, $v_\varepsilon\in\mathrm{CRE}(U_\varepsilon)$ of u, v resp., and set $f_\varepsilon:=u_\varepsilon+iv_\varepsilon$. Let $M=[M_\varepsilon]$ from (2.26). Proceeding by contradiction as in (2.11), we can prove that

$$\forall^0\varepsilon\,\forall z\in B^{\mathrm{E}}_{r_\varepsilon}(z_{0\varepsilon}):\ |f_\varepsilon(z)|<M_\varepsilon.$$

For these small ε, we can hence apply the classical Montel theorem to prove that $f_\varepsilon\in\mathcal{O}(B^{\mathrm{E}}_{r_\varepsilon}(z_{0\varepsilon}))$. We can finally argue as in (2.25). □

3. Embedding of distributions and other examples

The question of embedding suitable subspaces of $\mathcal{C}^0(\Omega)$, $\mathcal{C}^\infty(\Omega)$, $\mathcal{D}'(\Omega)$ (where $\Omega\subseteq\mathbb{R}^2$ is an open subset) in spaces of GHF arises naturally. The embedding of classical holomorphic functions $\mathcal{O}(\Omega)$ into GHF has already been considered just after Thm. 26 by the canonical map

$$\sigma:\mathcal{O}(\Omega)\to{}^\rho\mathcal{GH}(\langle\Omega\rangle_{\text{fin}})\quad f\mapsto[f(-)]_\rho \tag{3.1}$$

which is an injective homomorphism of algebras.

As we already mentioned above, even the embedding of Dirac delta δ and Heaviside function H would allow us to consider powers δ^k, H^h, k, $h\in\mathbb{N}$, and compositions such as $\delta\circ\delta$, $\delta^k\circ H^h$, $H^h\circ\delta^k$.

In this section, we emphasize that the type of mollification we need to perform depends on the regularization properties we aim to obtain for our problem at hand:

a) For example, possible standard holomorphic terms of our PDE are naturally embedded using (3.1).
b) Compactly supported distributions are embedded using a simple convolution with an *entire* mollifier μ (see Thm. 37 below).
c) More general distributions can be embedded by gluing together embedding of compactly supported ones by means of general sheaf theoretical methods. For this aim, we necessarily need to preserve supports, and this is guaranteed by considering a rapidly decreasing mollifier $\mu \in \mathcal{S}(\mathbb{R}^2)$, see Thm. 38.
d) If we need that the canonical embedding (3.1) coincides with the distributional one, it suffices to consider a mollifier with all null moments, see Thm. 38.
e) If we already know that our generalized functions satisfy the distributional CRE, or if we necessarily need to use a compactly supported mollifier k, we can use Thm. 39.
f) If we need results that do not depend on the used entire mollifier μ, this amounts to fixing the non-Archimedean properties of the embedding of the Dirac delta at the origin, see Thm. 41.

In other words, the present article introduces a mathematical structure of the form $(\rho, {}^{\rho}\mathcal{GH}(-), \mu, k)$, where the free variables ρ, μ and k have to be chosen depending on the needs we have; stating that some gauge ρ or mollifiers μ or k are always "better" than others risk to introduce a bias for suitable problems. See also Sec. 4 for a more complete account on this idea.

The first idea is to associate a net of holomorphic functions (f_ε) to a given distribution by a regularization process using an *entire* function $\mu \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \mu(x)\,\mathrm{d}x = 1. \tag{3.2}$$

For example, we can take $\mu := \mathcal{F}^{-1}(\beta) \in \mathcal{S}(\mathbb{R}^2)$ as the inverse Fourier transform of any smooth compactly supported function $\beta \in \mathcal{D}(\mathbb{R}^2)$ with $\beta(0) = 1$. The Schwartz-Paley-Wiener theorem implies that μ is an entire function that can be analytically extended to the whole complex space. We then define a δ-net as an approximation to the identity by

$$\mu_\varepsilon(z) := \rho_\varepsilon^{-2} \cdot \mu(\rho_\varepsilon^{-1} \cdot z) \quad \forall z \in \mathbb{C}\, \forall \varepsilon. \tag{3.3}$$

For every compactly supported complex valued distribution $T = (T_1, T_2) \in \mathcal{E}'(\Omega, \mathbb{C})$, the convolution $T * \mu_\varepsilon$ is well-defined and given by

$$(T * \mu_\varepsilon)(z) = \langle T, \mu_\varepsilon(z - \cdot)\rangle \quad \forall z \in \mathbb{C}.$$

This convolution defines an entire holomorphic function since T is compactly supported and μ_ε is an entire holomorphic function, see e.g. [56, Exercise 27.2 (4)]. Using a little abuse of language, we call *entire mollifier* any function $\mu \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that (3.2) (without any requirement about its support, positivity or symmetry).

Theorem 37. *Let $\mu \in \mathcal{C}^\infty(\mathbb{R}^2)$ be a entire mollifier, define the ρ-approximation to the identity (μ_ε) as above in (3.3). Then for each non-empty open set $\Omega \subseteq \mathbb{R}^2$, the map*

$$\iota_0 : T \in \mathcal{E}'(\Omega, \mathbb{C}) \mapsto [(T * \mu_\varepsilon)(-)] \in {}^{\rho}\mathcal{GH}(\langle\Omega\rangle_{\text{fin}})$$

is a linear monomorphism. Moreover, the embedding ι_0 preserves derivatives:

$$\iota_0(\partial^\alpha T) = \partial^\alpha \iota_0(T) \quad \forall \alpha \in \mathbb{N}^2, \tag{3.4}$$

and hence for all $z=x_0+iy_0\in{}^{\rho}\widetilde{\mathbb{C}}$*, we have*

$$\begin{aligned}\iota_0(T)'(z)&=\iota_0(\partial_1 T_1)(x_0,y_0)+i\cdot\iota_0(\partial_1 T_2)(x_0,y_0)\\&=\iota_0(\partial_2 T_2)(x_0,y_0)-i\cdot\iota_0(\partial_2 T_1)(x_0,y_0).\end{aligned}\tag{3.5}$$

