Functional analysis of infinite bounded operators

Applicant and PI: Paolo Giordano

Co-authors: M. Kunzinger, H. Vernaeve

Contents

It is well-known that many interesting linear operators $T : V \longrightarrow V$ are unbounded, i.e. there is no Lipschitz constant $C \in \mathbb{R}_{>0}$ such that $\|T(v)\| \le C \cdot \|v\|$ for all $v \in V$. A classical important example is clearly the derivative $T = (-)' : \mathcal{C}^\infty([a,b],\mathbb{C}) \longrightarrow \mathcal{C}^\infty([a,b],\mathbb{C})$. The standard way of coping with this problem is to extend the domain of T, e.g., to an appropriate Sobolev space and consider T as a densely defined unbounded operator. In this project, we suggest to instead consider T as a bounded operator, but with a bound given by an infinite number $C \in {}^\sigma\widetilde{\mathbb{R}}_{>0}$. Here ${}^\sigma\widetilde{\mathbb{R}}$ is the ring of Colombeau generalized numbers with gauge $\sigma = (\sigma_\varepsilon) \to 0$ (see below), and we can consider the derivative defined on $V = {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{C}}) \supseteq \mathcal{C}^\infty([a,b],\mathbb{C})$, where $\rho = (\rho_\varepsilon) \to 0$ is a second gauge, suitably related to σ (e.g. $\rho_\varepsilon = \varepsilon$ and $\sigma_\varepsilon = \varepsilon^{1/\varepsilon}$), and where ${}^{\rho}\mathcal{GC}^\infty([a,b], {}^{\rho}\widetilde{\mathbb{C}})$ is a space of generalized smooth functions (GSF[1]). The nonlinear theory of GSF has recently emerged as a minimal extension of Colombeau's theory that allows for more general domains for generalized functions, resulting in the closure with respect to composition, a better behavior on unbounded sets and new general existence results. Linear maps with an (infinite) Lipschitz constant $C \in {}^{\sigma}\widetilde{\mathbb{R}}_{>0}$ are called *infinite bounded operators* (IBO), and the main aim of the proposed project is to develop several topics of functional analysis for these operators. The work packages we propose to develop are:

1. Theory of infinite bounded operators;
2. Applications to analysis and quantum mechanics;
3. Universal properties of generalized functions and of basic IBO.

The proposal thus aims at showing the flexibility of a non-Archimedean framework, such as the Colombeau ring ${}^{\sigma}\widetilde{\mathbb{R}}$, in strongly extending classical results of functional analysis using a simpler setting, and in showing important applications in solving singular PDE and in quantum mechanics (QM).

1 Aims and research objectives

The main objective of the present research project is to develop the functional analysis of infinite bounded operators, so as to include in a simpler non-Archimedean framework, linear maps which are unbounded from the point of view of Archimedean analysis. Our main goals concern the proofs of classical results such as the open mapping theorem, the closed graph theorem, the Banach-Steinhaus uniform boundedness principle, the Riesz theorem, the Hahn-Banach theorem for *basic* IBO, the Lax-Milgram and the corresponding Galerkin theorem and the spectral theorem at least for IBO defined on spaces of the form ${}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{C}})$, where $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ is a functionally compact set (e.g. an interval $[a,b]^n$, where a, $b \in {}^{\rho}\widetilde{\mathbb{R}}$; note that if $a < 0 < b$ and a, b are infinite numbers, then $\mathbb{R}^n \subseteq [a,b]^n$; see Sec. 2.2). We plan to use [Gar05, Gar09, Ga-Vernaeve] as a blueprint for these generalizations since in these works the aforementioned results have already been obtained for linear maps with Lipschitz constant $C \in \widetilde{\mathbb{R}}$, i.e. for the case of only one gauge $\sigma_\varepsilon = \varepsilon$. As is characteristic for functional analysis, the style will be completely intrinsic and abstract, even if typical spaces we have in mind are ${}^{\rho}\widetilde{\mathbb{R}}$-graded Fréchet spaces of *generalized smooth functions* of the form ${}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{C}})$ (see Sec. 2.2). GSF are an extension of classical distribution theory which makes it possible to model nonlinear singular problems, while at the same time sharing a number of fundamental properties with ordinary smooth functions, such as the closure with respect to composition and several non trivial classical theorems of the calculus, see [Gio-Kun-Ver15, Gio-Kun-Ver19, Gio-Kun18a, Gio-Kun16, LL-Giordano16, L-Giordano19]. One could describe GSF as a methodological restoration of Cauchy-Dirac's original conception of generalized function (GF), see [Laug89, KatTal12]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. GSF are a minimal extension of Colombeau's theory of generalized functions (CGF), see [Col84, Col85, Col92, NePiSc98, Obe92, Pil94]. In fact, when the domain is the set $\widetilde{\Omega}_c$ of compactly supported generalized points in the open set $\Omega \subseteq \mathbb{R}^n$, then the two spaces of GF

[1]A complete list of acronyms can be found at the end of this document; To help the reader, Adobe Acrobat produces small windows (tooltips) near acronyms, equation references, figures and citations (near the end of the citation).

coincide, cf. [Gio-Kun-Ver15]. Therefore, we expect that the directions envisaged in the present project will also exert a considerable impact on Colombeau's theory. For these reasons, the department of Mathematics of the University of Vienna, and in particular the research group of Prof. M. Kunzinger (see http://www.mat.univie.ac.at/~mike/ and the included CV), constitute the ideal place where to implement the present research project, because of the group's specific competencies on generalized functions, functional analysis and partial differential equations (PDE). The close collaboration with Prof. H. Vernaeve (see https://cage.ugent.be/~hvernaev/ and the included CV) as co-author of the present proposal greatly increases its feasibility and chances of being successful, because of his expertise in functional analysis of $\widetilde{\mathbb{C}}$-Hilbert spaces and in non-standard analysis (NSA).

A concise presentation of the project's main aims is as follows (WP = work package):

WP 1: Theory of infinite bounded operators

Problems and motivations: Is it possible to develop the theory of unbounded operators as bounded linear maps with an *infinite* Lipschitz constant? What is the correct non-Archimedean framework to realize this idea?

The idea and the plan: In several works, cf. [Gio-Kun-Ver15, Gio-Kun16, Gio-Kun-Ver19, Giordano-L19, LL-Giordano16], we already generalized the classical Colombeau ring of generalized numbers $\widetilde{\mathbb{R}}$ into ${}^{\sigma}\widetilde{\mathbb{R}}$, where $\sigma = (\sigma_\varepsilon) : (0,1] \longrightarrow (0,1]$ is an increasing infinitesimal function (called *gauge*) that replaces the role of the classical net (ε). For example, in the case of the derivative operator, the idea is to consider *two* gauges ρ, σ, for example with σ^{-1} greater than any power ρ^{-N} (e.g. $\sigma_\varepsilon = \rho_\varepsilon^{1/\varepsilon}$). We can hence consider the subring

$$ {}^{\rho}_{\sigma}\widetilde{\mathbb{C}} := \{x \in {}^{\sigma}\widetilde{\mathbb{C}} \mid \exists N \in \mathbb{N} : \ |x| \le [\rho_\varepsilon]^{-N}\}, \tag{1} $$

and spaces of ρ-moderate GSFof the form

$$ {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}}) := \left\{ f \mid \partial^{\alpha} f \in {}^{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}}) \ \forall \alpha \in \mathbb{N}^n \right\}, \tag{2} $$

where $K \subseteq {}^{\rho}_{\sigma}\widetilde{\mathbb{R}}^n$ is a functionally compact set (see Sec. 2.2 for detailed explanations). In this way, we are working with an arbitrary gauge ρ and the additional gauge σ will serve as a tool to measure large infinities which are not ρ-moderate. For example, in this setting, the derivative acts on ${}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$, but will have a σ-moderate Lipschitz constant; we can hence start to extend the results of [Gar05, Gar09, Ga-Vernaeve].

