Order relation and geometrical representation of Fermat reals

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Abstract

Is it possible to define a powerful ring of actual infinitesimals using standard analysis only? Can we realize this goal without using a specific background of logic? Are we able to define this ring always keeping a good intuitive interpretation of the new numbers we are defining? The theory of Fermat reals was born trying to answer these questions. In this article we introduce axiomatically the ring of Fermat reals, we review its surprisingly simple definition and study its total order relation. We prove that a clear geometrical representation of this ring is possible, we characterize its skeleton group, generalize the existence of supremum and present its computer implementation.

Keywords: Fermat reals, actual infinitesimals, nilpotent infinitesimals, extensions of the real field.
MSC 2010 Subject Classifications: 06FXX, 06F25, 26E35, 14B10.

1 Axioms for infinitesimals

Several students experienced the unpleasant feeling to switch from the classical Calculus lectures, where usually everything is $\varepsilon-\delta$, to the Physics ones where the lecturer report, e.g., Einstein's formulas like

$$\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}=1+\frac{v^2}{2c^2}\qquad\qquad \sqrt{1-h_{44}(x)}=1-\frac{1}{2}h_{44}(x) \tag{1.1}$$

with explicit use of infinitesimals $v/c \ll 1$ or $h_{44}(x) \ll 1$, such that, e.g., $h_{44}(x)^2=0$. If student's passion is still alive, she can start a long journey passing through nonstandard analysis (NSA), Synthetic Differential Geometry (SDG), Surreal numbers, Levi-Civita fields, to mention only a few of

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possible stages of this trip. As researchers, the best we can hope for her is also to try her own solution, since two different languages ($\varepsilon-\delta$ and informal infinitesimals) which are able to describe a sufficiently large part of nature must have a stringent relationship among each other. Using only elementary analysis, after a couple of tens of years, she can also be successful. The solution ${}^\bullet\mathbb{R}$ is essentially unique, and indeed it can be described axiomatically.

Axiom, decomposition: ${}^\bullet\mathbb{R}$ is a commutative ring with unity, called *ring of Fermat reals*. Every Fermat reals $x \in {}^\bullet\mathbb{R}$ can be written, in a unique way, as

$$x = {}^\circ x + \sum_{i=1}^{N} a_i \cdot \mathrm{d}t_{\alpha_i}, \tag{1.2}$$

where ${}^\circ x$, α_i, $a_i \in \mathbb{R}$ are standard reals, $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 1$, $a_i \neq 0$. The term ${}^\circ x \in \mathbb{R}$ is called *standard part* of x, and $a_i =: {}^\circ x_i$ its i-th standard part.

For example $x = 2 - \log(3)\,\mathrm{d}t_2$ and $y = r \in \mathbb{R}$ are elements of ${}^\bullet\mathbb{R}$.

Axiom, base infinitesimals: The terms $\mathrm{d}t_a$ verify the following properties

$$\begin{aligned} \mathrm{d}t_a \cdot \mathrm{d}t_b &= \mathrm{d}t_{\frac{ab}{a+b}} \\ (\mathrm{d}t_a)^p &= \mathrm{d}t_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\geq 1} \\ \mathrm{d}t_a &= 0 \quad \forall a \in \mathbb{R}_{<1}. \end{aligned} \tag{1.3}$$

Therefore, among Fermat reals we also have *nilpotent infinitesimals*, like $x = 3\,\mathrm{d}t_2$, since $x^3 = 27\,\mathrm{d}t_{2/3} = 0$. These are exactly the same type of infinitesimals used by Einstein in formulas like (1.1). We will simply use the symbol $\mathrm{d}t$ for $\mathrm{d}t_1$.

Axiom, order of infinitesimals: The *order* $\omega(x) =: a_1$ (see (1.2)) can be interpreted as the leading term in the decomposition and hence it has the following expected properties

$$\begin{gathered} \omega(x+y) = \max\left[\omega(x), \omega(y)\right] \\ \frac{1}{\omega(x \cdot y)} = \frac{1}{\omega(x)} + \frac{1}{\omega(y)}, \end{gathered}$$

whenever x, y are infinitesimals such that $x + y \neq 0$, respectively $x \cdot y \neq 0$. In the decomposition (1.2), the term $\alpha_i =: \omega_i(x)$ will be called the i-th order of x.

Directly from (1.2) it is not hard to prove that if $k \in \mathbb{N}_{>1}$, then $x^k = 0$ iff $\omega(x) < k$. Nilpotent Fermat reals can be thought as non zero numbers which are so small that a suitable power of them gives zero.

Axiom, ideals of infinitesimals: For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the set

$$D_a := \{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0,\ \omega(x) < a + 1\}$$

is an ideal. Moreover, for $k \in \mathbb{N}_{\geq 1}$ we have that $D_k = \left\{x \in {}^\bullet\mathbb{R} \mid x^{k+1} = 0\right\}$.

We will simply use D for D_1. The ideal D_k is therefore the "ideal" k-th order infinitesimal neighbourhood of zero, where every k-th order Taylor formula doesn't have rest, since $x^{k+1}=0$. This is indeed the subject of the next

Axiom, Taylor formula: Set ${}^\bullet\mathbb{R}^d := {}^\bullet\mathbb{R}\times \overset{d}{\ldots\ldots} \times {}^\bullet\mathbb{R}$, then every ordinary smooth function $f : A \longrightarrow \mathbb{R}$ defined on an open set A of $\mathbb{R}^d$ can be extended to the set

$$
{}^\bullet A = \left\{ x \in {}^\bullet\mathbb{R}^d \mid {}^\circ x \in A \right\}, \tag{1.4}
$$

$$
{}^\bullet f : {}^\bullet A \longrightarrow {}^\bullet\mathbb{R},
$$

obviously obtaining a true extension, i.e. ${}^\bullet f(x) = f(x)$ if $x \in A$. Moreover, the following Taylor formula

$$
\forall h \in D_k^d : \ {}^\bullet f(x+h) = \sum_{\substack{j\in\mathbb{N}^d \\ |j|\le k}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x) \tag{1.5}
$$

holds, where $x \in A$ is a standard point, and $D_k^d = D_k \times \overset{d}{\ldots\ldots} \times D_k$.

Therefore, smooth functions becomes exactly equal to polynomials of degree k in the infinitesimal k-th order neighbourhood $x + D_k^d$. In particular $f(x+h) = f(x) + h \cdot f'(x)$ for $h \in D$, i.e. every smooth functions is equal to its tangent line in a first order infinitesimal neighbourhood. Einstein's formulas (1.1) are particular cases of this infinitesimal Taylor formula. Let us note that applying this formula to the function $f(x) = x^2$ in D we obtain that $h^2 = 0$ for every $h \in D$: if we want to write Einstein's formulas exactly as he did (i.e. with the equality sign and not with an approximate equality sign), we are necessarily forced to work in a ring with nilpotent infinitesimals and not in a field.

Axiom, cancellation laws: Let $h_1, \ldots, h_n \in D_\infty$, $i_1, \ldots, i_n \in \mathbb{N}$, $x \in {}^\bullet\mathbb{R}$, then we have

1. $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$ if and only if $\sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1$.
2. x is invertible if and only if ${}^\circ x \neq 0$.
3. If $x \cdot r = x \cdot s$ in ${}^\bullet\mathbb{R}$, where r, $s \in \mathbb{R}$ and $x \neq 0$, then $r = s$.

If you are scared by working in a ring instead of a field, these laws permit to effectively work with this type of infinitesimals. If x is invertible, and proceeding like in the case of formal power series, it is not hard to prove that

$$
\frac{1}{x} = \frac{1}{{}^\circ x} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_{i=1}^n \frac{{}^\circ x_i}{{}^\circ x} \cdot \mathrm{d}t_{a_i} \right)^j, \tag{1.6}
$$

where the series is really a finite sum due to nilpotency.

We will see later that the total order relation $x \le y$ can also be introduced axiomatically and effectively decided starting from the decomposition of x and y.

More advanced axioms have to be introduced to deal with smooth functions that are more general than extension of standard smooth functions, e.g. like the very simple $f(x) = x + \mathrm{d}t$, see [11, 7]. These axioms are formally very similar to the Reyes axiom, the integrability axiom and the constancy axiom of SDG ([14, 1, 15]). Therefore, the previously listed axioms are surely incomplete, on the one hand, but also redundant, from another point of view. However, they permit to characterize the structure $({}^\bullet\mathbb{R}, +, \cdot, {}^\circ(-), \mathrm{d}t_{(-)})$.

Theorem 1. *Let us suppose that the structure* $(R, \oplus, \odot, {}^\diamond(-), \delta t_{(-)})$ *verifies the previous axioms, then there exists one and only one ring isomorphism*

$$f : {}^\bullet\mathbb{R} \longrightarrow R$$

such that $f(a \cdot \mathrm{d}t_\alpha) = a \odot \delta t_\alpha$ *and* ${}^\diamond f(x) = {}^\circ x$. *Moreover, this isomorphism preserves also the order function* $\omega(-)$.