Proof. By Thm. 26, we have to show that for all $z=[z_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{C}}$, $T*\mu_\varepsilon$ defines a basic function of the type ${}^{\rho}\widetilde{\mathbb{C}}\longrightarrow{}^{\rho}\widetilde{\mathbb{C}}$, and $((T*\mu_\varepsilon)'(z_\varepsilon))\in\mathbb{C}_\rho$ for all $[z_\varepsilon]\in\langle\Omega\rangle_{\text{fin}}$. Since T is compactly supported, we can write $T=\sum_{|\alpha|\le m}\partial^\alpha g_\alpha$ for $m\in\mathbb{N}$ and compactly supported functions $g_\alpha\in\mathcal{C}^0(\mathbb{R}^2)$. We therefore have

$$\begin{aligned}(T*\mu_\varepsilon)(z_\varepsilon)&=\sum_{|\alpha|\le m}(-1)^{|\alpha|}\int_{\mathbb{R}^2}g_\alpha(x)\partial^\alpha\mu_\varepsilon(z_\varepsilon-x)\,\mathrm{d}x\\&=\sum_{|\alpha|\le m}(-1)^{|\alpha|}\int_{\mathbb{R}^2}g_\alpha(z_\varepsilon-y)\partial^\alpha\mu_\varepsilon(y)\,\mathrm{d}x\\&=\sum_{|\alpha|\le m}(-1)^{|\alpha|}\int_{\mathbb{R}^2}g_\alpha(z_\varepsilon-\rho_\varepsilon w)\rho_\varepsilon^{-|\alpha|}\partial^\alpha\mu(w)\,\mathrm{d}w\\&=\mathcal{O}(\rho_\varepsilon^{-m}).\end{aligned}\tag{3.6}$$

Since $(T*\mu_\varepsilon)'=T*\mu_\varepsilon'$, the same argument applies to the derivatives. Finally, from (3.6) it also follows independence from representatives since the same property holds for $[\mu_\varepsilon(\cdot-x)]$. To show injectivity, assume that $[(T*\mu_\varepsilon)(-)]=0$. Since $T*\mu_\varepsilon\to T$ in $\mathcal{D}'(\Omega)$, in order to prove that $T=0$, it suffices to demonstrate that $T*\mu_\varepsilon\to 0$ as $\varepsilon\to 0$ uniformly on compact subsets $L\Subset\Omega$. Suppose this were not the case, so that we could find some compact $L\Subset\Omega$, some $c>0$, a sequence $\varepsilon_k\downarrow 0$ and $z_k\in L$ such that $|(T*\mu_\varepsilon)(z_k)|\ge c$ for all k. Fixing any $z\in\Omega$ and setting $z_\varepsilon:=z_k$ for $\varepsilon=\varepsilon_k$ and $z_\varepsilon=z$ otherwise, we define an element $[z_\varepsilon]\in\langle\Omega\rangle_{\text{fin}}$ with $[(T*\mu_\varepsilon)(z_\varepsilon)]\ne 0$, a contradiction.

For (3.4), we have

$$\iota_0(\partial^\alpha T)_\varepsilon=(\partial^\alpha T)*\mu_\varepsilon=\partial^\alpha(T*\mu_\varepsilon)=\partial^\alpha\iota_0(T)_\varepsilon.$$

Therefore, (3.5) follows from (3.4) and the CRE for GHF, i.e. Thm. 34. □

In particular, if we assume that $T=\delta$, then we have a Dirac delta embedded as a GHF, defined by

$$\delta_\varepsilon(z):=(\delta*\mu_\varepsilon)(z)=\mu_\varepsilon(z)=\rho_\varepsilon^{-2}\mu\left(\rho_\varepsilon^{-1}z\right).$$

Since μ is an entire function, this also clearly confirms that Dirac delta is in fact a GHF given by $\delta(z)=\mathrm{d}\rho^{-2}\mu\left(\frac{z}{\mathrm{d}\rho}\right)\in{}^{\rho}\widetilde{\mathbb{C}}$ and defined on ${}^{\rho}\widetilde{\mathbb{C}}$, and it allows us to include in this theory a large family of interesting examples, sometime informally studied in physics (cf. [8, 3, 14, 20, 33, 40, 52]).

From (3.2) and the identity theorem for μ, it follows that $\int_{\mathbb{R}}\mu=:c\ne 0$. Therefore, we can also consider $\mu_1(x):=\frac{1}{c}\mu(x)$ for all $x\in\mathbb{R}$. Then $\delta_1(x):=\mathrm{d}\rho^{-1}\mu_1\left(\frac{x}{\mathrm{d}\rho}\right)$ for all $x\in{}^{\rho}\widetilde{\mathbb{R}}$ is defined by the 1-dimensional δ-net $\rho_\varepsilon^1\mu_1(\rho_\varepsilon^{-1}x)\to\delta$ in $\mathcal{D}'(\mathbb{R})$ (and it can also be considered the μ_1-embedding of the Dirac delta as a GSF; this formula firstly appears in [2]). This also yields the link to the GHF $\delta(z)$: in fact $\delta(x)=c\mathrm{d}\rho^{-1}\delta_1(x)$ for all $x\in{}^{\rho}\widetilde{\mathbb{R}}$.