Innovative features and deliverables: This new non-Archimedean scheme seems sufficiently simple to allow us to implement the aforementioned generalizations. On the other hand, it permits to naturally inscribe the notion of unbounded linear map into the conceptually simpler notion of continuous linear map (with respect to the sharp topology in ${}^{\sigma}\widetilde{\mathbb{C}}$).

WP 2: Applications to analysis and QM

Problems and motivations: Sometimes, theories such as the aforesaid one of IBO, produce theorems that require technical hypotheses to bypass the lacking of Archimedean properties, such as the existence of supremum or infimum of given sets. It is therefore particular important to illustrate the theory with relevant examples.

The idea and the plan: We plan to include in the theory significant examples from mathematical analysis and from QM. An important class of IBO has already been defined in [Ga-Vernaeve]: by definition, a *basic* operator T is always generated by a net $T_\varepsilon : V_\varepsilon \longrightarrow W_\varepsilon$ of ordinary linear maps, so that $T([v_\varepsilon(-)]) = [T_\varepsilon(v_\varepsilon)(-)]$, where e.g. $[v_\varepsilon(-)] : [x_\varepsilon] \in K = [K_\varepsilon] \mapsto [v_\varepsilon(x_\varepsilon)] \in {}^{\rho}_{\sigma}\widetilde{\mathbb{C}}$ is the GSF defined by the net of smooth functions $v_\varepsilon \in \mathcal{C}^\infty(K_\varepsilon, \mathbb{C})$. Examples we want to consider are essentially always of basic type: 1) Hilbert-Schmidt operators defined by multiplication and integration against a GSF kernel; 2) basic IBO defined using hyperfinite methods, i.e. using the set ${}^{\sigma}\widetilde{\mathbb{N}} \subseteq {}^{\sigma}\widetilde{\mathbb{R}}$ of infinite natural numbers, e.g. exponential of a basic operator using hyperseries, see Sec. 2.2 and [T-Giordano19]; 3) basic IBO defined by hyperfinite Picard-Lindelöf contractions (for ODE or PDE , see Sec. 2.2); 4) linear maps defined by Fourier and hyperfinite Fourier transform, cf. [M-Giordano19]; 5) differential IBO used in QM .

Innovative features and deliverables: All the listed example are of fundamental importance for mathematical analysis. The present proposal would also permit a development of QM similar to [BeLuSi19] (where ultrafunctions and NSA are used). The theory of GSF lends itself particularly well to problems from QM due to its built-in capacity of handling non-linear operations on distributions, because of the use of infinitesimal and infinite quantities in modeling physical systems and in integrating functions, and because physical measurements valued in ${}^{\rho}\widetilde{\mathbb{R}} \setminus \mathbb{R}$ (e.g. $\delta(0) \in {}^{\rho}\widetilde{\mathbb{R}} \setminus \mathbb{R}$) can be interpreted as idealized models as $\varepsilon \to 0$ corresponding to real ε-depending standard representatives.

WP 3: Universal properties of GF and basic IBO

Problems and motivations: A well-known interpretation of universal properties is that of "the simplest way to solve a given problem", see e.g. [ML98, LaSc12]. Discovering such properties for ${}^{\rho}\widetilde{\mathbb{R}}$, for GSF and for basic IBO, would result into a characterization of these spaces up to isomorphisms, and hence a comparison with [Tod11] is natural.

The idea and the plan: One may certainly expect that also the ring ${}^{\rho}\widetilde{\mathbb{R}}$ and the sheaf of GSF are the simplest way to solve suitable problems. See e.g. [Gio-Kun-Ver19] where the ring ${}^{\rho}\widetilde{\mathbb{R}}$ is introduced as a necessary consequence of natural conditions on its representatives (every representatives of zero is an infinitesimal function as $\varepsilon \to 0$), and where GSF are defined using minimal logical conditions. We can hence foresee that a suitable universal property holds also for basic IBO because, using the language of NSA, they are internal maps exactly as GSF. The universal property of the sheaf of Sobolev-Schwartz distributions, see [Giordano-M97, MelMun00] and Sec. 2.2, will be an important precedent and a useful benchmark for this WP.

Innovative features and deliverables: A conceptual category-theoretical characterization of spaces of GF and of basic IBO leads to focus on key properties and hence allows an intrinsic description. This will surely contribute to a deeper understanding of the relation between the Colombeau and the Sobolev-Schwartz approach to GF.

The present research project is designed for five co-workers: the applicant P. Giordano, co-authors M. Kunzinger, H. Vernaeve and two Ph.D. candidates. The CV of the applicant and the co-authors are included in this application. See also Sec. 5 for the organization of the research work.

2 State of the art

2.1 State of the art in the research field

Generalized functions:

J.F. Colombeau's theory of generalized functions allows one to perform non linear operations (of polynomial growth) between embedded distributions, avoiding the difficulty of the Schwartz impossibility theorem. See e.g. [Ned-Pil06, GrKuObSt01, NePiSc98, Pil94, Obe92, Col92] for an introduction with applications. This theory makes it possible to find generalized solutions of some well-known PDE which do not have solutions in the classical space of distributions, see [Obe92], and has manifold applications, e.g. to the theory of elasticity, fluid mechanics and in the theory of shock waves (see e.g. [Col92, Obe92]), to differential geometry and relativity theory [GrKuObSt01, Kun04, SteVic06] and to quantum field theory [CoGs08].

A new and fundamental step in the theory of generalized functions based on Colombeau generalized numbers, which presents several analogies with our present proposal, has first been achieved in [Ar-Fe-Ju05, Ar-Fe-Ju12]. In this work, the basic idea is to generalize the derivative as a limit of an incremental ratio taken with respect to the e-norm, [Ar-Fe-Ju05, Ar-Fe-Ju09] and with increments which are asymptotic to invertible infinitesimals of the form $[\varepsilon^r] \in {}^\rho\widetilde{\mathbb{R}}$, for $r \in \mathbb{R}_{\geq 0}$. This theory extends the usual classical notion of derivative and smoothness to set-theoretical functions on Colombeau generalized numbers, e.g. of the form $f : \widetilde{\mathbb{R}}^n \longrightarrow \widetilde{\mathbb{R}}^d$, and enables one to prove that every CGF is infinitely differentiable in this new sense. Several important applications have already been achieved (see [Ar-Fe-Ju12]) and hence the theory promises to be very relevant. As explained in greater detail in [Gio-Kun-Ver15, Gio-Kun16, Gio-Kun-Ver19], the notion of smoothness developed in [Ar-Fe-Ju05] includes functions like $i(x) = 1$ if x is infinitesimal and $i(x) = 0$ otherwise. This makes it impossible to prove classical theorems like the intermediate value one, whereas for GSF this theorem holds. The theory of GSF, on the other hand, while fully compatible with the approach in [Ar-Fe-Ju05], singles out a subclass of smooth functions with more favorable compatibility properties with respect to classical calculus and hence may be viewed as a refinement of that theory.