Proof: Let $x = r + \sum_{j=1}^N a_j \cdot \mathrm{d}t_{\alpha_j}$ be the decomposition of $x \in {}^\bullet\mathbb{R}$. The only possibility to define the searched isomorphism is obviously

$$f(x) := r \oplus \bigoplus_{j=1}^N a_j \odot \delta t_{\alpha_j} \in R. \tag{1.7}$$

We have to prove that f is a morphism of ordered ring. Let $y = s + \sum_{i=1}^M b_i \cdot \mathrm{d}t_{\beta_i}$ be the decomposition of $y \in {}^\bullet\mathbb{R}$, then to find the decomposition of $x + y$ and apply our definition (1.7), we firstly have to consider the sets of all the orders appearing in these decompositions:

$$\begin{aligned} O_x &:= \{\alpha_j \mid j = 1, \dots, N\} \\ O_y &:= \{\beta_i \mid i = 1, \dots, M\} \\ O &:= O_x \cup O_y. \end{aligned}$$

Secondly, we have to sum all the coefficients of the two decompositions corresponding to the same order: for every $q \in O$ set

$$c_q := \sum \left[\{a_j \mid j = 1, \dots, N \ , \ \alpha_j = q\} \cup \{b_i \mid i = 1, \dots, M \ , \ \beta_i = q\} \right].$$

Let us note that, by the definition of decomposition, if $q \in O_x \setminus O_y$, then $c_q = a_j$, where $j = 1, \dots, N$ is the unique index such that $\alpha_j = q$. Analogously if $q \in O_y \setminus O_x$, whereas, if $q \in O_x \cap O_y$, then $c_q = a_j + b_i$, where i and j are the unique indexes such that $\alpha_j = q = \beta_i$. Now, we have

$$x + y = r + s + \sum_{q \in O} c_q \cdot \mathrm{d}t_q.$$

Finally, we have to enumerate all the orders in $O \setminus \{q \in O \mid c_q = 0\}$ in increasing way

$$\{q_1, \ldots, q_K\} = O \setminus \{q \in O \mid c_q = 0\}$$

$$q_1 > q_2 > \cdots > q_K \geq 1.$$

Therefore

$$x + y = r + s + \sum_{k=1}^{K} c_{q_k} \cdot \mathrm{d}t_{q_k}$$

is the decomposition of $x + y$. Applying the definition (1.7) of f we get

$$\begin{aligned}
f(x+y) &= r \oplus s \oplus \bigoplus_{k=1}^{K} c_{q_k} \odot \delta t_{q_k} \\
f(x) + f(y) &= r \oplus \bigoplus_{j=1}^{N} a_j \odot \delta t_{\alpha_j} \oplus s \oplus \bigoplus_{i=1}^{M} b_i \odot \delta t_{\beta_i} = \\
&= r \oplus s \oplus \bigoplus_{q \in O} c_q \odot \delta t_q = \\
&= f(x+y).
\end{aligned}$$

For the product, we can proceed in a similar way:

$$x \cdot y = rs + \sum_{i=1}^{M} r b_i \cdot \mathrm{d}t_{\beta_i} + \sum_{j=1}^{N} s a_j \cdot \mathrm{d}t_{\alpha_j} + \sum_{i,j} a_j b_i \cdot \mathrm{d}t_{\alpha_j} \cdot \mathrm{d}t_{\beta_i}.$$

Now, we can use the property (1.3) obtaining $\mathrm{d}t_{\alpha_j} \cdot \mathrm{d}t_{\beta_i} = \mathrm{d}t_{\frac{\alpha_j \beta_i}{\alpha_j + \beta_i}}$. To the resulting sum, we can apply the method used above to obtain the decomposition of $x \cdot y$, i.e. the sum of all the coefficients corresponding to the same order, the deletion of the terms for which $c_q = 0$ or $q < 1$ and, finally, the ordering of the remaining summands. The proof proceed exactly as above for the sum, noting that we also have to use (1.3) for the structure $(R, \oplus, \odot, {}^{\diamond}(-), \delta t_{(-)})$, but in the reverse order with respect to the previous application.

Finally, the inverse morphism is necessarily defined as

$$g\left(r \oplus \bigoplus_{j=1}^{N} a_j \odot \delta t_{\alpha_j} \right) = r + \sum_{j=1}^{N} a_j \cdot \mathrm{d}t_{\alpha_j}.$$

Exactly as above, we can prove that g is indeed a morphism of ordered rings.

2 The model

Surprisingly, the model of the previous list of axioms is quite simple. For proofs and motivations related to this section, see [8].

We firstly need the following class of functions

Definition 2. We say that x *is a little-oh polynomial*, and we write $x \in \mathbb{R}_o[t]$ iff

1. $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$
2. We can write

$$x(t) = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad \text{as} \quad t \to 0^+$$

   for suitable

$$k \in \mathbb{N}$$

$$r, \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

$$a_1, \dots, a_k \in \mathbb{R}_{\geq 0}.$$

Hence, a little-oh polynomial $x \in \mathbb{R}_o[t]$ is a polynomial function with real coefficients, in the real variable $t \geq 0$, with generic positive powers of t, and up to a little-oh function as $t \to 0^+$.

Remark 3. Sometimes, but not always, we will use a notation like $h_t := h(t)$ for real functions of the real variable t. This permits to decrease the number of parenthesis used in formulas and to leave the classical notation $f(x)$ for functions of the form $f : {}^\bullet\mathbb{R} \longrightarrow {}^\bullet\mathbb{R}$. Moreover, we will use a slight modification of Landau's little-oh notation: writing $x_t = y_t + o(t)$ as $t \to 0^+$ we will always mean

$$\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0 \quad \text{and} \quad x_0 = y_0.$$

In other words, every little-oh function we will consider is continuous as $t \to 0^+$.

We can now define:

Definition 4. Let x, $y \in \mathbb{R}_o[t]$, then we say that $x \sim y$ or that $x = y$ in ${}^\bullet\mathbb{R}$ iff $x(t) = y(t) + o(t)$ as $t \to 0^+$. Because it is easy to prove that $\sim$ is an equivalence relation, we can define the quotient set ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$. Moreover, we define the standard part map as ${}^\circ(-) : x \in {}^\bullet\mathbb{R} \mapsto {}^\circ x = x(0) \in \mathbb{R}$, and $\mathrm{d}t_a := [t \in \mathbb{R}_{\geq 0} \mapsto t^{1/a} \in \mathbb{R}]_\sim \in {}^\bullet\mathbb{R}$.

Definition 5. Let A be an open subset of $\mathbb{R}^n$, $f : A \longrightarrow \mathbb{R}$ a smooth function. Define ${}^\bullet A$ like in (1.4) (or, equivalently, as ${}^\bullet A = A_o[t]/\sim$, where $A_o[t]$ is the set of little-oh polynomials taking values in the open set A) and for $x \in {}^\bullet A$ define

$${}^\bullet f(x) := f \circ x \text{ in } {}^\bullet\mathbb{R}.$$

In other words, using the notation $[x]_\sim \in {}^\bullet\mathbb{R}$ for the equivalence class generated by $x \in \mathbb{R}_o[t]$ modulo the relation $\sim$ defined in Definition 4, we can write the previous definition as ${}^\bullet f([x]_\sim) := [f \circ x]_\sim$.

It is not hard to show that this is indeed a model for our axioms (see [8]).

To see applications of this type of infinitesimals to elementary physics, see [9]. For meaningful (standard) metrics on ${}^\bullet\mathbb{R}$ and for roots of (nilpotent!) infinitesimals and their applications to fractional derivatives, see [12]. For the foundation of differential calculus for functions more general than extensions of ordinary smooth ones (like the previously mentioned $f(x) = x + \mathrm{d}t$), see [11].

We finally underscore that the theory of Fermat reals can be developed very far: every smooth manifold can be analogously extended using this type of infinitesimals; more generally, this extension is applicable to every diffeological space ([13]) obtaining a functor with very good preservation properties; the category of diffeological spaces is cartesian closed ([13, 10]) and embeds the category of smooth manifolds, so that these *Fermat extensions* can also be applied to infinite dimensional function spaces. For more details, see [11, 7].

3 Infinitesimals and order properties

Like in other disciplines, also in mathematics the layout of a work reflects the personal philosophical ideas of the authors. In particular, the present work is based on the idea that a good mathematical theory is able to construct a good dialectic between formal properties, proved in the theory, and their informal interpretations. The dialectic has to be, as far as possible, in both directions: theorems proved in the theory should have a clear and useful intuitive interpretation and, on the other hand, the intuition corresponding to the theory has to be able to suggest true sentences, i.e. conjectures or sketch of proofs that can be converted into rigorous proofs.

In a theory of new numbers, like the present one about Fermat reals, the introduction of an order relation can be a hard test of the excellence of this dialectic between formal properties and their informal interpretations. Indeed, if we introduce a new ring of numbers (like ${}^\bullet\mathbb{R}$) extending the real field $\mathbb{R}$, we want that the new order relation will extend the standard one on $\mathbb{R}$. This extension naturally leads to the wish of findings a geometrical representation of the new numbers, according to the above principle of having a good formal/informal dialectic.

For example, on the one hand in NSA the order relation on ${}^*\mathbb{R}$ has the best formal properties among all the theories of actual infinitesimals. On the other hand, the dialectic of these properties with the informal interpretations is not always good, due to the use of, e.g., an ultrafilter in the construction of ${}^*\mathbb{R}$. Indeed, in an ultrafilter on $\mathbb{N}$ we can always find a highly non constructive set $A \subset \mathbb{N}$ (see e.g. Appendix B in [7]); any sequence of reals $x : \mathbb{N} \longrightarrow \mathbb{R}$ which is constant to 1 on A is strictly greater than 0 in ${}^*\mathbb{R}$, but it seems not easy to give neither an intuitive interpretation nor a clear and meaningful geometric representation of the relation $x > 0$ in ${}^*\mathbb{R}$. In fact, it is also for motivations of this type that some approaches to give a constructive definition of a field similar to ${}^*\mathbb{R}$ have been attempted (see e.g. [16, 17, 18] and references therein).

In SDG we have a preorder relation (i.e. a reflexive and transitive relation, which is not necessarily anti-symmetric) with very poor properties, if compared with those of NSA. Nevertheless, the works developed in SDG (see e.g. [15]) exhibits that meaningful results can be obtained also in the differential geometry of infinite dimensional spaces, even if the order properties of the ground base ring are not so rich. Once again, the dialectic between formal properties and their intuitive interpretations represents a hard test for SDG too. E.g. it seems not so easy to interpret intuitively that every infinitesimal h in SDG verifies both $h \geq 0$ and $h \leq 0$. The lack of a total order, i.e. of the trichotomy law

$$x < y \quad \text{or} \quad y < x \quad \text{or} \quad x = y \tag{3.1}$$

makes really difficult, or even impossible, to have a geometrical representation of the infinitesimals of SDG.