Note also that $\delta(0) = \frac{\mu(0)}{\mathrm{d}\rho^2}$ is an infinite number. Moreover, if we take $\mu \in \mathcal{S}(\mathbb{R}^2)$ and $x \in {}^{\rho}\widetilde{\mathbb{R}}$ "far" from the origin, i.e. $|x| \geq r \in \mathbb{R}_{>0}$, then $\delta(x) = 0$ in ${}^{\rho}\widetilde{\mathbb{C}}$ because $|\delta(x)| \leq C \cdot r^{-n} \cdot \mathrm{d}\rho^{n-2}$ for some $C \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{N}$.

Integral properties of this Dirac delta will be considered in [47]; sufficient conditions for the identity theorem will be explored in [48] (see Sec. 4 below for some initial considerations).

The embedding of all distributions is a sufficiently general sheaf theoretical construction which starts from the embedding of compactly supported ones. After the works [31, 16], this is a well-known construction and needs only trivial reformulations in our setting. We clearly have to use that every distribution $T \in \mathcal{D}'(\Omega)$ can be seen as the maximal family (or colimit, or injective limit) of compactly supported distributions, see e.g. [35] and references therein, and we write

$$T = \operatorname*{Colim}_{\substack{\omega \in \mathcal{E}'(W) \\ T|_W = \omega}} \omega.$$

Theorem 38. *We have the following properties:*

(i) *With respect to the Euclidean topology defined by open subsets $\Omega \subseteq \mathbb{R}^2$, the map $\Omega \mapsto {}^{\rho}\mathcal{GH}(\langle\Omega\rangle_{\text{fin}})$ is a sheaf.*

(ii) *If the entire mollifier $\mu \in \mathcal{S}(\mathbb{R}^2)$, then it is well-defined the following*

$$\iota(T) := \operatorname*{Colim}_{\substack{\omega \in \mathcal{E}'(W) \\ T|_W = \omega}} \iota_0(\omega) \in {}^{\rho}\mathcal{GH}(\langle\Omega\rangle_{\text{fin}}) \quad \forall T \in \mathcal{D}'(\Omega). \tag{3.7}$$

In this way, we obtain a linear sheaf monomorphism

$$\iota : \mathcal{D}'(\Omega, \mathbb{C}) \longrightarrow {}^{\rho}\mathcal{GH}(\langle\Omega\rangle_{\text{fin}})$$

which coincides with ι_0 on compactly supported distributions: $\iota|_{\mathcal{E}'(\Omega)} = \iota_0$.

(iii) *This embedding preserves derivatives, i.e. $\iota(\partial^\alpha T) = \partial^\alpha \iota(T)$ for all $\alpha \in \mathbb{N}^2$.*

(iv) *If $\mu \in \mathcal{S}(\mathbb{R}^2)$ has all null moments, i.e. $\int x^\alpha \mu(x)\,\mathrm{d}x = 0$ for all $|\alpha| \geq 1$, then the embedding ι coincides with the canonical one (3.1) on holomorphic functions: $\iota|_{\mathcal{O}(\Omega)} = \sigma$.*

Proof. Property (i) is a direct consequence of [28, Cor. 87]. Exactly as proved in [31, Prop. 1.2.12], since μ is rapidly decreasing, the embedding ι_0 of compactly supported distributions preserves supports (where the standard support of $f \in {}^{\rho}\mathcal{GH}(\langle\Omega\rangle_{\text{fin}})$ is defined as in [28, pag. 23]). This implies that the family $W \mapsto \{\iota_0(\omega) \mid \omega \in \mathcal{E}'(W),\ T|_W = \omega\}$ is a coherent one. Therefore, (3.7) is well-defined. The remaining properties can be proved exactly as in [31] or [16]. □

A second way, inspired by [30], to get a meaningful class of examples of GHF is to regularize a locally integrable function convolving with a *compactly supported* mollifier. However, since necessarily we are not smoothing with an entire mollifier, we have to assume that distributional CRE hold:

Theorem 39. *Let Ω be an open subset of $\mathbb{C}$. Let $k \in \mathcal{D}(\Omega)$ with $\int_{\mathbb{R}^2} k = 1$, and define $k_\varepsilon(z) := \rho_\varepsilon^{-2} \cdot k(\rho_\varepsilon^{-1} \cdot z)$ for all $z \in \mathbb{C}$. If f is locally integrable on Ω and, as a distribution, satisfies the CRE, then f can be identified with the GHF defined by*

$$\jmath(f) : [z_\varepsilon] \in d_f \mapsto [(f * k_\varepsilon)(z_\varepsilon)] \in {}^{\rho}\widetilde{\mathbb{C}},$$

in the sense that $\jmath(f)=\jmath(g)$ implies $f=g$. We recall that the domain d_f has been defined in Thm. 26 (in the present case with $U_\varepsilon=\Omega$). This embedding preserves weak derivatives of f, assuming that they exist: $\jmath(\partial^\alpha f)=\partial^\alpha\jmath(f)$.
Moreover, if $f\in\mathcal{C}^0(\Omega)$, then

(i) The standard part of this embedding yields back the given function, which necessarily is holomorphic: $\jmath(f)^\circ|_\Omega=f\in\mathcal{O}(\Omega)$.
(ii) If the embedding $\jmath(f)$ does not attain non-Archimedean values, i.e. $\jmath(f)|_\Omega\subseteq\mathbb{C}$, then $\jmath(f)|_\Omega=f\in\mathcal{O}(\Omega)$.

Proof. We have to use again Thm. 26, but in order to show that $f_\varepsilon:=f*k_\varepsilon\in\mathcal{O}(B^{\mathrm{E}}_{\rho^q_\varepsilon}(z_\varepsilon))$ we have to use the ideas of [30] (and [59]) and the assumption that CRE holds for weak derivatives of f. For the sake of completeness, we quickly repeat here the arguments of [30].