Theory of $\widetilde{\mathbb{C}}$-modules

The theory of locally convex topological $\widetilde{\mathbb{C}}$-modules is the key reference for the first part of the present project, see [Gar05, Gar05b, Gar09, Ga-Vernaeve]. In these works, very general theorems for bounded operators (with Lipschitz constant in $\widetilde{\mathbb{R}}$) such as projection theorem, open mapping theorem, the closed graph theorem, the Banach-Steinhaus uniform boundedness principle, the Riesz theorem, a suitable version of the Hahn-Banach theorem and the Lax-Milgram theorem have been proved, under suitable assumptions. Frequently, these conditions originate from the lacking of the existence in $\widetilde{\mathbb{R}} \cup \{\pm\infty\}$ of infimum for arbitrary subsets of $\widetilde{\mathbb{R}}$ or from the existence of non-invertible elements in $\widetilde{\mathbb{R}}$. This leads to important new notions such as those of *reachable* and *edged* subset, and of the *normalization property* (see [Ga-Vernaeve] for details).

For the Hahn-Banach theorem in non-Archimedean valued fields, see [Ing52]. For a version of the Hahn-Banach theorem framed in subfields of $\widetilde{\mathbb{C}}$, see [May07]. The impossibility of a general Hahn-Banach theorem for $\widetilde{\mathbb{C}}$-functionals has been proved in [Vernaeve10]. Therefore, note that the possibility to prove a general Hahn-Banach theorem for $\widetilde{\mathbb{C}}$-*basic* functionals, i.e. defined by a suitable net (T_ε) of classical functionals, is still an open problem. Moreover, some of the aforementioned conditions can be avoided in the NSA approach to Colombeau theory, where the ring of generalized numbers ${}^{\rho}\mathbb{R}$ is actually a field, and where the Hahn-Banach theorem can be proved (see e.g. [Vernaeve10, TodVer08, Tod13] and references therein).

Todorov axiomatic approach to GF

In [Tod13, Tod11] an axiomatic approach to NSA-based Colombeau-type GF is presented. This description aims at introducing GF with improved properties of generalized scalars (which in this approach is an algebraically closed Cantor complete field), with more general theoretical results, such as the aforementioned Hahn-Banach theorem from [TodVer08], and finally to axiomatically describe GF using algebraic and functional analytic tools. If the generalized continuum hypothesis is assumed, these axioms characterize this space of GF up to isomorphism. In particular, we note that Axiom 10 (Maximality principle) of [Tod13] may point to a possible universal property.

Ultrafunctions approach to QM

Ultrafunctions represent an interesting class of GF extending Schwartz's distributions of finite order. The approach uses methods from NSA , mainly through the use of Λ-limits (see [Ben16, BeLB14, BeLB15a, BeLB15b, BeLuSq20]). As for GSF (and in a certain sense also for CGF , [Ar-Fe-Ju05]), one of the key points is the possibility to view these GF as set-theoretical maps defined on a superreal non-Archimedean field. Another positive feature is that this approach extensively uses hyperfinite methods in treating this kind of GF.

For the aims of the present proposal, we emphasize in particular the approach to QM using ultrafunctions as presented in [BeLuSi19]. Here, the space of ultrafunctions is introduced as a richer non-Archimedean framework for a description of physical systems in QM, and the solution for the Schrödinger equation for a Hamiltonian with the delta function potential is studied. Finally, five classical axioms to approach QM are reformulated in this NSA-based framework, plus a sixth axiom stating: "*In a laboratory only the states associated to a finite expectation value of the physically relevant quantities can be realized. These states are called physical states, the rest of the states is called ideal states*". In Sec. 3, we will compare this concept with our proposal.

2.2 State of the art in applicant's research

In this section, we briefly introduce some of the key notions of the present research proposal.

Some basic notations we will use in the following are: nets in the variable $\varepsilon \in I := (0,1]$ are written as (x_ε); if (x_ε) is a net of real numbers, $x = [x_\varepsilon]$ denotes the corresponding equivalence class with respect to the equivalence relation $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ iff $|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^m)$ for every $m \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

The ring of Colombeau ${}^{\rho}\widetilde{\mathbb{R}}$

Given an increasing net $\rho = (\rho_\varepsilon) : I \longrightarrow I$ such that $\rho_\varepsilon \to 0$ as $\varepsilon \to 0^+$ (which is called a *gauge*), the ring ${}^{\rho}\widetilde{\mathbb{R}}$ is the quotient of the ring of *ρ-moderate nets* ($\exists N \in \mathbb{N} : \ x_\varepsilon = O(\rho_\varepsilon^{-N})$) modulo *$\rho$-negligible nets* ($\forall n \in \mathbb{N} : \ x_\varepsilon = O(\rho_\varepsilon^{n})$). The point of view of GSF is frequently that of a theory where ${}^{\rho}\widetilde{\mathbb{R}}$ acts as the ring of scalars for all the subsequent constructions. For example, the sharp topology is preferably defined using the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^{\rho}\widetilde{\mathbb{R}}$ and the balls $B_r(x) := \{y \in {}^{\rho}\widetilde{\mathbb{R}}^d \mid |y - x| < r\}$, where $r > 0$ means being a *strictly positive* generalized number, i.e. $r \in {}^{\rho}\widetilde{\mathbb{R}}_{\geq 0}$ and r is invertible (see [Ar-Fe-Ju09, Ar-Ju01]). In this proposal, we use the notation $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$.

Generalized smooth functions as a category of smooth set-theoretical maps

If $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ are arbitrary subsets of generalized numbers, a GSF $f \in {}^{\rho}\mathcal{GC}^{\infty}(X,Y)$ can be simply defined as a set-theoretical map $f : X \longrightarrow Y$ such that

$$\exists (f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I \, \forall [x_\varepsilon] \in X \, \forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \text{ is } \rho-\text{moderate and } f(x) = [f_\varepsilon(x_\varepsilon)], \tag{3}$$

see [Gio-Kun-Ver15, Gio-Kun-Ver19]. If (3) holds, we say that the net (f_ε) *defines* f. If $X = \widetilde{\Omega}_c$, the set of compactly supported points in the open set $\Omega \subseteq \mathbb{R}^n$, then ${}^{\rho}\mathcal{GC}^{\infty}(\widetilde{\Omega}_c, {}^{\rho}\widetilde{\mathbb{R}})$ coincides exactly with the set-theoretical maps induced by all the CGF of the algebra $\mathcal{G}^s(\Omega)$, see [Gio-Kun-Ver15]. The greater flexibility in the choice of the domains X leads e.g. to the closure of GSF with respect to composition, to the extreme value theorem on closed intervals bounded by infinite numbers, to purely infinitesimal solutions of ODE or also to inverses of given GSF, see [Gio-Kun-Ver15, Gio-Kun-Ver19, L-Giordano19, Gio-Kun16]. Classical theorems like the chain rule, existence and uniqueness of primitives, integration by change of variables, the intermediate value theorem, mean value theorems, the extreme value theorem, Taylor's theorem in several forms for the remainder, suitable sheaf properties, the local inverse and implicit function theorems, some global inverse function theorems, the Banach fixed point theorem, the Picard-Lindelöf theorem and several results in the classical theory of the calculus of variations, hold for these GSF, see [Gio-Kun-Ver15, Gio-Kun-Ver19, Gio-Kun16, L-Giordano19, LL-Giordano16]. One of the peculiar properties of GSF is that these extensions of classical theorems for smooth functions have very natural statements, formally similar to the classical ones. All this underscores the different philosophical approach as compared to [Ar-Fe-Ju05] (and to the more classical Colombeau theory), which constitutes a more general approach, but where some of these classical theorems do not hold.