We want to start this section showing that in our setting there is a strong connection between some order properties and some algebraic properties. In particular, we will show that it is not possible to have good order properties and at the same time a uniqueness without limitations in the first order infinitesimal Taylor formula, i.e.

$$\exists!\, m \in {}^\bullet\mathbb{R} : \ \forall h \in D : \ f(h) = f(0) + h \cdot m, \tag{3.2}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth. We start observing that in ${}^\bullet\mathbb{R}$ the product of two first order infinitesimals $h, k \in D$ is always zero: $h \cdot k = 0$ (see "Axiom, cancellation laws"). In the following theorem, we can see that the property $h \cdot k = 0$ is a general consequence of the hypothesis to have a total order on D. The idea of this theorem can be glimpsed at from the figure 3.1, where it is represented that if we neglect h^2 and k^2, because we consider them zero, then we have strong reasons to expect that also $h \cdot k$ will also be zero

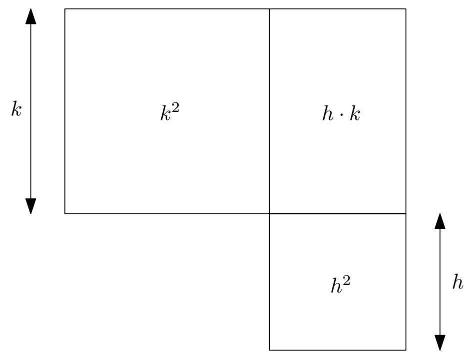

Figure 3.1: How to guess that $h \cdot k = 0$ for two first order infinitesimals h, $k \in D$

From this picture comes the idea to find a formal demonstration based on the implication

$$h, k \geq 0\ ,\ \ h \leq k \quad \Longrightarrow \quad 0 \leq hk \leq k^2 = 0$$

All these ideas conduct toward the following theorem.

Theorem 6. *Let $(R, \leq)$ be a generic ordered ring and $D \subseteq R$ a subset of this ring, such that*

1. $0 \in D$
2. $\forall h \in D :\ \ h^2 = 0$ *and* $-h \in D$
3. $(D, \leq)$ *is a total order*

then

$$\forall h, k \in D :\ \ h \cdot k = 0 \tag{3.3}$$

This theorem implies that if we want a total order in our theory of infinitesimal numbers, and if in this theory we consider $D = \{h \,|\, h^2 = 0\}$, then we must accept that the product of any two elements of D must be zero. For example, if we think that a geometric representation of infinitesimals cannot be possible if we do not have, at least, the trichotomy law, then in this theory we must also have that the product of two first order infinitesimals is zero. Finally, because in SDG property (3.3) is false, this theorem also implies that in SDG it is not possible to define a total order (and not only a preorder) on the set D of first order infinitesimals compatible with ring operations.

Proof of Theorem 6:

Let h, $k \in D$ be two elements of the subset D. By hypotheses $0, -h, -k \in D$, hence all these elements are comparable with respect to the order relation $\leq$, because, by hypotheses, this relation is total (i.e. (3.1) is true). E.g.

$$h \leq k \quad \text{or} \quad k \leq h.$$

We will consider only the case $h \leq k$, because analogously we can deal with the case $k \leq h$, simply exchanging everywhere h with k and vice versa.

First sub-case: $k \geq 0$. By multiplying both sides of $h \leq k$ by $k \geq 0$ we obtain

$$hk \leq k^2. \tag{3.4}$$

If $h \geq 0$ then, multiplying by $k \geq 0$ we have $0 \leq hk$, so from (3.4) we have $0 \leq hk \leq k^2 = 0$, and hence $hk = 0$.
If $h \leq 0$ then, multiplying by $k \geq 0$ we have

$$hk \leq 0. \tag{3.5}$$

If, furthermore, $h \geq -k$, then multiplying by $k \geq 0$ we have $hk \geq -k^2$, hence form (3.5) $0 \geq hk \geq -k^2 = 0$, hence $hk = 0$.
If, otherwise, $h \leq -k$, then multiplying by $-h \geq 0$ we have $-h^2 = 0 \leq hk \leq 0$ from (3.5), hence $hk = 0$. This concludes the discussion of the case $k \geq 0$.

Second sub-case: $k \leq 0$. In this case we have $h \leq k \leq 0$. Multiplying both inequalities by $h \leq 0$ we obtain $h^2 = 0 \geq hk \geq 0$ and hence $hk = 0$.

Property (3.3) is incompatible with the uniqueness in a possible formula like (3.2) framed in a ring R of Theorem 6. In fact, if a, $b \in D$ are two elements of the subset $D \subseteq R$, then both a and b play the role of $m \in R$ in for the linear function

$$f : h \in D \mapsto h \cdot a = 0 \in R.$$

So, if (3.2) applies to linear functions (or less, to constant functions), the uniqueness part of this formula cannot hold in the ring R.

In the next section, we will introduce a natural and meaningful total order relation on ${}^\bullet\mathbb{R}$. Since we will also see that the order relation permits to have a geometric representation of Fermat reals, we can summarize the conclusions of this section saying that the uniqueness in (3.2) is incompatible with a natural geometric interpretation of Fermat reals and hence with a good dialectic between formal properties and informal interpretations in this theory.

4 Order relation

From the previous sections one can draw the conclusion that the ring of Fermat reals ${}^\bullet\mathbb{R}$ is essentially "the little-oh" calculus. On the other hand the

Fermat reals give us more flexibility than this calculus: working with ${}^\bullet\mathbb{R}$, we do not have to bother ourselves with remainders made of "little-oh", but we can neglect them and use the useful algebraic calculus with nilpotent infinitesimals. Moreover, thinking the elements of ${}^\bullet\mathbb{R}$ as new numbers, and not simply as "little-oh functions", permits to treat them in a different and new way, for example to define on them an order relation with a clear geometrical interpretation.

First of all, let us introduce the useful notation

$$\forall^0 t \geq 0 : \; \mathcal{P}(t)$$

and we will read the quantifier $\forall^0 t \geq 0$ saying *"for every $t \geq 0$ (sufficiently) small"*, to indicate that the property $\mathcal{P}(t)$ is true for all t in some right neighborhood of $t = 0$ (recall that, by Definition 2, our little-oh polynomials are defined on $\mathbb{R}_{\geq 0}$.), i.e.

$$\exists\, \delta > 0 : \; \forall t \in [0, \delta) : \; \mathcal{P}(t)$$

The first heuristic idea to define an order relation is the following

$$x \leq y \iff x - y \leq 0 \iff \exists\, z : \; z = 0 \text{ in } {}^\bullet\mathbb{R} \quad \text{and} \quad x - y \leq z. \tag{4.1}$$

More precisely, if x, $y \in {}^\bullet\mathbb{R}$ are two little-oh polynomials, we want to ask locally that x_t is less than or equal to y_t, but up to a $o(t)$ for $t \to 0^+$, where the little-oh function $o(t)$ depends on x and y. Where it will be useful to simplify notations, we will write "$x = y$ in ${}^\bullet\mathbb{R}$" instead of $x \sim y$, and we will talk directly about the elements of $\mathbb{R}_o[t]$ instead of their equivalence classes; for example we can say that $x = y$ in ${}^\bullet\mathbb{R}$ and $z = w$ in ${}^\bullet\mathbb{R}$ imply $x + z = y + w$ in ${}^\bullet\mathbb{R}$. The only notion of equality between little-oh polynomials is, of course, the equivalence relation of Definition 4 and, as usual, we must always prove that our relations between little-oh polynomials are well defined. Formally, the idea (4.1) corresponds to the following

Definition 7. Let x, $y \in {}^\bullet\mathbb{R}$, then we say

$$x \leq y$$

iff we can find $z \in {}^\bullet\mathbb{R}$ such that $z = 0$ in ${}^\bullet\mathbb{R}$ and

$$\forall^0 t \geq 0 : \; x_t \leq y_t + z_t.$$

Recall that $z = 0$ in ${}^\bullet\mathbb{R}$ is equivalent to $z_t = o(t)$ for $t \to 0^+$. It is immediate to see that we can equivalently define $x \leq y$ if and only if we can find $x' = x$ and $y' = y$ in ${}^\bullet\mathbb{R}$ such that $x_t \leq y_t$ for every t sufficiently small. From this it also follows that the relation $\leq$ is well defined on ${}^\bullet\mathbb{R}$, i.e. if $x' = x$ and $y' = y$ in ${}^\bullet\mathbb{R}$ and $x \leq y$, then $x' \leq y'$. As usual, we will use the notation $x < y$ for $x \leq y$ and $x \neq y$.

Theorem 8. *The relation $\leq$ is an order, i.e. is reflexive, transitive and anti-symmetric. It extends the order relation of $\mathbb{R}$, and with it $({}^\bullet\mathbb{R}, \leq)$ is an ordered ring. Finally the following sentences are equivalent:*

1. *$h \in D_\infty$, i.e. h is an infinitesimal*
2. *$\forall r \in \mathbb{R}_{>0} : \; -r < h < r$.*

Hence, an infinitesimal can be thought of as a number with standard part zero, or as a number smaller than every standard positive real number and greater than every standard negative real number. Thus, it has in this sense the same property as an infinitesimal both in NSA and in SDG (in the latter case with real numbers of type $\frac{1}{n}$ ($n \in \mathbb{N}_{>0}$) only).

Proof: It is immediate to prove that the relation is reflexive. To prove transitivity, if $x \leq y$ and $y \leq w$, then we have

$$\forall^0 t \geq 0 : \; x_t \leq y_t + z_t \quad \text{and} \quad \forall^0 t \geq 0 : \; y_t \leq w_t + z'_t,$$

and these imply

$$\forall^0 t \geq 0 : \; x_t \leq y_t + z_t \leq w_t + z_t + z'_t,$$

showing that $x \leq w$. To prove that it is also anti-symmetric, take $x \leq y$ and $y \leq x$, then we have

$$x_t \leq y_t + z_t \quad \forall t \in [0, \delta_1) \tag{4.2}$$

$$y_t \leq x_t + z'_t \quad \forall t \in [0, \delta_2) \tag{4.3}$$

$$\lim_{t \to 0^+} \frac{z_t}{t} = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{z'_t}{t} = 0$$

because z and z' are equal to zero in ${}^\bullet\mathbb{R}$, i.e. are $o(t)$ for $t \to 0^+$. Hence, from (4.2) and (4.3), for $\delta := \min\{\delta_1, \delta_2\}$ we have

$$-\frac{z'_t}{t} \leq \frac{x_t - y_t}{t} \leq \frac{z_t}{t} \quad \forall t \in [0, \delta)$$

and hence $\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0$, that is $x = y$ in ${}^\bullet\mathbb{R}$.

If r, $s \in \mathbb{R}$ and $r \leq s$ as real numbers, then it suffices to take $z_t = 0$ for every $t \geq 0$ in Definition 7 to obtain that $r \leq s$ also in ${}^\bullet\mathbb{R}$. Vice versa, if $r \leq s$ in ${}^\bullet\mathbb{R}$, then for some $z = 0$ in ${}^\bullet\mathbb{R}$ we have

$$\forall^0 t \geq 0 : \; r \leq s + z_t$$

and hence, for $t = 0$ we have $r \leq s$ in $\mathbb{R}$ because $z = 0$ and hence $z_0 = 0$. This proves that the order relation $\leq$ defined in ${}^\bullet\mathbb{R}$ extends the order relation on $\mathbb{R}$.