The function f_ε is smooth because k_ε is smooth. Let $\frac{\partial}{\partial\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$, so that asserting that f as a distribution satisfies CRE means that

$$\begin{aligned}\frac{\partial f_\varepsilon}{\partial\bar{z}}&=\frac{\partial}{\partial\bar{z}}\int_{\mathbb{R}^2}f(\zeta)\mu_\varepsilon(z-\zeta)\,\mathrm{d}\zeta=\int_{\mathbb{R}^2}f(\zeta)\frac{\partial}{\partial\bar{z}}\mu_\varepsilon(z-\zeta)\,\mathrm{d}\zeta\\&=-\int_{\mathbb{R}^2}f(\zeta)\frac{\partial}{\partial\bar{\zeta}}\mu_\varepsilon(z-\zeta)\,\mathrm{d}\zeta=0,\end{aligned}$$

because $\mu_\varepsilon\in\mathcal{D}(\Omega)$ is a test function. Thus, f_ε is a $\mathcal{C}^\infty$-function satisfying the CRE and hence is a holomorphic function. We can now proceed exactly as in Thm. 37 in order to apply Thm. 26. Also the preservation of weak derivatives follows like in Thm. 37 .

If $f\in\mathcal{C}^0(\Omega)$, then $f_\varepsilon\to f$ in Ω as $\varepsilon\to0^+$. Consider a standard point $z\in\Omega$ and taking as γ the boundary of any $B^{\mathrm{E}}_r(z)\subseteq\Omega$, by Cauchy's integral formula, we have

$$f(z)=\lim_{\varepsilon\to0^+}f_\varepsilon(z)=\lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_\gamma\frac{f_\varepsilon(\zeta)}{\zeta-z}\,\mathrm{d}\zeta=\frac{1}{2\pi i}\int_\gamma\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta,$$

so that $f\in\mathcal{O}(\Omega)$. Finally, $\jmath(f)^\circ(z)=\jmath(f)(z)^\circ=[\lim_{\varepsilon\to0}f_\varepsilon(z)]=f(z)$ and hence, if $\jmath(f)(z)\in\mathbb{C}$, then $\jmath(f)^\circ(z)=\jmath(f)(z)=f(z)$. □

Remark 40.

(i) It is important to note that $\jmath(f)(z)=[f_\varepsilon(z_\varepsilon)]$ is very different from its standard part, exactly as $\delta|_{\mathbb{R}}=0$ whereas $\delta(0)$ is an infinite number. The embedded GHF $\jmath(f)$ keeps all the smooth non-Archimedean information gained with the regularization $f_\varepsilon=f*k_\varepsilon$. This is also confirmed by property (ii) of Thm. 39: if $\jmath(f)|_\Omega$ does not have any non-Archimedean information, then the smoothing process is useless and $\jmath(f)$ is only an extension of f to generalized numbers of $d_f\subseteq{}^\rho\widetilde{\mathbb{C}}$, i.e. it equals the canonical embedding (3.1).
(ii) Clearly, Thm. 37 allows us to also embed bump functions or a nowhere analytic smooth function multiplied by a bump which constantly equals 1 on a neighborhood of a given Ω. In the former case, in [48] we will show that any Taylor hyperseries of a GHF *always* has a strictly positive radius of convergence $r\in{}^\rho\widetilde{\mathbb{R}}_{>0}$. In case the center of the hyperseries is a flat point, then the radius $r\approx0$ is infinitesimal: indeed, a bump function identically equals 0 (in ${}^\rho\widetilde{\mathbb{C}}$) infinitely close to flat points. In the latter case, the regularization process with holomorphic functions allows one to overcome the limitation given by

estimates (ii) of Thm. 26, which does not hold for nowhere analytic smooth functions.

Finally, as we will explore in the subsequent paper of this series ([47]), the path integral can be used to define a primitive of generalized holomorphic functions, which opens up the possibility of another way to embed distributions such as a Heaviside like function.

4. Comparison with Colombeau theory and conclusions

In Sec. 2.3, by comparing Thm. 26 with [49, Thm. 2], we already showed that Colombeau generalized holomorphic functions are all included as GHF. The family of examples given in the previous section shows that this inclusion in meaningfully strict. See [15, 16] for the relations about how to include periodic hyperfunctions in Colombeau type algebras. In this closing section, we want to underscore some analogies and differences between our construction and Colombeau theory.

First of all, in both theories, Schwartz impossibility theorem about multiplication of distributions (see e.g. [31] and references therein) is bypassed because only the pointwise product of ordinary holomorphic functions is preserved, not that of continuous functions. Using Thm. 37, in general we only have $\iota(f)\cdot\iota(g)\approx\iota(f\cdot g)$ if f, g are continuous and compactly supported, where $\approx$ has been defined in Def. 2. This is clearly very natural if one thinks at the regularization process of any embedding and hence the related augmented non-Archimedean information.

A direct comparison between generally recognized technical drawbacks of Colombeau theory and the theory of GHF is as follows:

(i) Colombeau generalized functions are not closed with respect to composition, and only a sufficient condition is possible (see [31, Def. 1.2.7]), whereas GHF are closed with respect to composition, see Thm. 24;

(ii) In general, Colombeau generalized functions can be defined only on finite points of $\langle\Omega\rangle_{\text{fin}}$, whereas GHF can be defined on more general domains, see Def. 20. This implies that we can consider solutions of differential equations e.g. defined on infinitesimal interval and that cannot be extended or defined in domains containing infinite numbers. For example, even in the real case of GSF, consider $y'=-\frac{t}{1+y}\cdot\frac{1}{h}$, $y(0)=0$, where h is a positive invertible infinitesimal, and whose real solution is $y(t)=-1+\sqrt{1-\frac{t^2}{h}}$, $t\in(-\sqrt{h},\sqrt{h})$. This limitation of Colombeau theory is essentially due to the bias of defining a sheaf of generalized functions starting from open sets $\Omega\subseteq\mathbb{R}^n$ in order to have a more direct comparison with classical distribution theory. In our construction, we started assuming a non-Archimedean point of view by changing the ring of scalars.