Particularly interesting for the present research proposal are the structure of graded ${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet space on a solid functionally compact set and the notion of hyperseries. Firstly, a functionally compact set is a sharply bounded internal sets $K = [K_\varepsilon] = \{[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in K_\varepsilon \text{ for } \varepsilon \text{ small}\} \subseteq B_R(0)$, for some $R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, generated by a net $K_\varepsilon \Subset \mathbb{R}^n$ of compact sets. Secondly, a *solid set* is a set $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ whose interior in the sharp topology is dense in S; the latter allows us to deal with partial derivatives at boundary points. For example, every closed interval $[a,b] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ is functionally compact and solid. On functionally compact sets, GSF satisfy the extreme value theorem and hence on every closed interval they can be integrated $\int_a^b f \in {}^{\rho}\widetilde{\mathbb{R}}$. The space ${}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ shares many properties with the classical Fréchet spaces of ordinary smooth functions defined on a compact set. In particular, these spaces are sharply Cauchy complete and their sharp topology can be defined using a countable family

$$\|f\|_m := \left[\max_{\substack{|\alpha| \le m \\ 1 \le k \le d}} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f_\varepsilon^k(x))| \right] \in {}^{\rho}\widetilde{\mathbb{R}} \quad \forall m \in \mathbb{N}, \tag{4}$$

of ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms, see [Gio-Kun18a, L-Giordano19]. We also proved a generalization of the Banach fixed point theorem and of the corresponding Picard-Lindelöf theorem that are applicable to any Cauchy problem with a normal generalized PDE, see [Giordano-L19]. The basic idea is the notion of loss of derivatives: if $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ is a solid functionally compact set, and $y_0 \in X \subseteq {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$, then we say that $P : X \longrightarrow X$ is a *contraction on* X *with loss of derivatives* $L \in \mathbb{N}$ *starting from* y_0 if

$$\forall i \in \mathbb{N} \, \exists \alpha_i \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \ \|P(u) - P(v)\|_i \le \alpha_i \cdot \|u - v\|_{i+L} \ \forall u, v \in X$$

and

$$\lim_{\substack{n,m\to+\infty\\ n\le m}} \alpha_{i+mL}^{n}\cdot\|P(y_0)-y_0\|_{i+mL}=0, \tag{5}$$

where the limit is taken with respect to the sharp topology and with n, $m\in\mathbb{N}$. We proved that if $\alpha_i\le\alpha_{i+1}$ and X is sharply Cauchy complete, then P is sharply continuous, $\exists\lim_{n\to+\infty}P^n(y_0)=:y$ and $P(y)=y$. Note explicitly that, in general, we don't have the uniqueness of the fixed point y, exactly because we can have a loss of $L>0$ derivatives. If $T\subseteq{}^{\rho}\widetilde{\mathbb{R}}$, $S\subseteq{}^{\rho}\widetilde{\mathbb{R}}^n$ are solid functionally compact sets, $Y\subseteq{}^{\rho}\mathcal{GC}^\infty(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ and the set-theoretical map $F:T\times S\times Y\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}^d$ satisfies $F(-,-,y)\in{}^{\rho}\mathcal{GC}^\infty(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ for all $y\in Y$, then we say that F is *uniformly Lipschitz on* Y *with constants* $(\Lambda_i)_{i\in\mathbb{N}}\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}^{\mathbb{N}}$ *and loss of derivatives* $L\in\mathbb{N}$ *if*

$$\forall i\in\mathbb{N}\,\forall u,v\in Y:\ \|F(-,-,u)-F(-,-,v)\|_i\le\Lambda_i\cdot\|u-v\|_{i+L}.$$

We can prove that any PDE of the form $\partial_t y(t,x)=G\left[t,x,\partial_x y(t,x)\right]$ (this type of DE are called of *normal type*; note that the classical Lewy counter-example, [Lew57], corresponds to two real PDE which are *not* in normal form), where G is a GSF, defines a uniformly Lipschitz map on the space

$$Y=\left\{y\in{}^{\rho}\mathcal{GC}^\infty(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)\mid\|y-y_0\|_i\le r_i\ \forall i\in\mathbb{N}\right\}. \tag{6}$$

We finally proved the following generalization of the Picard-Lindelöf theorem: let $t_0\in{}^{\rho}\widetilde{\mathbb{R}}$, α, $r_i\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $T_\alpha:=[t_0-\alpha,t_0+\alpha]$. Let $y_0\in{}^{\rho}\mathcal{GC}^\infty(S,H)$, where $H\subseteq{}^{\rho}\widetilde{\mathbb{R}}^d$ is a sharply closed set such that $\overline{B_r(y_0(x))}\subseteq H$ for all $x\in S$. Define Y_α as in (6), but using T_α instead of T, and assume that F is uniformly Lipschitz on Y_α with constants $(\Lambda_i)_{i\in\mathbb{N}}$ and loss of derivatives L. Finally, assume that

$$\begin{aligned}
&\Lambda_i\le\Lambda_{i+1}\quad\forall i\in\mathbb{N}\\
&\|F(-,-,y)\|_i\le M_i(y)\quad\forall y\in Y_\alpha\\
&\alpha\cdot M_i(y)\le r_i\quad\forall i\in\mathbb{N}\\
&\lim_{\substack{n,m\to+\infty\\ n\le m}}\alpha^{n+1}\cdot\Lambda_{i+mL}^{n}\cdot\|F(-,-,y_0)\|_{i+mL}=0\\
&\exists s\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}\,\forall m\in\mathbb{N}:\ \alpha\cdot\Lambda_{i+mL}<1-s.
\end{aligned}$$

Then there exists a solution $y\in{}^{\rho}\mathcal{GC}^\infty(T_\alpha\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ of the Cauchy problem

$$\begin{cases}\partial_t y(t,x)=F(t,x,y) & \forall(t,x)\in T_\alpha\times S\\ y(0,x)=y_0(x) & \forall x\in S\end{cases}$$

Note explicitly that this is only an existence result and nothing is stated about the uniqueness of the solution. A generalization of these results to k-th order PDE can easily be proved because, due to the missing closure of GSF with respect to composition, every higher order PDE can be reduced to a system of first order PDE, see [Giordano-L19]. For example, by taking $S=[-\mathrm{d}\rho^{-1},\mathrm{d}\rho^{-1}]^n\supseteq\mathbb{R}^n$ it is possible to prove the existence, local in the normal variable t, but global in the "space" variable $x\in S\supseteq\mathbb{R}^n$ of every polynomial PDE with real coefficients, i.e. where $G\in\mathbb{R}[t,x,d]$. This includes an infinite class of PDE that cannot even be formulated e.g. within the theory of Sobolev-Schwartz distributions. Note also that this kind of results is not possible for CGF either due to the missing of closure with respect to arbitrary compositions and because generally speaking the domain $T_\alpha\times S$

also includes non compactly supported points (an arbitrary CGF can be evaluated only on this kind of points).