The relationships between the ring operations and the order relation can be stated as

$$\begin{aligned} x \le y \quad &\Longrightarrow \quad x + w \le y + w \\ x \le y \quad &\Longrightarrow \quad -x \ge -y \\ x \le y \ , \ w \ge 0 \quad &\Longrightarrow \quad x \cdot w \le y \cdot w. \end{aligned}$$

The first two are immediate consequences of the Definition 7. To prove the last one, let us suppose that

$$\begin{aligned} x_t \le y_t + z_t \quad \forall^0 t \ge 0 \\ w_t \ge z'_t \quad \forall^0 t \ge 0. \end{aligned} \tag{4.4}$$

Then $w_t - z'_t \ge 0$ for every t small and hence from (4.4)

$$x_t \cdot (w_t - z'_t) \le y_t \cdot (w_t - z'_t) + z_t \cdot (w_t - z'_t) \quad \forall^0 t \ge 0$$

from which it follows

$$x_t \cdot w_t \le y_t \cdot w_t + (-x_t z'_t - y_t z'_t + z_t w_t - z_t z'_t) \quad \forall^0 t \ge 0.$$

But $-xz' - yz' + zw - zz' = 0$ in ${}^\bullet\mathbb{R}$ because $z = 0$ and $z' = 0$ and hence the conclusion follows.

Finally, we know (see "Axiom, ideals of infinitesimals") that $h \in D_\infty$ if and only if ${}^\circ h = 0$ and this is equivalent to

$$\forall r \in \mathbb{R}_{>0} : \ -r < {}^\circ h < r. \tag{4.5}$$

But if, e.g., ${}^\circ h < r$, then

$$\forall^0 t \ge 0 : \ h_t \le r$$

because the function $t \to h_t$ is continuous, and hence we also have $h \le r$ in ${}^\bullet\mathbb{R}$. Analogously, from (4.5) we can prove that $-r \le h$ for all $r \in \mathbb{R}_{>0}$. Of course, $r \notin D_\infty$ if $r \in \mathbb{R}$, so it cannot be that $h = r$.

Vice versa, if

$$\forall r \in \mathbb{R}_{>0} : \ -r < h < r$$

then, e.g., $h_t \le r + z_t$ for t small. Hence, for $t = 0$ we have $-r \le {}^\circ h = h_0 \le r$ for every $r > 0$, and so ${}^\circ h = 0$.

Example. We have e.g. $\mathrm{d}t > 0$ and $\mathrm{d}t_2 - 3\,\mathrm{d}t > 0$ because for $t \ge 0$ sufficiently small $t^{1/2} > 3t$ and hence

$$t^{1/2} - 3t > 0 \quad \forall^0 t \ge 0.$$

From examples like these ones we can guess that our little-oh polynomials are always locally comparable with respect to pointwise order relation, and this is the first step to prove that the trichotomy law holds. In the following statement we will use the notation $\forall^0 t > 0 : \ \mathcal{P}(t)$, that naturally means

$$\forall^0 t \geq 0 : \ t \neq 0 \quad \Longrightarrow \quad \mathcal{P}(t),$$

where $\mathcal{P}(t)$ is a generic property depending on t.

Lemma 9. *Let $x,\ y \in {}^\bullet\mathbb{R}$, then*

1. $^\circ x < {}^\circ y \quad \Longrightarrow \quad \forall^0 t \geq 0 : \ x_t < y_t$
2. *If $^\circ x = {}^\circ y$, then*

$$\left(\forall^0 t > 0 : \ x_t < y_t\right) \quad or \quad \left(\forall^0 t > 0 : \ x_t > y_t\right) \quad or \quad \left(x = y \ in \ {}^\bullet\mathbb{R}\right)$$

Proof:
1.) Let us suppose that $^\circ x < {}^\circ y$, then the continuous function $t \geq 0 \mapsto y_t - x_t \in \mathbb{R}$ assumes the value $y_0 - x_0 > 0$, hence is locally positive, i.e.

$$\forall^0 t \geq 0 : \ x_t < y_t$$

2.) Now let us suppose that $^\circ x = {}^\circ y$, and introduce the notations

$$x_t = {}^\circ x + \sum_{i=1}^{N} \alpha_i \cdot t^{a_i} + z_t \quad \forall t \geq 0$$
$$y_t = {}^\circ y + \sum_{j=1}^{M} \beta_j \cdot t^{b_j} + w_t \quad \forall t \geq 0$$

where $x = {}^\circ x + \sum_{i=1}^{N} \alpha_i \cdot \mathrm{d}t_{1/a_i}$ and $y = {}^\circ y + \sum_{j=1}^{M} \beta_j \cdot \mathrm{d}t_{1/b_j}$ are the decompositions of x and y (hence $0 < \alpha_i < \alpha_{i+1} \leq 1$ and $0 < \beta_j < \beta_{j+1} \leq 1$), whereas w and z are little-oh polynomials such that $z_t = o(t)$ and $w_t = o(t)$ for $t \to 0^+$.
Case: $a_1 < b_1$. In this case the least power in the two decompositions is $\alpha_1 \cdot t^{a_1}$, and hence we expect that the second alternative of the conclusion is the true one if $\alpha_1 > 0$, otherwise the first alternative will be the true one if $\alpha_1 < 0$ (recall that always $\alpha_i \neq 0$ in a decomposition). Indeed, let us analyze, for $t > 0$, the condition $x_t < y_t$: the following formulas are all equivalent to it

$$\sum_{i=1}^{N} \alpha_i \cdot t^{a_i} < \sum_{j=1}^{N} \beta_j \cdot t^{b_j} + w_t - z_t$$

$$t^{a_1} \cdot \left[\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1}\right] < t^{a_1} \cdot \left[\sum_{j=1}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}\right]$$

$$\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1} < \sum_{j=1}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}.$$

Therefore, let us consider the function

$$f(t) := \sum_{j=1}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1} - \alpha_1 - \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1} \quad \forall t \geq 0$$

We can write

$$(w_t - z_t) \cdot t^{-a_1} = \frac{w_t - z_t}{t} \cdot t^{1-a_1}$$

and $\frac{w_t - z_t}{t} \to 0$ as $t \to 0^+$ because $w_t = o(t)$ and $z_t = o(t)$. Furthermore, $a_1 \leq 1$ hence t^{1-a_1} is bounded in a right neighborhood of $t = 0$. Therefore, $(w_t - z_t) \cdot t^{-a_1} \to 0$ and the function f is continuous at $t = 0$ too, because $a_1 < a_i$ and $a_1 < b_1 < b_j$. By continuity, the function f is locally strictly positive if and only if $f(0) = -\alpha_1 > 0$, hence

$$\begin{aligned} \left(\forall^0 t > 0 : \ x_t < y_t\right) &\iff \alpha_1 < 0 \\ \left(\forall^0 t > 0 : \ x_t > y_t\right) &\iff \alpha_1 > 0. \end{aligned}$$

Case: $a_1 > b_1$. We can argue in an analogous way with b_1 and β_1 instead of a_1 and α_1.
Case: $a_1 = b_1$. We shall exploit the same idea used above and analyze the condition $x_t < y_t$. The following are equivalent ways to express this condition

$$t^{a_1} \cdot \left[\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1}\right] < t^{a_1} \cdot \left[\beta_1 + \sum_{j=2}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}\right]$$

$$\alpha_1 + \sum_{i=2}^{N} \alpha_i \cdot t^{a_i - a_1} < \beta_1 + \sum_{j=2}^{N} \beta_j \cdot t^{b_j - a_1} + (w_t - z_t) \cdot t^{-a_1}.$$

Hence, exactly as we have demonstrated above, we can state that

$$\begin{aligned} \alpha_1 < \beta_1 &\implies \forall^0 t > 0 : \ x_t < y_t \\ \alpha_1 > \beta_1 &\implies \forall^0 t > 0 : \ x_t > y_t. \end{aligned}$$

Otherwise $\alpha_1 = \beta_1$ and we can restart with the same reasoning using a_2, b_2, α_2, β_2, etc. If $N = M$, i.e. the number of addends in the decompositions are equal, using this procedure we can prove that

$$\forall t \geq 0 : \ x_t = y_t + w_t - z_t,$$

that is $x = y$ in ${}^\bullet\mathbb{R}$.

It remains to consider the case, e.g., $N < M$. In this hypotheses, using the previous procedure we would arrive at the following equivalent way to express the condition $x_t < y_t$:

$$0 < \sum_{j>N} \beta_j \cdot t^{b_j} + w_t - z_t$$

$$0 < t^{b_{N+1}} \cdot \left[\beta_{N+1} + \sum_{j>N+1} \beta_j \cdot t^{b_j - b_{N+1}} + (w_t - z_t) \cdot t^{-b_{N+1}} \right]$$

$$0 < \beta_{N+1} + \sum_{j>N+1} \beta_j \cdot t^{b_j - b_{N+1}} + (w_t - z_t) \cdot t^{-b_{N+1}}.$$

Hence

$$\beta_{N+1} > 0 \quad \Longrightarrow \quad \forall^0 t > 0 : \; x_t < y_t$$

$$\beta_{N+1} < 0 \quad \Longrightarrow \quad \forall^0 t > 0 : \; x_t > y_t.$$

This lemma can be used to find an equivalent formulation of the order relation.

Theorem 10. *Let $x,\, y \in {}^\bullet\mathbb{R}$, then*

1. $x \le y \quad \Longleftrightarrow \quad \big(\forall^0 t > 0 : \; x_t < y_t\big)$ *or* $(x = y$ *in* ${}^\bullet\mathbb{R})$
2. $x < y \quad \Longleftrightarrow \quad \big(\forall^0 t > 0 : \; x_t < y_t\big)$ *and* $(x \ne y$ *in* ${}^\bullet\mathbb{R})$

Proof:
1. $\Rightarrow$: If ${}^\circ x < {}^\circ y$ then, from the previous Lemma 9 we can derive that the first alternative is true. If ${}^\circ x = {}^\circ y$, then from Lemma 9 we have

$$\big(\forall^0 t > 0 : \; x_t < y_t\big) \quad \text{or} \quad (x = y \text{ in } {}^\bullet\mathbb{R}) \quad \text{or} \quad \big(\forall^0 t > 0 : \; x_t > y_t\big) \tag{4.6}$$

In the first two cases we have the conclusion. In the third case, from $x \le y$ we obtain

$$\forall^0 t \ge 0 : \; x_t \le y_t + z_t, \tag{4.7}$$

with $z_t = o(t)$. Hence, from the third alternative of (4.6) we have

$$0 < x_t - y_t \le z_t \quad \forall^0 t > 0$$

and hence $\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0$, i.e. $x = y$ in ${}^\bullet\mathbb{R}$.