(iii) Colombeau theory naturally assumes the possibility to take arbitrary derivatives and requires, by definition, that generalized holomorphic functions satisfy the CRE. On the contrary, we made an effort of asking only a first order condition, so that CRE, Goursat, Looman-Menchof and Montel theorems of Sec. 2.4 can be proved. Similarly, a more flexible criterion of holomorphicity is also provable, Thm. 26.

(iv) Colombeau theory of generalized holomorphic functions, see e.g. [57], uses ordinary series $\sum_{n\in\mathbb{N}}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ and hence, as we already explained in Sec. 1.1, good convergence result are not possible. The use of hyperseries

${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$, i.e. of series extended over *infinite* generalized hypernatural numbers, in [48] solves this type of problems.

(v) For the sake of completeness, we only mention here that the notion of Fourier transform of Colombeau generalized functions presents several limitations in important theorems, such as the inversion one. On the contrary, with the notion of hyperfinite Fourier transform, we can consider the Fourier transform of any GSF, without restriction to tempered type, and these limitations are removed, see [46].

Concerning the intrinsic embedding of distributions, which is frequently presented as a drawback of Colombeau theory, first of all we have to explicitly state that this objection is wrong, because it is well-known that considering a suitable index set instead of $I=(0,1]$, allows one to have an intrinsic embedding, see [31].

On the other hand, in the present paper we tried to underscore a different point of view: at least since the work of Leray [39], smoothing possible singularities using the dependence from a suitable parameter ε is now a natural and frequently used method (see e.g. [5, 6, 44, 4, 41] for recent works using similar methods). Therefore, in our opinion, the present theory of GHF has to be considered as the introduction of a mathematical structure of the form $(\rho,{}^{\rho}\mathcal{GH}(-),\mu,k)$, where μ is any entire mollifier for the embedding of compactly supported distributions, and k is a compactly supported function with integral 1 for the embedding of locally integrable functions satisfying the distributional CRE. The free variables ρ, μ and k need to be chosen depending on the regularization problem we have to solve. Searching for hypothetical "best or intrinsic" values of these variables would imply having less freedom in this method.

However, it is interesting to note that this freedom is already necessarily constrained by the following uniqueness result of the embedding of Thm. 37:

Theorem 41. *Let μ_1, $\mu_2\in\mathcal{C}^\infty(\mathbb{R}^2)$ be entire mollifiers and $\frac{\delta_1^{(n)}(0)}{n!}=[\delta_{1n,\varepsilon}]$, $\frac{\delta_2^{(n)}(0)}{n!}=[\delta_{2n,\varepsilon}]$ be the coefficients of the Taylor formula at 0 of the corresponding Dirac delta GHF, i.e. $\delta_1(z)=\mathrm{d}\rho^{-2}\mu_1(\mathrm{d}\rho^{-1}z)$ and $\delta_2(z)=\mathrm{d}\rho^{-2}\mu_2(\mathrm{d}\rho^{-1}z)$ respectively. If these coefficients are* strongly ρ-equivalent*, denoted by $(\delta_{1n,\varepsilon})\simeq_\rho(\delta_{2n,\varepsilon})$ as in* [55, Def. 3.(ii)]*, i.e.*

$$\forall q,r\in\mathbb{N}\,\forall^0\varepsilon\,\forall n\in\mathbb{N}:|\delta_{1n,\varepsilon}-\delta_{2n,\varepsilon}|\le\rho_\varepsilon^{nq+r}, \tag{4.1}$$

then $\mu_1=\mu_2$. In other words, if the embedding $\iota(-)$ of Thm. 37 preserves the Taylor formula of Dirac delta, then it does not depend on the entire mollifier μ.

This theorem will be fully clear only after the introduction, in [48], of the notion of equality between hyperseries, which is given exactly by (4.1), (see also [55] for the case of generalized real analytic functions). However, it can already be glimpsed looking at (ii) of Thm. 26: the upper bound

$$\exists Q,R\in\mathbb{R}_{>0}\,\forall^0\varepsilon\,\forall n\in\mathbb{N}:\left|\frac{f_\varepsilon^{(n)}(z_\varepsilon)}{n!}\right|\le\rho_\varepsilon^{-nQ-R}$$

is analogous to the moderateness condition in Def. 1, and hence (4.1) corresponds to the related negligibility condition.

Proof of Thm. 41. For all $n\in\mathbb{N}$, we have

$$\delta_{1n,\varepsilon}=\frac{\delta_{1\varepsilon}^{(n)}(0)}{n!}=\frac{\rho_\varepsilon^{-n-2}\mu_1^{(n)}(0)}{n!}\text{ and }\delta_{2n,\varepsilon}=\frac{\delta_{2\varepsilon}^{(n)}(0)}{n!}=\frac{\rho_\varepsilon^{-n-2}\mu_2^{(n)}(0)}{n!}.$$

Setting $q=1$ and $r=0$ in (4.1), for ε small and for all $n\in\mathbb{N}$, we have

$$\left|\frac{\mu_1^{(n)}(0)}{n!}-\frac{\mu_2^{(n)}(0)}{n!}\right|\le\rho_\varepsilon^{nq-n+2+r}\le\rho_\varepsilon^2.$$

Taking $\varepsilon\to0$, we can conclude that $\frac{\mu_1^{(n)}(0)}{n!}=\frac{\mu_2^{(n)}(0)}{n!}$, for all $n\in\mathbb{N}$. Then $\mu_1=\mu_2$ since μ_1 and μ_2 are entire functions. □