To introduce the notion of hyperlimit and hyperseries, we first consider the set of *hypernatural numbers* in ${}^{\rho}\widetilde{\mathbb{R}}$, i.e. ${}^{\rho}\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\}$. If σ, ρ are *two* gauges and $a : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ is a σ-generalized sequence of ρ-generalized numbers, then the hyperlimit $l = {}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n$ is simply the limit of this sequence in the sharp topology, i.e.

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in \mathbb{N}_\sigma : \ n \geq M \ \Rightarrow \ |a_n - l| < \mathrm{d}\rho^q.$$

The importance of considering two gauges lies in the fact that if $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$, then ${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{\log n} = 0 \in {}^{\rho}\widetilde{\mathbb{R}}$, whereas $\nexists\, {}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{\log n}$ for all gauges ρ. The notion of hyperseries is a particular case: let $a : \mathbb{N} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be a sequence of ${}^{\rho}\widetilde{\mathbb{R}}$ and let $s \in {}^{\rho}\widetilde{\mathbb{R}}$. Assume that the partial sums with summands $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ can be extended to ${}^{\sigma}\widetilde{\mathbb{N}}$ as $N \in {}^{\sigma}\widetilde{\mathbb{N}} \mapsto \sum_{n=0}^{N} a_n := \left[\sum_{n=0}^{\mathrm{nint}(N)_\varepsilon} a_{n\varepsilon}\right] \in {}^{\rho}\widetilde{\mathbb{R}}$ (here $\mathrm{nint}(k)$ is the nearest integer function), then we set ${}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}} a_n := {}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N} a_n$ whenever this hyperlimit exists. For example, one can easily prove that ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} k^n = \frac{1}{1-k}$ for all $k \in {}^{\rho}\widetilde{\mathbb{R}}_{<1}$ and ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$ for all $x \in {}^{\rho}\widetilde{\mathbb{R}}$ finite.

We finally mention that the sheaf property for GSF, see [Gio-Kun-Ver19], can be used to prove that the functor ${}^{\rho}\mathcal{GC}^{\infty}(-, T)$ belongs to a suitable Grothendieck topos ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$ of sheaves. This topos can be considered a Cartesian closed universe of generalized sets and functions which is closed with respect to set theoretical operations such as $X \cup Y$, $X \cap Y$, Y^X, $X \times Y$, $\mathcal{P}(X)$, subsets, etc. This allows us to consider a framework of infinite dimensional spaces of generalized functions like ${}^{\rho}\mathcal{GC}^{\infty}(C, D)^{{}^{\rho}\mathcal{GC}^{\infty}(A,B)} = {}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}({}^{\rho}\mathcal{GC}^{\infty}(A, B), {}^{\rho}\mathcal{GC}^{\infty}(C, D))$.

Closed supremum and infimum

Infimum and supremum are basic important concepts frequently used in functional analysis; it is therefore indispensable to have a sufficiently complete understanding of these notions in a more general non-Archimedean setting like ${}^{\rho}\widetilde{\mathbb{R}}$, where infimum and supremum can also not exist. The notion of infimum of subsets of $\widetilde{\mathbb{R}}$ has been firstly studied in [Ga-Vernaeve], where it is recognized that the idea of *closed infimum* $\overline{\inf}(S) \in \widetilde{\mathbb{R}}$

$$\left(\forall s \in S : \ s \leq \overline{\inf}(S)\right) \quad \text{and} \quad \forall q \in \mathbb{N}\, \exists \bar{s} \in S : \ \overline{\inf}(S) - [\varepsilon]^q \leq \bar{s},$$

is actually more adapted to properties related to the sharp topology. This has been confirmed in [MTA-Giordano19], where we proved that: 1) every topology on ${}^{\rho}\widetilde{\mathbb{R}}$ generated by a set of radii (the sharp topology with radii ${}^{\rho}\widetilde{\mathbb{R}}_{>0}$, and the Fermat one with radii $\mathbb{R}_{>0}$, are particular cases of this notion, see [Gio-Kun13, Gio-Kun-Ver15]) defines corresponding different notions of closed infimum and supremum; 2) we introduced the notion of a set $S \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ *well sewed from below*, and we proved that if S is of this type, then greatest lower bound and closed infimum coincide. For example, the set $S = \left[-\frac{1}{2}, 1\right] \cup \left\{\left[\mathrm{sgn}\left(\sin\frac{1}{\varepsilon}\right)\right]\right\} \subseteq \widetilde{\mathbb{R}}$ is *not* well sewed from below because of the *fringe point* $\left[\mathrm{sgn}\left(\sin\frac{1}{\varepsilon}\right)\right] \in \widetilde{\mathbb{R}}$, whereas its *sewing* $\overline{S}^{\mathrm{f}} := \{m \mid m \text{ is a fringe point of } S\} = \{-1\} \cup \left[-\frac{1}{2}, 1\right]$ is well sewed from below; 3) a lower bound of S is called *Archimedean* if $\exists n \in \mathbb{N}_{>0}\, \exists \bar{s} \in S : \ \frac{M}{n} \geq \bar{s}$. We proved that if S is well sewed from below, then the existence of the closed infimum is equivalent to the existence of an Archimedean lower bound; this serves as a convenient substitute for the existence of a finite infimum of subsets $S \subseteq \mathbb{R}$ which are bounded from below; 4) finally, in [MTA-Giordano19] we showed that the notions of closed infimum and supremum are the correct ones when dealing with hyperlimits of monotone hypersequences. All these notions generalize what we normally have in $\mathbb{R}$.

Multidimensional integration

A necessary background to obtain a spectral-like theorem for (basic) IBO, is an integration theory for GSF over subsets of ${}^{\rho}\widetilde{\mathbb{R}}^{n}$ sufficiently similar to Lebesgue one. We achieved this result in [Gio-Kun-Ver19], by defining integration over measurable functionally compact sets $K=[K_\varepsilon]\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{n}$. For this multidimensional integration, we can ε-wise integrate any GSF $f=[f_\varepsilon(-)]$ (and clearly, we have independence from representatives (K_ε) and (f_ε)); any interval $[a,b]^n$ is measurable, even if it is unbounded; we have the usual integration by substitution formula; a suitable useful additivity property, but also a very powerful continuity property stating that if a hypersequence of GSF converges pointwise on K, then the convergence is actually uniform and integral and hyperlimit can be exchanged. This notion of integral is closely related to integration over membranes developed in [Ar-Fe-Ju12], even if the latter is not a theory of integration over subsets of $\widetilde{\mathbb{R}}^{n}$ because different representatives of $[0,1]\subseteq\widetilde{\mathbb{R}}$ can define different membranes and hence different integrals. Moreover, a membrane is always bounded by definition, whereas functionally compact sets can be unbounded.

In view of a possible extension of the Riesz-Markov theorem, these results are particularly important when applied to arbitrary intervals $[a,b]^n$. In particular, we want to mention the proof of this theorem given in [RS1, Thm. IV.4], which holds for the space $X=[0,1]\subseteq\mathbb{R}$ by a simple application of the Hahn-Banach theorem. Due to its simplicity, this proof seems generalizable to any interval $[a,b]\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ and to basic IBO.