1. $\Leftarrow$: This follows immediately from the reflexive property of $\le$ or from the Definition 7.

2. $\Rightarrow$: From $x < y$ we have $x \le y$ and $x \ne y$, so the conclusion follows from the previous *1.*

2. $\Leftarrow$: From $\forall^0 t > 0 : x_t < y_t$ and from *1.* it follows $x \le y$ and hence $x < y$ from the hypotheses $x \ne y$.
Now we can prove that our order is total

Corollary 11. *Let x, $y \in {}^\bullet\mathbb{R}$, then in ${}^\bullet\mathbb{R}$ we have*

1. $x \le y \quad$ *or* $\quad y \le x \quad$ *or* $\quad x = y$
2. $x < y \quad$ *or* $\quad y < x \quad$ *or* $\quad x = y$

Proof:
1.) If $^\circ x < {}^\circ y$, then from Lemma 9 we have $x_t < y_t$ for $t \ge 0$ sufficiently small. Hence, from Theorem 10 we have $x \le y$. We can argue in the same way if $^\circ x > {}^\circ y$. Also the case $^\circ x = {}^\circ y$ can be handled in the same way using *2.* of Lemma 9.

2.) This part is a general consequence of the previous one. Indeed, if we have $x = y$, then we have the conclusion. Otherwise we have $x \ne y$, and using the previous *1.* we can deduce strict inequalities from inequalities because $x \ne y$.

From the proof of Lemma 9 and from Theorem 10 we can deduce the following

Theorem 12. *Let x, $y \in {}^\bullet\mathbb{R}$. If $^\circ x \ne {}^\circ y$, then*

$$x < y \iff {}^\circ x < {}^\circ y.$$

Otherwise, if $^\circ x = {}^\circ y$, then

1. *If $\omega(x) > \omega(y)$, then $x > y$ iff $^\circ x_1 > 0$*
2. *If $\omega(x) = \omega(y)$, then*

$$\begin{aligned} {}^\circ x_1 > {}^\circ y_1 \quad &\Longrightarrow \quad x > y \\ {}^\circ x_1 < {}^\circ y_1 \quad &\Longrightarrow \quad x < y. \end{aligned}$$

This statement can be taken for an axiomatic definition of the order. Using it, it is not hard to prove that the isomorphism of Theorem 1 preserves also this order relation.

Example. The previous Theorem gives an effective criterion to decide whether $x < y$ or not. Indeed:

1. first of all $x < y$ is equivalent to $0 < y - x$, so we can describe the algorithm for the case $0 < x$, $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, only. If the standard part $^\circ x \ne 0$, then the order relation can be decided on the basis of this standard part only. E.g. $2 + \mathrm{d}t_2 > 0$ and $1 + \mathrm{d}t_2 < 3 + \mathrm{d}t$.

2. Otherwise, if the standard part $^\circ x = 0$, we look at the order $\omega(x)$ and at the first standard part $^\circ x_1$, which is the coefficient of the biggest infinitesimals in the decompositions of x: because $\omega(x) > \omega(0) = 0$, we have that $x > 0$ iff $^\circ x_1 > 0$. E.g. $3\,\mathrm{d}t_2 > 0$; $\mathrm{d}t_2 > a\,\mathrm{d}t$ for every $a \in \mathbb{R}$; $\mathrm{d}t < \mathrm{d}t_2 < \mathrm{d}t_3 < \ldots < \mathrm{d}t_k$ for every $k > 3$, and $\mathrm{d}t_k > 0$.

4.1 Absolute value

Having a total order, we can define the absolute value

Definition 13. Let $x \in {}^\bullet\mathbb{R}$, then

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Exactly like for the real field $\mathbb{R}$, we can prove the usual properties of the absolute value:

$$\begin{aligned} & |x| \geq 0 \\ & |x+y| \leq |x| + |y| \\ & -|x| \leq x \leq |x| \\ & ||x| - |y|| \leq |x-y| \\ & |x| = 0 \iff x = 0. \end{aligned}$$

Moreover, also the following cancellation law is provable.

Theorem 14. *Let $h \in {}^\bullet\mathbb{R} \setminus \{0\}$ and $r,\ s \in \mathbb{R}$, then*

$$|h| \cdot r \leq |h| \cdot s \quad \Longrightarrow \quad r \leq s$$

Proof: In fact if $|h| \cdot r \leq |h| \cdot s$ then from Theorem 10 we obtain that either

$$\forall^0 t > 0 : \ |h_t| \cdot r \leq |h_t| \cdot s \tag{4.8}$$

or $|h| \cdot r = |h| \cdot s$. But $h \neq 0$ so

$$\left(\forall^0 t > 0 : \ h_t > 0\right) \quad \text{or} \quad \left(\forall^0 t > 0 : \ h_t < 0\right)$$

hence we can always find a $\bar{t} > 0$ such that $|h_{\bar{t}}| \neq 0$ and to which (4.8) is applicable. Therefore, in the first case we must have $r \leq s$. In the second one we have

$$|h| \cdot r = |h| \cdot s$$

but $h \neq 0$, hence $|h| \neq 0$ and so the conclusion follows from "Axiom, cancellation laws".

4.2 Powers and logarithms

In this section we will tackle definition and properties of powers x^y and logarithms $\log_x y$. Due to the presence of nilpotent elements in ${}^\bullet\mathbb{R}$, we cannot define these operations without any limitation. For example, we cannot define the square root having the usual properties, like

$$x \in {}^\bullet\mathbb{R} \quad \Longrightarrow \quad \sqrt{x} \in {}^\bullet\mathbb{R} \tag{4.9}$$
$$x = y \text{ in } {}^\bullet\mathbb{R} \quad \Longrightarrow \quad \sqrt{x} = \sqrt{y} \text{ in } {}^\bullet\mathbb{R} \tag{4.10}$$
$$\sqrt{x^2} = |x|$$

because they are incompatible with the existence of $h \in D$ such that $h^2 = 0$, but $h \neq 0$ (nonetheless, it is possible to define arbitrary roots of our nilpotent infinitesimals with sufficient good properties, see [12]). We can obtain a property like (4.9) (i.e. the closure of ${}^\bullet\mathbb{R}$ with respect to a given operation) only for smooth functions, because they have to take little-oh polynomials to little-oh polynomials. Moreover, the Definition 5 states that to obtain a well defined operation we need a locally Lipschitz function. For these reasons, we will limit x^y to $x > 0$ and x invertible only, and $\log_x y$ to x, $y > 0$ and both x, y invertible.

Definition 15. Let x, $y \in {}^\bullet\mathbb{R}$, with x strictly positive and invertible, then

1. $x^y := [t \geq 0 \mapsto x_t^{y_t}]_{= \text{ in } {}^\bullet\mathbb{R}}$
2. If $y > 0$ and y is invertible, then $\log_x y := [t \geq 0 \mapsto log_{x_t} y_t]_{= \text{ in } {}^\bullet\mathbb{R}}$.

Because of Theorem 10, from $x > 0$ we have

$$\forall^0 t > 0 : \; x_t > 0$$

so that the previous operations are well defined in ${}^\bullet\mathbb{R}$ because ${}^\circ x \neq 0 \neq {}^\circ y$. Directly from the definition of equality in ${}^\bullet\mathbb{R}$ (Definition 4) the usual properties follow:

$$\begin{aligned}
(x^y)^z &= x^{y \cdot z} \\
x^y \cdot x^z &= x^{y+z} \\
x^n &= x \cdot \overset{n}{\ldots\ldots} \cdot x \quad \text{if} \quad n \in \mathbb{N} \\
\log_x (x^y) &= y \\
x^{\log_x y} &= y \\
\log(x \cdot y) &= \log x + \log y \\
\log_x (y^z) &= z \cdot \log_x y \\
x^{\log y} &= y^{\log x}.
\end{aligned}$$

About the monotonicity properties, it suffices to use Theorem 10 to prove immediately the usual properties (where x, y and w are invertible)

$$\begin{aligned}
z > 0 \, , \, x \geq y > 0 \quad &\Longrightarrow \quad x^z \geq y^z \\
z < 0 \, , \, x \geq y > 0 \quad &\Longrightarrow \quad x^z \leq y^z \\
w > 1 \, , \, x \geq y > 0 \quad &\Longrightarrow \quad \log_w x \geq \log_w y \\
0 < w < 1 \, , \, x \geq y > 0 \quad &\Longrightarrow \quad \log_w x \leq \log_w y
\end{aligned}$$

Analogous implications, but with strict equalities, are true if we suppose $x > y$.

Finally, it can be useful to state here the *elementary transfer theorem for inequalities*, whose proof follows immediately from the definition of $\le$ and from Theorem 10:

Theorem 16. *Let A be an open subset of $\mathbb{R}^n$, and τ, $\sigma : A \longrightarrow \mathbb{R}$ be smooth functions. Then*

$$\forall x \in {}^\bullet A : \ {}^\bullet\tau(x) \le {}^\bullet\sigma(x)$$

iff

$$\forall r \in A : \ \tau(r) \le \sigma(r).$$

4.3 Solution of linear equation

In this section we want to face the existence problem concerning 1-dimensional linear equations.