Stating this uniqueness result in different words: if we need to choose the entire mollifier μ in order to have specific smoothing property, this corresponds to choose a particular form of the Taylor formula of the corresponding Dirac delta. Therefore, the freedom in choosing μ corresponds to the freedom of selecting a matched Dirac delta embedded as a GHF. This is also natural if we think at the infinite amount of non-Archimedean properties that $\delta(-)\in{}^{\rho}\mathcal{GH}({}^{\rho}\widetilde{\mathbb{C}})$ satisfies whilst the classical "macroscopic" version $(\varphi\in\mathcal{D}(\mathbb{R}^2)\mapsto\varphi(0)\in\mathbb{R})\in\mathcal{D}'(\mathbb{R}^2)$ does not.

For the sake of clarity, we close this section showing that the classical ideas to prove the identity theorem for holomorphic function can also be repeated for GHF. However, the identity principle does not hold in our framework (see also [55, Thm. 39 and 40] for the case of generalized real analytic functions) exactly because we are in a non-Archimedean setting: e.g. every interval of ${}^{\rho}\widetilde{\mathbb{R}}$ is not connected in the sharp topology because the set of all the infinitesimals is a clopen set, see e.g. [23]. Therefore, repeating the classical proof of the identity theorem leads us to state that the set of points where two given GHF agree is "only" a clopen set.

In [48] we will be able to prove the following natural

Lemma 42. *Let f, $g\in{}^{\rho}\mathcal{GH}(U,{}^{\rho}\widetilde{\mathbb{C}})$ and $c\in U$. If for all $n\in\mathbb{N}$ we have $f^{(n)}(c)=g^{(n)}(c)$ then*

$$\exists r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}:f\,|_{B_r(c)}=g\,|_{B_r(c)}\,.$$

We can also think at the following theorem as a result that does not depend on [48], if we take the property stated in this Lemma 42 as a natural assumption.

Theorem 43. *Let $U\subseteq{}^{\rho}\widetilde{\mathbb{C}}$ be an open set and f, $g\in{}^{\rho}\mathcal{GH}(U,{}^{\rho}\widetilde{\mathbb{C}})$. Then the set*

$$\mathcal{O}:=\operatorname{int}\{z\in U\mid f(z)=g(z)\}$$

is clopen in the sharp topology.

Proof. For simplicity, considering $f-g$, without loss of generality we can assume $g=0$. We only have to show that the set $\mathcal{O}$ is closed in U. Assume that c is in the closure of $\mathcal{O}$ in U, i.e.

$$c\in U\;\forall r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}\;\exists\bar{c}\in B_r(c)\cap\mathcal{O}. \tag{4.2}$$

We have to prove that $c\in\mathcal{O}$. We first note that for each $\bar{c}\in\mathcal{O}$, we have $B_p(\bar{c})\subseteq\mathcal{O}$ for some $p\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and hence

$$f(\bar{z})=0\quad\forall\bar{z}\in B_p(\bar{c}). \tag{4.3}$$

Now, fix $n\in\mathbb{N}$ in order to prove that $f^{(n)}(c)=0$. From (4.2), for all $r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ we can find $\bar{c}_r\in B_r(c)\cap\mathcal{O}$ such that $f^{(n)}(\bar{c}_r)=0$ from (4.3). From sharp continuity of $f^{(n)}$, we have $f^{(n)}(c) = \lim_{r\to 0^+} f^{(n)}(\bar{c}_r) = 0$ for all $n \in \mathbb{N}$. Therefore, since $f \in {}^\rho\mathcal{GH}(U, {}^\rho\widetilde{\mathbb{C}})$ and $c \in U$, by Lemma 42, we can hence find $\delta > 0$ such that $f(z) = 0$ for all $z \in B_\delta(c)$, i.e. $c \in \mathcal{O}$. □

The lacking of a general identity theorem is hence a feature of GHF because it enables the inclusion of a wide range of interesting generalized functions, paving also the way for a more general version of the Cauchy-Kowalevski theorem. As we already mentioned above, since functions in Sobolev spaces can be approximated using Taylor series, see [7, 10, 53], and GHF solutions of PDE equals their Taylor hyperseries, we also hope to find a way to understand when a GHF solution is actually the embedding of a function in a Sobolev space.

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References

[1] J. Aragona, R. Fernandez, S.O. Juriaans, Differential calculus and integration of generalized functions over membranes, Monatshefte für Mathematik (2012) 166 1–18. https://doi.org/10.1007/s00605-010-0275-z

[2] J. Aragona, R. Fernandez, S.O. Juriaans, A discontinuous colombeau differential calculus, Monatshefte für Mathematik (2005) 144(1) 13–29. https://doi.org/10.1007/s00605-004-0257-0

[3] F. Bagarello, Fourier transforms, fractional derivatives, and a little bit of quantum mechanics, Rocky Mountain J. Math. (2020) 50(2) 415-428. https://doi.org/10.1216/rmj.2020.50.415

[4] F.B. Belgacem, P.E. Jabin, Compactness for nonlinear continuity equations, Journal of Functional Analysis (2013) 264(1) 139–168. https://doi.org/10.1016/j.jfa.2012.10.005

[5] C. Bertucci, J.M. Lasry, P.L. Lions, On lipschitz solutions of mean field games master equations. Journal of Functional Analysis (2024) 287(5) 110486. https://doi.org/10.1016/j.jfa.2024.110486

[6] F. Bethuel, Asymptotics for vectorial Allen–Cahn type problems. Journées équations aux dérivées partielles (2023), Talk no. 1, 16 p. https://doi.org/10.5802/jedp.672

[7] B. Bojarski, Taylor expansion and Sobolev spaces, Bull. Georgian Natl. Acad. Sci. (2011) 2(2) 5–10.