3 Work program

In this section, we describe the methods we plan to employ in carrying out the research program sketched above. For each of the three parts of the research project, we will also give a (subjective) judgment of its feasibility. Of course, this qualitative judgment of feasibility will be justified and will also be used to quantify and support the project's time planning.

3.1 WP 1: Theory of infinite bounded operators

As we already outlined above, the idea of IBO can be explained by firstly stating e.g. that any derivative operator $\partial^\alpha(-)$ can be thought of as a map

$$\partial^\alpha(-):{}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K,{}^{\rho}_{\sigma}\widetilde{\mathbb{C}})\longrightarrow{}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K,{}^{\rho}_{\sigma}\widetilde{\mathbb{C}}),$$

where $\sigma\le\rho$ are two gauges, K is a functionally compact solid subset of ${}^{\sigma}\widetilde{\mathbb{R}}^{n}$, the subring ${}^{\rho}_{\sigma}\widetilde{\mathbb{C}}\subseteq{}^{\sigma}\widetilde{\mathbb{C}}$ (see (1)) is the set of ρ-moderate numbers in ${}^{\sigma}\widetilde{\mathbb{C}}$, and the space of GSF ${}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K,{}^{\rho}_{\sigma}\widetilde{\mathbb{C}})\subseteq{}^{\sigma}\mathcal{GC}^{\infty}(K,{}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$ (see (2)) is the set of all the σ-GSF whose derivatives are ρ-moderate. It is not hard to prove that if

$$\exists M\in\mathbb{N}\,\forall N\in\mathbb{N}:\ \mathrm{d}\rho^{-N}\le\mathrm{d}\sigma^{-M},\tag{7}$$

then

$$\exists C\in{}^{\sigma}\widetilde{\mathbb{R}}_{>0}\,\forall f\in{}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K,{}^{\rho}_{\sigma}\widetilde{\mathbb{C}})\,\forall m\in\mathbb{N}:\ \|f\|_0>0\ \Rightarrow\ \|\partial^\alpha f\|_m\le C\cdot\|f\|_0\tag{8}$$

(recall (4)). We have hence a property closely reminiscent of the usual one for bounded operators, if we measure the Lipschitz constant of $\partial^{\alpha}(-)$ using an infinite number in ${}^{\sigma}\widetilde{\mathbb{R}}$. For example, from (8) the continuity of $\partial^{\alpha}(-)$ with respect to the sharp topology in ${}^{\sigma}\widetilde{\mathbb{R}}$ follows. Note that the classical counter example $f_n(x) = \sin(nx)$, $x \in [0, 2\pi]$, satisfies $\|f_n\|_0 = 1$ and $\|f_n'\|_1 = n$, so it does not represent a counter-example of (8) if $n \in {}^{\rho}\widetilde{\mathbb{N}}$, i.e. if $f_n \in {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$ (whereas, if $n \in {}^{\sigma}\widetilde{\mathbb{N}}$, then $f_n \notin {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$).

Our plan for this WP can be summarized as follows:

1. Definition of IBO using (8) as a typical case. Definition of basic IBO T by following the general definition of [Ga-Vernaeve] (i.e. T is generated by a net $T_{\varepsilon} : V_{\varepsilon} \longrightarrow W_{\varepsilon}$ of ordinary linear maps, so that $T([v_{\varepsilon}]) = [T_{\varepsilon}(v_{\varepsilon})]$).

2. As mentioned above, we want to proceed using an intrinsic and abstract approach, i.e. considering an arbitrary ${}^{\sigma}\widetilde{\mathbb{C}}$-module $\mathcal{G}$ instead of the particular space ${}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$. To this end, it could be useful to consider the condition that $\left\{ v \in \mathcal{G} \mid \|v\|_0 \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} \right\}$ is dense in $\mathcal{G}$, and to prove that this condition holds for $\mathcal{G} = {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$ by generalizing Lem. 4.23 and Thm. 4.25 of [Ar-Ju01]. It could be important to note that if $0 \in \bar{S} \subseteq [0, 1] \subseteq \mathbb{R}$, and $e_S := [1|_S] \in {}^{\sigma}\widetilde{\mathbb{R}}$ is the generalized number defined by the characteristic function of S, then $\|f\|_m \cdot e_S = 0$ implies $f \cdot e_S = 0$.

3. We note here that the natural notion of solution of DE for GSF is rather strong, because we have the closure with respect to composition. Essentially for this reason, we are interested in graded ${}^{\sigma}\widetilde{\mathbb{C}}$-Fréchet spaces, i.e. of the form $(\mathcal{G}, (\| - \|_m)_{m \in \mathbb{N}})$, with a ${}^{\sigma}\widetilde{\mathbb{C}}$-module and a countable family of ${}^{\sigma}\widetilde{\mathbb{R}}$-valued norms such as (4). Therefore, we plan to consider *graded* ${}^{\sigma}\widetilde{\mathbb{C}}$-Hilbert spaces of the type $\left(H, (\langle -, - \rangle_m)_{m \in \mathbb{N}}\right)$, where e.g. $\langle f, g \rangle_m := \sum_{|\alpha| \le m} \int_K \overline{\partial^{\alpha} f(x)} \partial^{\alpha} g(x) \, \mathrm{d}x \in {}^{\sigma}\widetilde{\mathbb{R}}$ in case of spaces of GSF .

4. Prove that any IBO T defined on the space ${}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$ is an arrow (i.e. a smooth map) in the Grothendieck topos ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$ of GSF , i.e. if $U \subseteq {}^{\sigma}\widetilde{\mathbb{R}}^u$ is a σ-sharply open set and $\varphi : U \longrightarrow {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$ satisfies that $\varphi^{\vee}(u, k) := \varphi(u)(k)$ is a GSF of the type $U \times K \longrightarrow {}^{\rho}_{\sigma}\widetilde{\mathbb{C}}$ (Cartesian closure property of the topos), then $T \circ \varphi \in {}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}(U, {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}(K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}}))$, i.e. $(T \circ \varphi)^{\vee} \in {}^{\sigma}\mathcal{GC}^{\infty}(U \times K, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$.

5. From bounded operators to IBO using [Ga-Vernaeve, Gar09, Gar05, Gar05b] as a blueprint: open mapping theorem, closed graph theorem, Banach-Steinhaus uniform boundedness principle, Riesz theorem, existence and uniqueness of adjoint Hermitian operators, Lax-Milgram theorem and the corresponding Galerkin theorem.

6. Hahn-Banach theorem for basic IBO; note, for example, that the proof of Hahn-Banach in [AMR88] seems to be repeatable for basic IBO because the key step where one has to use supremum and infimum of suitable subsets of $\mathbb{R}$ has now to be repeated ε-wise.

7. Proof of the Riesz-Markov theorem by generalizing [RS1, Thm. IV.4] and its consequences on multidimensional integration of GSF. The spectrum for IBO and the spectral theorem on functionally compact set starting from the proof of [RS1].