Theorem 17. *If a, b, $c \in {}^\bullet\mathbb{R}$ and $a < c < a + b$, then*

$$\exists x \in {}^\bullet\mathbb{R} : \ 0 \le x \le 1 \text{ and } a + x \cdot b = c$$

Proof: We can suppose $a = 0$, because $a + x \cdot b = c$ is equivalent to $x \cdot b = c - a$. If ${}^\circ b \ne 0$, then b is invertible and it suffices to set $x := \frac{c}{b}$ to have the conclusion. Otherwise, ${}^\circ b = 0$ and hence also ${}^\circ c = 0$. Let us consider the decompositions of c and b

$$c = \sum_{i=1}^{k} {}^\circ c_i \cdot \mathrm{d}t_{\omega_i(c)}$$

$$b = \sum_{j=1}^{h} {}^\circ b_j \cdot \mathrm{d}t_{\omega_j(b)}.$$

We have to find a number $x = {}^\circ x + \sum_{n=1}^{N} {}^\circ x_n \cdot \mathrm{d}t_{\omega_n(x)}$ such that $x \cdot b = c$. Since $a = 0$, our hypotheses $a < c < a + b$ becomes $0 < c < b$, so that $\forall^0 t > 0 : \ b_t > 0$ and hence for $t > 0$ sufficiently small, we can form the ratio between the corresponding little-oh polynomials

$$\begin{aligned} \frac{c_t}{b_t} &= \frac{\sum_{i=1}^{k} {}^\circ c_i \cdot t^{\frac{1}{\omega_i(c)}}}{\sum_{j=1}^{h} {}^\circ b_j \cdot t^{\frac{1}{\omega_j(b)}}} \\ &= \frac{t^{\frac{1}{\omega_1(b)}} \cdot \sum_{i=1}^{k} {}^\circ c_i \cdot t^{\frac{1}{\omega_i(c)} - \frac{1}{\omega_1(b)}}}{t^{\frac{1}{\omega_1(b)}} \cdot \sum_{j=1}^{h} {}^\circ b_j \cdot t^{\frac{1}{\omega_j(b)} - \frac{1}{\omega_1(b)}}}. \end{aligned} \tag{4.11}$$

Let us note that from $c > 0$ we have ${}^\circ c_1 > 0$, and $\omega(b) = \omega_1(b) \ge \omega(c) \ge \omega_i(c)$ because $0 < c < b$. From (4.11) and (1.6) we have

$$\frac{c_t}{b_t} = \frac{\sum_{i=1}^{k} {}^\circ c_i \cdot t^{\frac{1}{\omega_i(c)} - \frac{1}{\omega_1(b)}}}{{}^\circ b_1 \cdot \left(1 + \sum_{j=2}^{h} \frac{{}^\circ b_j}{{}^\circ b_1} \cdot t^{\frac{1}{\omega_j(b)} - \frac{1}{\omega_1(b)}} \right)}$$

$$= \frac{1}{{}^\circ b_1} \cdot \sum_{i=1}^{k} {}^\circ c_i \cdot t^{\frac{1}{\omega_i(c)} - \frac{1}{\omega_1(b)}} \cdot \sum_{k=0}^{+\infty} (-1)^k \cdot \left(\sum_{j=2}^{h} \frac{{}^\circ b_j}{{}^\circ b_1} \cdot t^{\frac{1}{\omega_j(b)} - \frac{1}{\omega_1(b)}} \right)^k .$$

Writing, for simplicity, $a \ominus b := \left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$, we can write the previous little-oh polynomial using the common notation with $\mathrm{d}t_a$:

$$x_t := \frac{c_t}{b_t} = \frac{1}{{}^\circ b_1} \sum_{i=1}^{k} {}^\circ c_i \, \mathrm{d}t_{\omega_i(c) \ominus \omega_1(b)} \sum_{k=0}^{+\infty} (-1)^k \cdot \left(\sum_{j=2}^{h} \frac{{}^\circ b_j}{{}^\circ b_1} \mathrm{d}t_{\omega_j(b) \ominus \omega_1(b)} \right)^k . \tag{4.12}$$

As usual, the series in this formula is really a finite sum if it is interpreted in ${}^\bullet\mathbb{R}$, because D_∞ is an ideal of nilpotent infinitesimals. Going back in these passages, it is quite easy to prove that the previously defined $x \in {}^\bullet\mathbb{R}$ verifies the desired equality $x \cdot b = c$. Moreover, from the Definition 7 of order, and from $0 \le c \le b$, the relations $0 \le x \le 1$ follow.

Let us note that we cannot have uniqueness of solutions, due to nilpotency. For example, if $a = 0$, $c = \mathrm{d}t_2 + \mathrm{d}t$ and $b = \mathrm{d}t_3$, then $x = \mathrm{d}t_6 + \mathrm{d}t_{3/2}$ is a solution of $a \cdot x + b = c$, but $x + \mathrm{d}t$ is another solution. Moreover, let us note that this theorem is not in contradiction with the non Archimedean property of ${}^\bullet\mathbb{R}$ (let $a = 0$ and $b \in D_\infty$) because of the inequalities that c must verify to have a solution.

4.4 The skeleton group of ${}^\bullet\mathbb{R}$

The skeleton group is a typical instrument used to study non-Archimedean totally ordered rings. For the sake of completeness, we include here its definition.

Definition 18. Let $(R, +, \cdot, <)$ be a totally ordered ring, and define in it the absolute value as usual. Let a, $b \in R_{\neq 0}$, then we say

$$a \ll b \quad :\Longleftrightarrow \quad \forall n \in \mathbb{N} : \; n \cdot a < b$$

and we will read it a *is infinitely smaller than* b. Moreover, we will say

$$a \sim b \quad :\Longleftrightarrow \quad \neg(|a| \ll |b|) \text{ and } \neg(|b| \ll |a|).$$

The relation $\sim$ is an equivalence relation, and we will denote by

$$S_R := \{[a]_\sim \mid a \in R_{\neq 0}\}$$

the set of all its equivalence classes. Moreover, it is possible to prove that the following definitions are correct:

$$[a]_\sim \cdot [b]_\sim := [a \cdot b]_\sim$$
$$[a]_\sim^{-1} := [a^{-1}]_\sim$$
$$[a]_\sim < [b]_\sim \quad :\iff \quad \forall n \in \mathbb{N} : \; n|a| < |b|.$$

It is possible to prove that $(S_R, \cdot, <)$ is a totally ordered group, called the *skeleton group of* R. This notion is naturally tied with the notion of non-Archimedean field. Indeed, the skeleton group of the real field is trivial $S_\mathbb{R} = \{[1]_\sim\}$, but it is not so for non-Archimedean fields, where we always have $\mathbb{Z} \subseteq S_R$, and $\mathbb{Q} \subseteq S_R$ in case R admits roots of positive elements.

The following theorem characterizes the skeleton group of the ring of Fermat reals.

Theorem 19. *Let a, $b \in {}^\bullet\mathbb{R}_{\ge 0}$, then we have:*

1. $b \notin D_\infty \quad \Longrightarrow \quad (a \ll b \iff a \in D_\infty)$
2. $b \in D_\infty \quad \Longrightarrow \quad (a \ll b \iff a \in D_\infty$ *and* $\omega(a) < \omega(b))$
3. $a \sim b \iff {}^\circ a,\ {}^\circ b \neq 0$ *or* $\omega(a) = \omega(b)$
4. *It results*
$$[a]_\sim < [b]_\sim$$
if and only if
$$(a \in D_\infty \text{ and } b \notin D_\infty) \text{ or } (a, b \in D_\infty \text{ and } \omega(a) < \omega(b)) \, .$$

Therefore, the skeleton group of the ring of Fermat reals is $\{[1]_\sim\} \oplus D_\infty/\omega$.

Proof: To prove 1, assume that ${}^\circ b \neq 0$. If $a \ll b$, then it cannot be ${}^\circ a \neq 0$, because, otherwise, we would find $n \in \mathbb{N}$ such that $n{}^\circ a > {}^\circ b$ and $na > b$. Therefore, $a \in D_\infty$. Vice versa, if ${}^\circ a = 0$, then for every $n \in \mathbb{N}$, we have $n{}^\circ a = 0 < {}^\circ b$ and hence $na < b$, that is $a \ll b$.

If ${}^\circ b = 0$ and $a \ll b$, then ${}^\circ a = 0$ can be proved as before. Moreover, from Theorem 12 it follows that it cannot be $\omega(a) > \omega(b)$ because $a > 0$ and these would imply $a > b$, whereas $a < b$ from $a \ll b$. On the other hand, it cannot be $\omega(a) = \omega(b)$ because ${}^\circ a_1 > 0$ and hence $n{}^\circ a_1 > {}^\circ b$ for a sufficiently big $n \in \mathbb{N}$. This would implies $na > b$, whereas $a \ll b$. Therefore, it must be $\omega(a) < \omega(b)$. Once again from Theorem 12 the opposite implication follows directly.

To prove 3, let us assume $\neg\,(a \ll b)$ and $\neg\,(b \ll a)$ and ${}^\circ b = 0$, analogously we can proceed if ${}^\circ a = 0$. From 2 we get that $a \notin D_\infty$ or $\omega(a) \geq \omega(b)$. The first case is impossible, because it implies $b \ll a$ from 1. Therefore, we obtain that $a \in D_\infty$ and $\omega(a) \geq \omega(b)$, and reverting the role of a, b we get the conclusion.

Finally, property 4 follows directly from 1 and 2.

4.5 Supremum and infimum

Obviously, the ring of Fermat reals is not Dedekind complete, and the typical example of bounded set without a least upper bound is the set D_∞ of all the infinitesimals. However, a stronger notion of set bounded from above, which works on every standard part and on the orders, may guarantee the existence of "the" supremum for this kind of sets, that we will call *strongly bounded* from above. A first idea, partially wrong, is to evaluate, for a given set $B \subset {}^\bullet\mathbb{R}$, the possibility to consider a recursive definition of the following form. We start considering the supremum of the standard parts of numbers in B:

$$s_0 := \sup \left\{ {}^\circ x \mid x \in B \right\}.$$

If there are no number in B with s_0 as standard part, then $s := s_0$ is our candidate for "the" least upper bound. Otherwise, we consider those numbers having s_0 as standard part:

$$B_0^\circ := \left\{ x \in B \mid {}^\circ x = s_0 \right\},$$

and the supremum of the corresponding orders:

$$\sigma_1 := \sup \left\{ \omega_1(x) \mid x \in B_0^\circ \right\} \quad \text{if } B_0^\circ \neq \emptyset.$$

If there are no number in B_0° of order σ_1, then $s := s_0 + \mathrm{d}t_{\sigma_1}$ is our candidate. Otherwise, we continue the recursive process:

$$\begin{aligned} B_1^\omega &:= \left\{ x \in B_0^\circ \mid \omega(x) = \sigma_1 \right\} \\ s_1 &:= \sup \left\{ {}^\circ x_1 \mid x \in B_1^\omega \right\} \quad \text{if } B_1^\omega \neq \emptyset, \end{aligned}$$

in this case $s := s_0 + s_1 \, \mathrm{d}t_{\sigma_1}$ is our candidate. In general we will have:

$$\begin{aligned} B_i^\circ &:= \left\{ x \in B_i^\omega \mid {}^\circ x = s_i \right\} \\ \sigma_{i+1} &:= \sup \left\{ \omega_{i+1}(x) \mid x \in B_i^\circ \right\} \quad \text{if } B_i^\circ \neq \emptyset \\ B_{i+1}^\omega &:= \left\{ x \in B_i^\circ \mid \omega(x) = \sigma_{i+1} \right\} \\ s_{i+1} &:= \sup \left\{ {}^\circ x_{i+1} \mid x \in B_{i+1}^\omega \right\} \quad \text{if } B_{i+1}^\omega \neq \emptyset \end{aligned}$$

$$s = s_0 + \sum_i s_i \, \mathrm{d}t_{\sigma_i}. \tag{4.13}$$

The idea to define a stronger notion of set B, bounded from above, is, of course, to select those sets B for which these least upper bounds exist.