[8] R.A. Brewster, J.D. Franson, Generalized delta functions and their use in quantum optics. Journal of Mathematical Physics (2018) 59(1), 012102. https://doi.org/10.1063/1.4985938

[9] A. Bryzgalov, K. Islami, P. Giordano, Infinitesimal and infinite numbers in mathematical physics. Nonlinear Dyn. (2024) 112 20573–20609. https://doi.org/10.1007/s11071-024-10223-8

[10] P. Cavaliere, A. Cianchi, Classical and approximate Taylor expansions of weakly differentiable functions, Annales Academiæ Scientiarum Fennicae Mathematica (2014) 39 527–544. https://doi.org/10.5186/aasfm.2014.3933

[11] J. Colombeau, *New generalized functions and multiplication of distributions*, North-Holland, Netherlands, 1984.

[12] J. Colombeau, J. Galé, Holomorphic generalized functions, Journal of Mathematical Analysis and Applications (1984) 103(1) 117–133. https://doi.org/10.1016/0022-247X(84)90162-8

[13] N. Cutland, C. Kessler, E. Kopp, D. Ross, On Cauchy's notion of infinitesimal, British J. Philos. Sci. (1988) 39(3) 375–378. https://doi.org/10.1093/bjps/39.3.375

[14] M. Davidson, Classical Charged Particle Models Derived from Complex Shift Methods. Int J Theor Phys. (2023) 62, 154. https://doi.org/10.1007/s10773-023-05411-y

[15] A. Delcroix, M.F. Hasler, S. Pilipovic, V. Valmorin, Embeddings of ultradistributions and periodic hyperfunctions in Colombeau type algebras through sequence spaces, Mathematical Proceedings of the Cambridge Philosophical Society (2004) 137 (3) 697-708. http://dx.doi.org/10.1017/S0305004104007923

[16] A. Delcroix. Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions. Novi Sad Journal of Mathematics (2005) 35(2) 27-40.

[17] P.A.M. Dirac, The physical interpretation of the quantum dynamics, Proc. R. Soc. Lond. A. (1927) 113 621–641. https://doi.org/10.1098/rspa.1927.0012

[18] P. Ehrlich, The Rise of non-Archimedean Mathematics and the Roots of a Misconception I: The Emergence of non-Archimedean Systems of Magnitudes, Arch. Hist. Exact Sci. (2006) 60 1–121. http://dx.doi.org/10.1007/s00407-005-0102-4

[19] L. Evans, *Partial differential equations*, American Mathematical Society, Providence, R.I., 2010.

[20] V.P. Frolov, J. Pinedo Soto, Nonlocal Modification of the Kerr Metric, Symmetry (2023) 15(9), 1771. http://dx.doi.org/10.3390/sym15091771

[21] C. Garetto, H. Vernaeve, Hilbert C-modules: structural properties and applications to variational problems, Transactions of The American Mathematical Society (2011) 363(4) 2047–2090. http://dx.doi.org/10.1090/S0002-9947-2010-05143-8

[22] S. Gastão, P. Giordano, A. Bryzgalov, M. Lazo, Calculus of variations and optimal control for generalized functions. Nonlinear Analysis (2022) 216, 112718, . https://doi.org/10.1016/j.na.2021.112718

[23] P. Giordano, M. Kunzinger, New topologies on Colombeau generalized numbers and the Fermat Reyes theorem, J. Math. Anal. Appl. (2013) 399 229–238. http://doi.org/10.1016/j.jmaa.2012.10.005

[24] P. Giordano, L. Luperi Baglini, Asymptotic gauges: Generalization of Colombeau type algebras, Mathematische Nachrichten (2016) 289 247-274. https://doi.org/10.1002/mana.201400278

[25] P. Giordano, M. Kunzinger, *Inverse Function Theorems for Generalized Smooth Functions*, in: Oberguggenberger, M., Toft, J., Vindas, J., Wahlberg, P. (eds) Generalized Functions and Fourier Analysis. Operator Theory: Advances and Applications (2017), vol 260. Birkhäuser, Cham pp. 95-114. http://dx.doi.org/10.1007/978-3-319-51911-1_7

[26] P. Giordano, M. Kunzinger, A convenient notion of compact set for generalized functions. Proceedings of the Edinburgh Mathematical Society (2018) 61(1) 57–92. https://doi.org/10.1017/S0013091516000559

[27] P. Giordano, M. Kunzinger, H. Vernaeve, Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications (2015) 422(1) 56–71. https://doi.org/10.1016/j.jmaa.2014.08.036

[28] P. Giordano, M. Kunzinger, H. Vernaeve, A Grothendieck topos of generalized functions I: basic theory, Dissertationes Mathematicae (2024) 592 74. http://dx.doi.org/10.4064/dm230920-7-3

[29] P. Giordano, L. Luperi Baglini, H. Vernaeve, A Grothendieck topos of generalized functions III: normal PDE (preprint). See https://www.mat.univie.ac.at/~giordap7/ToposIII.pdf.

[30] J.D. Gray, S.A. Morris, When is a Function that Satisfies the Cauchy-Riemann Equations Analytic? The American Mathematical Monthly (1978) 85(4) 246-256. https://doi.org/10.1080/00029890.1978.11994567

[31] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer, *Geometric theory of generalized functions with applications to general relativity,* Mathematics And Its Applications (vol. 537), Kluwer Academic Publishers, 2001.

[32] L. Hörmander, *The analysis of linear partial differential operators,* vol. I, 2nd Ed., Springer-Verlag, 1990.