Risks and solutions: The use of two gauges to incorporate the classical notion of unbounded linear map into that of IBO with infinite Lipschitz constant is the key idea of this WP. Since we start from the solid background of [Ga-Vernaeve, Gar09, Gar05, Gar05b], we can foresee that several results can be generalized without notable problems. In case of unforeseen problems, our study of closed infimum/supremum and of multidimensional integration will surely be of great help.

Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *high* assessment of feasibility.

3.2 WP 2: Applications to analysis and QM

We plan to consider a relevant set of important applications of the previous theory of IBO, both in mathematical analysis and in QM.

1. Similarly to what we presented above for the derivative operator $\partial^\alpha(-)$, we can define the exponential function by considering two gauges $\rho \geq \sigma$ and the subring of ${}^\sigma\widetilde{\mathbb{R}}$ defined by

   $$ {}^\rho_\sigma\widetilde{\mathbb{R}} := \{x \in {}^\sigma\widetilde{\mathbb{R}} \mid \exists N \in \mathbb{N} : \ |x| \leq \mathrm{d}\rho^{-N}\}. $$

   If we have

   $$ \forall N \in \mathbb{N}\, \exists M \in \mathbb{N} : \ \mathrm{d}\rho^{-N} \leq -M \log \mathrm{d}\sigma, \tag{9} $$

   then $e^{(-)} : [x_\varepsilon] \in {}^\rho_\sigma\widetilde{\mathbb{R}} \mapsto [e^{x_\varepsilon}] \in {}^\sigma\widetilde{\mathbb{R}}$ is well defined. For example, if $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{1/\varepsilon}\right)$, then $\sigma \leq \rho$ and both (7) and (9) holds for $M = 1$. Note that the natural ring morphism $[x_\varepsilon]_{\sim_\sigma} \in {}^\rho_\sigma\widetilde{\mathbb{R}} \mapsto [x_\varepsilon]_{\sim_\rho} \in {}^\rho\widetilde{\mathbb{R}}$ is surjective but in general not injective. If T is a basic IBO on $\mathcal{G} = {}^\rho_\sigma\mathcal{GC}^\infty(K, {}^\rho_\sigma\widetilde{\mathbb{C}})$, we want to study the properties of $e^T : [v_\varepsilon] \in \mathcal{G} \mapsto \left[e^{T_\varepsilon(v_\varepsilon)(-)}\right] \in \mathcal{G}$. More generally, the exponential function is a particular case of a real hyperanalytic GSF, i.e. a GSF f which equals its Taylor hyperseries. We therefore intend to consider the basic IBO $f(T)$ similarly defined using an hyperseries. Considering the embedding of Sobolev-Schwartz distributions used e.g. in [GrKuObSt01] (convolution with an *entire* Colombeau mollifier), we conjecture that the embedding of every Sobolev-Schwartz distribution is actually a hyperanalytic GSF. This would imply a strong extension of the continuous functional calculus of operators.

2. Prove that each Picard-Lindelöf n-th iteration of a ${}^\sigma\widetilde{\mathbb{C}}$-linear PDE defines a basic IBO I_n. We also want to consider the interesting case where $n \in {}^\rho\widetilde{\mathbb{N}}$, i.e. IBO corresponding to hyperfinite Picard-Lindelöf contractions. This would allow us to consider a more natural hyperlimit instead of a classical limit in the definition of contraction, see (5). We have already proved that, once again, the iteration I_n takes GSF into GSF if we choose the gauge σ so that $n \in {}^\rho\widetilde{\mathbb{N}}$ implies $n^n \in {}^\sigma\widetilde{\mathbb{N}}$.

3. Prove that every GSF $f \in \mathcal{G} = {}^\rho_\sigma\mathcal{GC}^\infty(K, {}^\rho_\sigma\widetilde{\mathbb{C}})$ defines a basic IBO via $T_f : \varphi \in {}^\rho_\sigma\mathcal{GD}_K({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho_\sigma\widetilde{\mathbb{C}}) \mapsto \int_K f \cdot \varphi \in {}^\rho_\sigma\widetilde{\mathbb{C}}$, where ${}^\rho_\sigma\mathcal{GD}({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho_\sigma\widetilde{\mathbb{C}})$ is the space of GSF compactly supported in K (see [Gio-Kun18a]). The fundamental lemma of the calculus of variations, see [LL-Giordano16, Lem. 37], entails that f is uniquely determined by this functional. We want to investigate whether every *basic smooth operator* (where "smooth arrow" has already been explained above using the Cartesian closure of the topos ${}^\rho\mathrm{T}\mathcal{GC}^\infty$) is of the form T_f for some GSF f. The idea to recover the density f from the basic smooth functional T is to consider $f(x) := T(\delta_x)$, where δ_x is the Dirac delta centered at $x \in {}^\rho\widetilde{\mathbb{R}}^n$ and to use the sheaf property proved in [Gio-Kun-Ver19].

4. Classical examples, such as:

   a) The space ℓ^2 of ${}^\rho\widetilde{\mathbb{R}}$ numbers, but using absolutely convergent hyperseries;

b) Sobolev spaces of GSF: note that a natural transposition of the classical definition in case of functionally compact sets $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ gives

$$W^{k,p}(K) := \left\{ u \in {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}}) \mid \int_K |\partial^{\alpha} u|^p \in {}^{\rho}\widetilde{\mathbb{R}} \ \forall |\alpha| \le k \right\} = {}^{\rho}_{\sigma}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$$

because every GSF is always differentiable and integrable on K (even if K could be unbounded, e.g. $u(x) = e^{ikx}$, $x \in [-\mathrm{d}\rho^{-1}, \mathrm{d}\rho^{-1}] \supseteq \mathbb{R}$). We hence plan to say that $u \in {}^{\rho}_{\sigma}\mathcal{GD}(U, {}^{\rho}_{\sigma}\widetilde{\mathbb{C}})$ is p-integrable if $\exists \overline{\sup}_{x \in U} |u(x)|^p$, where U is a strongly internal set (see [Gio-Kun-Ver15]) whose Lebesgue measure $\lambda(U) \in {}^{\rho}\widetilde{\mathbb{R}}$ is given by $\lambda(U) := \sum_{q \in \mathbb{N}} \lambda(K_q)$, where $U = \bigcup_{q \in \mathbb{N}} K_q$, $K_q \subseteq \mathrm{int}\,(K_{q+1})$, is an exhaustion of U by functionally compact sets (and $\lambda(U)$ does not depend on this exhaustion). We can hence also consider $W^{k,p}(U)$ for these measurable strongly internal sets U.

c) Give the general definition of differential operator and prove that it is an IBO.

d) IBO defined by the Fourier transform and the hyperfinite Fourier transform of GSF, see [M-Giordano19].

e) Classical operators used in QM, such as: position, momentum, kinetic and potential energy, Hamiltonian, angular momentum and spin angular momentum, etc.

5. Reformulate the axioms of QM, e.g. those used in [BeLuSi19], using GSF and IBO. In particular, reformulation of the sixth axiom as follows: "In a laboratory only the states associated to a *near-standard* expectation value of the physically relevant quantities can be realized. These states are called physical states, the rest of the states *can be interpreted as idealized models as* $\varepsilon \to 0$, *corresponding to real* ε*-depending physical states*". Solution of the Schrödinger equation for a Hamiltonian with an arbitrary GSF as potential and using the Picard-Lindelöf theorem for PDE. We are particularly interested in the solution of the stationary Schrödinger equation for an infinite rectangular potential well (a case that cannot be formalized using Sobolev-Schwartz distributions, see e.g. [GaPa90]) or a rectangular potential well with periodic barriers changing at infinite frequency, and application to high frequency laser pulses acting on quantum objects (see e.g. [VKRKPW]). Comparison with the solution given in [BeLuSi19] in case of the delta function.