A first problem is that in (4.13) we can have infinite summands. For example, if

$$B = \left\{ \sum_{i=1}^N \mathrm{d}t_{a_i} \mid N \in \mathbb{N}\ ,\ 1 \le a_i \le 1 + \frac{1}{i}\ \forall i = 1, \ldots, N \right\},$$

then our idea (4.13) would get into

$$s = 0 + \mathrm{d}t_2 + \mathrm{d}t_{1+\frac{1}{2}} + \mathrm{d}t_{1+\frac{1}{3}} + \dots \tag{4.14}$$

which has no meaning in our context.

A second problem is that our least upper bounds are unique only up to infinitesimals of some order: we say that $x,\ y \in {}^\bullet\mathbb{R}$ are equal up to an infinitesimal of k-th order, and we will write $x =_k y$ iff ${}^\circ x = {}^\circ y$ and $\omega(x - y) \le k$. This notion finds several applications in the development of the calculus on ${}^\bullet\mathbb{R}$ for functions defined on infinitesimal sets like D_a (see [7]; for some application of this notion to the deduction of the wave equation, see [9]), and this explain our quotation marks around some definite article. For example if

$$B = \left\{ 1 - \frac{1}{n} \,|\, n \in \mathbb{N}_{>0} \right\},$$

then our recursive process stops at the first step, because $B_0^\circ = \emptyset$. However, our previous candidate $s = s_0 = 1$ is surely an upper bound, but also $1 - h$, where h is any strictly positive infinitesimal, is another upper bound. Note, hence, that there is no least upper bound, exactly because there is no infimum in D_∞. Analogously, if

$$B = \left\{ 1 - \frac{1}{n} \,|\, n \in \mathbb{N}_{>0} \right\} \cup \left\{ 1 + \mathrm{d}t_{2-\frac{1}{n}} \,|\, n \in \mathbb{N}_{>0} \right\},$$

then our recursive process stops at the second step, because $B_1^\omega = \emptyset$. Therefore, $s = 1 + \mathrm{d}t_2$ is an upper bound, but $1 + \mathrm{d}t_2 - h$, where $h \in D$, is a lesser upper bound.

In the first definition we solve the problem (4.14).

Definition 20. Let $B \subseteq {}^\bullet\mathbb{R}$, then we say that B *is strongly bounded from above* if and only if there exists $n \in \mathbb{N}$ such that

$$B_n^\omega = \emptyset \quad \text{or} \quad B_n^\circ = \emptyset$$

and for every $i \in \mathbb{N}$, the sets of real numbers

$$\left\{ \omega_{i+1}(x) \,|\, x \in B_i^\circ \right\} \quad , \quad \left\{ {}^\circ x_{i+1} \,|\, x \in B_{i+1}^\omega \right\}$$

are bounded from above.

Therefore, in the definition of set strongly bounded from above, we have added the condition that our previous recursive process eventually stops. Let us note, that our sets B_i° and B_i^ω are defined for every $i \in \mathbb{N}$, even if they can be empty (of course, the empty set can be bounded from above by every real number).

In the second definition, we relax the uniqueness condition in the definition of supremum.

Definition 21. Let $B \subseteq {}^\bullet\mathbb{R}$, $s \in {}^\bullet\mathbb{R}$ and $a \in \mathbb{R}_{>0} \cup \{\infty\}$, then we say that *s is a supremum of B up to* D_a if and only if the following conditions hold:

1. $\forall x \in B : \ x \le s$
2. $\forall m \in {}^\bullet\mathbb{R} : \ (\forall x \in B : \ x \le m) \ , \ m \le s \quad \Longrightarrow \quad m =_a s$

Theorem 22. *For every set of Fermat reals, strongly bounded from above, there exists* $a \in \mathbb{R}_{>0} \cup \{\infty\}$ *and a least upper bound up to* D_a*.*

Proof: Exactly as in the informal discussion above, we firstly consider the case $B_0^\circ = \emptyset$. This means that

$$\forall x \in B : \ {}^\circ x < s_0,$$

and hence also $x < s := s_0$. Moreover if $m \in {}^\bullet\mathbb{R}$ is an upper bound of B, with $m \le s$, then ${}^\circ m \le {}^\circ s$. It cannot be ${}^\circ m < {}^\circ s$ because, otherwise, we would find a number $x \in B$ such that ${}^\circ m < {}^\circ x < {}^\circ s$, and this would imply $m < x$. Therefore, ${}^\circ m = {}^\circ s$ and we can set $a = \infty$ to obtain our conclusion.

We can now suppose $B_0^\circ \ne \emptyset$ and therefore, we can consider the greatest $n \in \mathbb{N}$ such that $B_n^\circ \ne \emptyset$. We set

$$s := \begin{cases} s_0 + \sum_{i=1}^{n-1} s_i \, \mathrm{d}t_{\sigma_i} + \mathrm{d}t_{\sigma_n} & \text{if } B_n^\omega = \emptyset \\ s_0 + \sum_{i=1}^{n} s_i \, \mathrm{d}t_{\sigma_i} & \text{if } B_n^\omega \ne \emptyset. \end{cases}$$

In both cases it results $x \le s$ by construction and by Theorem 12. Finally, setting $a = \sigma_n - 1$ we have the conclusion, once again using Theorem 12 and proceeding as above.

5 Geometrical representation of Fermat reals

At the beginning of this article, we argued that one of the conducting idea in the construction of Fermat reals is to maintain always a clear intuitive meaning. More precisely, we always tried, and we will always try, to keep a good dialectic between provable formal properties and their intuitive meaning. In this direction we can see the possibility to find a geometrical representation of Fermat reals.

The idea is that to any Fermat real $x \in {}^\bullet\mathbb{R}$ we can associate the function

$$t \in \mathbb{R}_{\ge 0} \mapsto {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} \in \mathbb{R} \tag{5.1}$$

where N is, of course, the number of addends in the decomposition of x. Therefore, a geometric representation of this function is also a geometric representation of the number x, because different Fermat reals have different decompositions, see "Axiom, decomposition". Finally, we can guess that, because the notion of equality in ${}^\bullet\mathbb{R}$ depends only on the germ generated by each little-oh polynomial (see Definition 4), we can represent each $x \in {}^\bullet\mathbb{R}$ with only the first small part of the function (5.1).

Definition 23. If $x \in {}^\bullet\mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$, then

$$\mathit{graph}_\delta(x) := \left\{ ({}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)}, t) \,|\, 0 \le t < \delta \right\} \tag{5.2}$$

where N is the number of addends in the decomposition of x.

Note that the value of the function are placed in the abscissa position, so that the correct representation of $\mathrm{graph}_\delta(x)$ is given by the figure 5.1. This inversion of abscissa and ordinate in the $\mathrm{graph}_\delta(x)$ permits to represent this graph as a line tangent to the classical straight line $\mathbb{R}$ and hence to have a better graphical picture (see the following figures). Finally, note that if $x \in \mathbb{R}$ is a standard real, then $N = 0$ and the $\mathrm{graph}_\delta(x)$ is a vertical line passing through ${}^\circ x = x$, i.e. they are "ticks on axis".

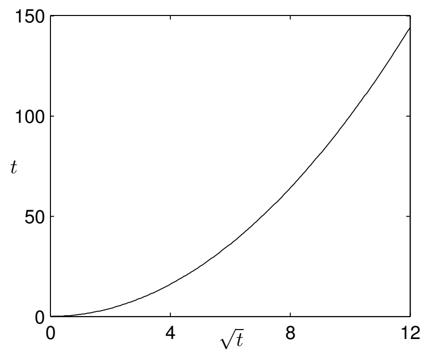

Figure 5.1: The geometrical representation of $\mathrm{d}t_2$

The following theorem permits to represent geometrically the Fermat reals

Theorem 24. *If $\delta \in \mathbb{R}_{>0}$, then the function*

$$x \in {}^\bullet\mathbb{R} \mapsto \mathrm{graph}_\delta(x) \subset \mathbb{R}^2$$

is injective. Moreover if x, $y \in {}^\bullet\mathbb{R}$, then we can find $\delta \in \mathbb{R}_{>0}$ (depending on x and y) such that

$$x < y$$

if and only if

$$\forall p, q, t : \ (p, t) \in \mathrm{graph}_\delta(x) \ , \ (q, t) \in \mathrm{graph}_\delta(y) \quad \Longrightarrow \quad p < q \tag{5.3}$$

Proof: The application $\rho(x) := \mathrm{graph}_\delta(x)$ for $x \in {}^\bullet\mathbb{R}$ is well defined because it depends on the terms ${}^\circ x$, ${}^\circ x_i$ and $\omega_i(x)$ of the decomposition of x. Now, suppose that $\mathrm{graph}_\delta(x) = \mathrm{graph}_\delta(y)$, then

$$\forall t \in [0,\delta) :\ {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} = {}^\circ y + \sum_{j=1}^{M} {}^\circ y_j \cdot t^{1/\omega_j(y)} \tag{5.4}$$

Let us consider the Fermat reals generated by these functions, i.e.

$$x' := \left[t \geq 0 \mapsto {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} \right]_{=\text{ in } {}^\bullet\mathbb{R}}$$
$$y' := \left[t \geq 0 \mapsto {}^\circ y + \sum_{j=1}^{M} {}^\circ y_j \cdot t^{1/\omega_j(y)} \right]_{=\text{ in } {}^\bullet\mathbb{R}}$$

then the decompositions of x' and y' are exactly the decompositions of x and y

$$x' = {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \,\mathrm{d}t_{\omega_i(x)} = x \tag{5.5}$$
$$y' = {}^\circ y + \sum_{j=1}^{M} {}^\circ y_j \,\mathrm{d}t_{\omega_j(y)} = y. \tag{5.6}$$

From (5.4) it follows $x' = y'$ in ${}^\bullet\mathbb{R}$, and hence also $x = y$ from (5.5) and (5.6).