[33] G. Kaiser, Physical wavelets and their sources: Real physics in complex spacetime. J. Phys. A Mathematical Gen. (2003) 36(30), R291. https://doi.org/10.1088/0305-4470/36/30/201

[34] M.G. Katz, D. Tall, A Cauchy-Dirac Delta Function, Found Sci. (2013) 18 107–123. https://doi.org/10.1007/s10699-012-9289-4

[35] D. Kebiche, P. Giordano, Universal properties of spaces of generalized functions, Journal of Mathematical Analysis and Applications (2025) 551(2)1 129687 https://doi.org/10.1016/j.jmaa.2025.129687

[36] N. Koblitz. *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Graduate Texts in Mathematics (Book 58). Springer, 2nd Ed., 1996.

[37] D. Laugwitz, Definite values of infinite sums: aspects of the foundations of infinitesimal analysis around 1820, Arch. Hist. Exact Sci. (1989) 39(3) 195–245. https://doi.org/10.1007/BF00329867

[38] A. Lecke, L. Luperi Baglini, P. Giordano, The classical theory of calculus of variations for generalized functions, Advances in Nonlinear Analysis (2019) 8(1) 779-808. https://doi.org/10.1515/anona-2017-0150

[39] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. (1934) 63 193–248. https://doi.org/10.1007/BF02547354

[40] I.V. Lindell, Delta function expansions, complex delta functions and the steepest descent method, Am. J. Phys. (1993) 61 438-42. https://doi.org/10.1119/1.17238

[41] P. Luo, K. Pan, S. Peng, Y. Zhou, Local uniqueness of concentrated solutions and some applications on nonlinear Schrödinger equations with very degenerate potentials. Journal of Functional Analysis 2023 284(12), 109921. https://doi.org/10.1016/j.jfa.2023.109921

[42] L. Luperi Baglini, P. Giordano, D. Kebiche, H. Vernaeve, A Grothendieck topos of generalized functions II: ODE (preprint). See https://www.mat.univie.ac.at/~giordap7/ToposII.pdf.

[43] J.A. Marti, (C, ε, P)-*Sheaf Structures and Applications*, in: Oberguggenberger, M., Grosser, M., Kunzinger, M., & Hormann, G. (1st Eds.), Nonlinear Theory of Generalized Functions. Routledge, 1999, pp. 175-186. http://dx.doi.org/10.1201/9780203745458-16

[44] M. Montenegro, E.V. Teixeira, Gradient estimates for viscosity solutions of singular fully nonlinear elliptic equations, Journal of Functional Analysis (2010) 259(2) 428–452. https://doi.org/10.1016/j.jfa.2010.03.021

[45] A. Mukhammadiev, D. Tiwari, G. Apaaboah, Supremum, infimum and hyperlimits in the non-Archimedean ring of Colombeau generalized numbers, Monatshefte für Mathematik (2021) 196 163–190. https://doi.org/10.1007/s00605-021-01590-0

[46] A. Mukhammadiev, D. Tiwari, P. Giordano, A Fourier transform for all generalized functions. Accepted in Dissertationes Mathematicae (2023).

[47] S. Nugraheni, P. Giordano, Path integration of generalized holomorphic functions (preprint), 2025. See http://ugm.id/GHF2

[48] S. Nugraheni, P. Giordano, Hyper power series for generalized holomorphic functions (preprint), 2025. See http://ugm.id/GHF3

[49] M. Oberguggenberger, S. Pilipović, V. Valmorin, Global Representatives of Colombeau Holomorphic Generalized Functions, Monatshefte für Mathematik (2007) 151 67–74. https://doi.org/10.1007/s00605-006-0416-6

[50] M. Oberguggenberger, H. Vernaeve, Internal sets and internal functions in Colombeau theory, Journal of Mathematical Analysis and Applications (2008) 341(1) 649–659. https://doi.org/10.1016/j.jmaa.2007.10.030

[51] W.M. Rodrigues, A.R.G. Garcia, S.O. Juriaans, J.C. Silva. Sets of Uniqueness for holomorphic Nets (preprint), 2020. http://dx.doi.org/10.13140/RG.2.2.36425.98405

[52] V.A. Smagin, Complex delta function and its information application. Automatic Control and Computer Sciences (2014) 48(1), 1016. https://doi.org/10.3103/S0146411614010064

[53] D. Spector, L^p-Taylor approximations characterize the Sobolev space $W^{1,p}$, C.R. Acad. Sci. Paris, Ser. I. (2015) 353 327–332. https://doi.org/10.1016/j.crma.2015.01.010

[54] D. Tiwari, P. Giordano, Hyperseries in the non-Archimedean ring of Colombeau generalized numbers, Monatshefte für Mathematik (2022) 197 193–223. https://doi.org/10.1007/s00605-021-01647-0

[55] D. Tiwari, A. Mukhammadiev, P. Giordano, Hyper-power series and generalized real analytic functions, Monatshefte für Mathematik (2024) 203 475–508. https://doi.org/10.1007/s00605-023-01849-8

[56] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, United Kingdom, 1967.

[57] H. Vernaeve, Generalized analytic functions on generalized domains (preprint), 2008. https://doi.org/10.48550/arXiv.0811.1521

[58] S. Winitzki, The Dirac delta function cannot be applied to complex arguments. See https://www.academia.edu/32968278

[59] L. Zalcman, Real Proofs of Complex Theorems (and Vice Versa), The American Mathematical Monthly (1974) 81(2) 115-137. https://doi.org/10.1080/00029890.1974.11993518

Sekar Nugraheni
Faculty of Mathematics, University of Vienna, Austria
Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Indonesia
Email address: sekar.nugraheni@ugm.ac.id

Paolo Giordano
Faculty of Mathematics, University of Vienna, Austria
Email address: paolo.giordano@univie.ac.at