Now suppose that $x < y$, then, using the same notations used above, we have also $x' = x$ and $y' = y$ and hence

$$x' = {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} < {}^\circ y + \sum_{j=1}^{M} {}^\circ y_j \cdot t^{1/\omega_j(y)} = y' \quad \text{in } {}^\bullet\mathbb{R}.$$

We apply Theorem 10 obtaining that locally $x'_t < y'_t$, i.e.

$$\exists\, \delta > 0 :\ \forall t \in [0,\delta) :\ {}^\circ x + \sum_{i=1}^{N} {}^\circ x_i \cdot t^{1/\omega_i(x)} < {}^\circ y + \sum_{j=1}^{M} {}^\circ y_j \cdot t^{1/\omega_j(y)}.$$

This is an equivalent formulation of (5.3), and, because of Theorem 10, it is equivalent to $x' = x < y' = y$.

Example. In figure 5.2 we have the representation of some first order infinitesimals.

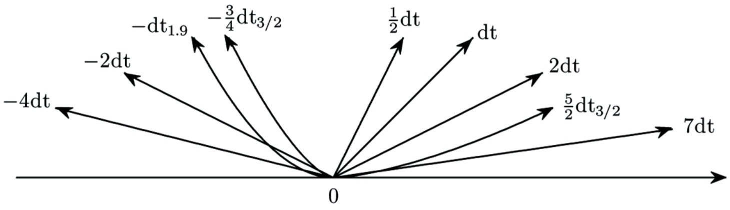

Figure 5.2: Some first order infinitesimals

The arrows are justified by the domain $\mathbb{R}_{\geq 0}$ of the function (5.1), so that the representing lines have a first point and a direction. The smaller is $\alpha \in (0, 1)$ and the nearer is the representation of the product $\alpha\, dt$ to the vertical line passing through zero, which is the representation of the standard real $x = 0$. Finally, recall that $\mathrm{d}t_k \in D$ if and only if $1 \leq k < 2$.

If we multiply two infinitesimals we obtain a smaller number, hence one whose representation is nearer to the vertical line passing through zero, as represented in the figure 5.3

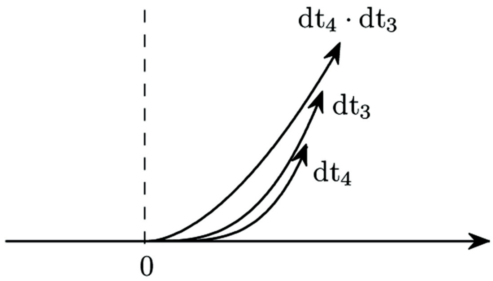

Figure 5.3: The product of two infinitesimals

In figure 5.4 we have a representation of some infinitesimals of order greater than 1. We can see that the greater is the infinitesimal $h \in D_a$ (with respect to the order relation $\leq$ defined in ${}^\bullet\mathbb{R}$) and the higher is the order of intersection of the corresponding line $\mathrm{graph}_\delta(h)$.

Finally, in figure 5.5 we represent the order relation on the basis of Theorem 24.

Intuitively, the method to see if $x < y$ is to look at a suitably small neighborhood (i.e. at a suitably small $\delta > 0$) at $t = 0$ of their representing lines $\mathrm{graph}_\delta(x)$ and $\mathrm{graph}_\delta(y)$: if, with respect to the horizontal directed straight line, the curve $\mathrm{graph}_\delta(x)$ comes before the curve $\mathrm{graph}_\delta(y)$, then x is less than y in the ring ${}^\bullet\mathbb{R}$ of Fermat reals.

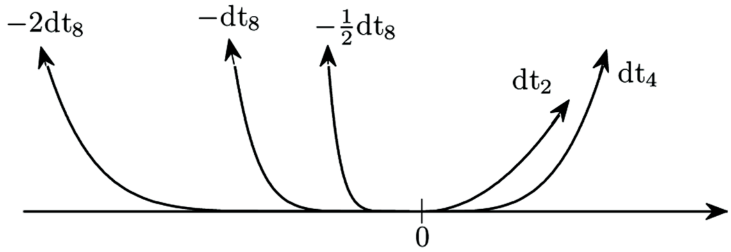

Figure 5.4: Some higher order infinitesimals

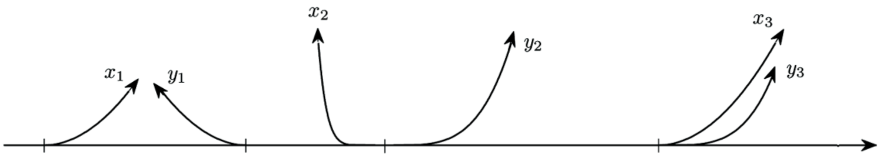

Figure 5.5: Different cases in which $x_i < y_i$

6 Computer implementation

The definition of the ring of Fermat reals is highly constructive. Therefore, using object oriented programming, it is not hard to write a computer code corresponding to ${}^\bullet\mathbb{R}$. We realized a first version of this software using Matlab R2010b.

The constructor of a Fermat real is `x=FermatReal(s,w,r)`, where `s` is the $n+1$ double vector of standard parts (`s(1)` is the standard part ${}^\circ x$) and `w` is the double vector of orders (`w(1)` is the order $\omega(x)$ if $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$, otherwise `w=[]` is the empty vector). The last input `r` is a logical variable and assume value `true` if we want that the display of the number `x` is realized using the Matlab `rats` function for both its standard parts and orders. In this way, the number will be displayed using continued fraction approximations and therefore, in several cases, the calculations will be exact. These inputs are the basic methods of every Fermat real, and can be accessed using the `subsref`, and `subsasgn`, notations `x.stdParts`, `x.orders`, `x.rats`. The function `w=orders(x)` gives exactly the double vector `x.orders` if $x \in {}^\bullet\mathbb{R}\setminus\mathbb{R}$ and `0` otherwise.

The function `dt(a)`, where `a` is a double, construct the Fermat real $\mathrm{d}t_a$. Because we have overloaded all the algebraic operations, like `x+y`, `x*y`, `x-y`, `-x`, `x==y`, `x~=y`, `x<y`, `x<=y`, `x^y`, we can define a Fermat real e.g. using an expression of the form `x=2+3*dt(2)-1/3*dt(1)`, which corresponds to `x=FermatReal([2 3 -1/3],[2 1],true)`.

We have also realized the function `y=decomposition(x)`, which gives the decomposition of the Fermat real `x`, `abs(x)`, `log(x)`, `exp(x)`, `isreal(x)`, `isinfinitesimal(x)`, `isinvertible(x)`.

The function `plot(t,x)` shows the curve (5.2) at the given input `t` of double, and the logical function `v=eqUpTo(k,x,y)` corresponds to $x =_k y$.

The ratio `x/y` has been implemented using the formula (4.12), if `x` and `y` are infinitesimals and `y~=0`, or the formula (1.6) if `y` is invertible. Finally, the function `y=ext(f,x)`, corresponds to ${}^\bullet f(x)$ and has been realized using the evaluation of the symbolic Taylor formula of the inline function `f`.

Using these tools, we can easily find, e.g., that

$$\frac{\sin(\mathrm{d}t_3 + 2\,\mathrm{d}t_2)}{\cos(-\,\mathrm{d}t_4 - 4\,\mathrm{d}t)} = \mathrm{d}t_3 + 2\,\mathrm{d}t_2 - \frac{1}{2}\,\mathrm{d}t_{\frac{6}{5}} + \frac{5}{6}\,\mathrm{d}t.$$

This corresponds to the following Matlab code:

```
>> x=dt(3)+2*dt(2)
   x =
   dt_3 + 2*dt_2
>> y=-dt(4)-4*dt(1)
   y =
   -dt_4 - 4*dt
>> g=inline('cos(y)')
   g =
   Inline function:  g(y) = cos(y)
   >> f=inline('sin(x)')
   f =
   Inline function:  f(x) = sin(x)
>> decomposition(ext(f,x)/ext(g,y))
   ans =
   dt_3 + 2*dt_2 + 1/2*dt_6/5 + 5/6*dt
```

Up to now, this code has been written only to show concretely the possibilities of the ring ${}^\bullet\mathbb{R}$. On the other hand, it is clear that it is possible to write it with a more specific aim. For example, like in case of the Levi-Civita field ([4, 19]) possible applications of a specifically rewritten code can be in automatic differentiation theory. Let us note that, even if the theory of Fermat reals applies to smooth functions, a full treatment of right and left derivatives is possible ([7]), so that the theory can be applied consistently also to piecewise smooth functions. Finally, the use of nilpotent elements permits to fully justify that every derivative estimation of a computer function ([19]) reduces to a finite number of algebraic calculations.

The Matlab source code is freely available under open-source license, and can be requested to the author of the present article.

7 Conclusions

Actual infinitesimals has been used, and are still used, to discover non trivial truths of the real world. Historically, standard $\varepsilon-\delta$ calculus took the place of infinitesimals and, probably, it is a more general way to face the problems of calculus, keeping at the same time a very good intuitive interpretation. Because they are both good instruments to solve real-world problems, it seems natural to conjecture that there could be a strict relation between actual infinitesimals and potential infinitesimals of standard calculus. In fact, several authors tried to reduce the rigorous construction of actual infinitesimals using the standard calculus. The first powerful solutions have been NSA and SDG, but a discussion about their intuitive meaning or about their constraints started (see e.g. [3, 6, 2]). For example, some researchers state the impossibility to draw infinitesimals of NSA because from any non zero infinitesimal is possible, without using the axiom of choice, to construct a non measurable set (see [5, 7]). We have seen that a rigorous definition of a ring of actual infinitesimals having very good intuitive interpretation is possible. In particular, in this paper we have studied the total order relation in the ring ${}^\bullet\mathbb{R}$ of Fermat reals and see that a geometrical representation is possible. Being highly constructive, we implemented the ring ${}^\bullet\mathbb{R}$ using Matlab object oriented programming, so that it is possible to foresee potential useful application in automatic differentiation theory, formal solution of differential equations or in perturbation theory.

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