Risks and solutions: Some of the listed examples are partly tied to unsolved conjectures, such as those in 1, 3, 4b above. In case of unforeseen problems in proving these conjectures, we can anyway develop an interesting and relevant part of them. All the other applications do not present foreseeable risks.

Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *high* assessment of feasibility.

3.3 WP 3: Universal properties of GF and IBO

In order to discover universal properties of ${}^{\rho}\widetilde{\mathbb{R}}$, of the sheaf of GSF and of IBO, our starting point will be the co-universal property of the sheaf of Sobolev-Schwartz distributions (see [Giordano-M97, MelMun00]): Let $\mathcal{D}'(\Omega)$ be the space of distributions on the open set $\Omega \subseteq \mathbb{R}^n$. Let $\lambda_{\Omega}(f) \in \mathcal{D}'(\Omega)$ be the usual embedding of the continuous functions $f \in \mathcal{C}^0(\Omega)$, and let $D(T) \in \mathcal{D}'(\Omega)$ be the derivative of the distribution $T \in \mathcal{D}'(\Omega)$. Then we have the following properties:

1. $\mathcal{D}' : \Omega \mapsto \mathcal{D}'(\Omega)$ is a sheaf of vector spaces defined on the open sets of $\mathbb{R}^n$ and $\lambda : \mathcal{C}^0 \longrightarrow \mathcal{D}'$ is a sheaf morphism.

2. the distributional derivative D is compatible with derivative $(-)'$ of $\mathcal{C}^1$ functions, i.e. we have

$$D \circ \lambda \circ i = \lambda \circ (-)',$$

   where $i_\Omega : \mathcal{C}^1(\Omega) \hookrightarrow \mathcal{C}^0(\Omega)$ is the inclusion.

3. $(\mathcal{D}', D, \lambda)$ is a co-universal solution of the previous properties 1, 2, i.e. if $(\overline{\mathcal{D}'}, \bar{D}, \bar{\lambda})$ also satisfy 1 and 2, then there exists one and only one sheaf morphism $\psi : \mathcal{D}' \longrightarrow \overline{\mathcal{D}'}$ such that

$$\psi \circ \lambda = \bar{\lambda}, \qquad D \circ \psi = \bar{D} \circ \psi.$$

As for any (co-)universal property, this characterizes the entire sheaf $\mathcal{D}'$ up to sheaf morphisms of vector spaces. Note that, as is typical in considering universal properties, this characterization necessarily singles out the particular choices which a particular construction depends on, such as the embedding λ of continuous functions.

We can hence plan the present WP as follows:

1. Characterize, up to ordered ring isomorphisms, ${}^\rho\widetilde{\mathbb{R}}$ as a quotient ordered ring with a maximal equivalence relation and a distinguished invertible element $\mathrm{d}\rho \in {}^\rho\widetilde{\mathbb{R}}$ such that

$$\forall [w_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} : \ [w_\varepsilon] = 0 \ \Rightarrow \ \lim_{\varepsilon \to 0^+} w_\varepsilon = 0,$$
$$\forall x \in {}^\rho\widetilde{\mathbb{R}}\, \exists N \in \mathbb{N} : \ -\mathrm{d}\rho^{-N} \le x \le \mathrm{d}\rho^{-N}.$$

   See also [Gio-Kun-Ver19, Sec. 2].

2. Characterize, up to smooth isomorphisms $\psi : {}^\rho\mathcal{GC}^\infty(X, Y) \longrightarrow {}^\rho\mathcal{GC}^\infty(X, Y)$ in the topos ${}^\rho\mathrm{T}\mathcal{GC}^\infty$, the space ${}^\rho\mathcal{GC}^\infty(X, Y)$ of GSF as the simplest (co-universal) space of internal functions (in the sense of NSA) closed with respect to arbitrary derivatives in ${}^\rho\widetilde{\mathbb{R}}$ and taking $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ to $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. Explore the possibility to characterize the entire sheaf ${}^\rho\mathcal{GC}^\infty$ of GSF in the topos ${}^\rho\mathrm{T}\mathcal{GC}^\infty$. Explore the possibility to find a universal property of the sheaf of special CGF considering the particular choice of the sup-norms used in its definition. We also want to analyze the possibility to characterize GSF similarly to the previous co-universal property of Sobolev-Schwartz distributions, i.e. by formalizing the idea that they are the simplest sheaf where continuous functions are embedded using regularization through a given Colombeau mollifier.

3. For simplicity, let $\mathcal{G} := {}^\rho_\sigma\mathcal{GC}^\infty(K, {}^\rho_\sigma\widetilde{\mathbb{C}})$ and let $\mathcal{B}(\mathcal{G}, \mathcal{G}) \subseteq {}^\rho\mathrm{T}\mathcal{GC}^\infty(\mathcal{G}, \mathcal{G})$ be the space of basic internal operators on $\mathcal{G}$. Characterize, up to smooth isomorphisms $\Psi : \mathcal{B}(\mathcal{G}, \mathcal{G}) \longrightarrow \mathcal{B}(\mathcal{G}, \mathcal{G})$ in the topos ${}^\rho\mathrm{T}\mathcal{GC}^\infty$, the space $\mathcal{B}(\mathcal{G}, \mathcal{G})$ of basic IBO as the simplest space of internal functions which are ${}^\sigma\widetilde{\mathbb{C}}$-linear, ${}^\sigma\widetilde{\mathbb{R}}$-bounded and mapping $\mathcal{G}$ to itself.

4. Compare the previous (co-)universal properties with the axioms of [Tod13, Tod11], highlighting what are the pros and cons of the two approaches.

Risks: Due to the expertise of the applicant in this topic, and due to the right topos-theoretical framework to formulate them, we do not foresee particular risks in this WP.

Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *very high* assessment of feasibility.

4 Scientific relevance, originality and expected benefits for potential users

The present research proposal takes place in the following international research frameworks:

- It fits well in current threads of Austrian research, in particular those of the DIANA group of Prof. M. Kunzinger at the University of Vienna, who is also one of the main developers of the theory of GSF.
- It also fits very well into the research interest of Ghent University, in particular those of Prof. H. Vernaeve and of Prof. M. Ruzhansky (see https://ruzhansky.org/). We also plan to initiate a collaboration with the highly active research group of Prof. M. Ruzhansky (even if this collaboration can be considered as an added value and it is not essential for the development of the proposal. For this reason, we do not include a corresponding collaboration letter).
- It also fits well into the research interests of the international community of CGF, where the interest for functional analytic tools and applications of Colombeau's theory to physics was clearly voiced in several conferences and monographs.

Originality, innovations and benefits of the present proposal:

- The theory of IBO represents a clear example of strong simplification due to a fundamental non-Archimedean property. It will surely stimulate further research of functional analysis in this framework.
- Several of the presented applications and examples represent strong generalization of well-known results (like, e.g., of the continuous functional calculus of operators which can certainly be extended at least to hyperanalytic GSF), but with a much simpler approach.
- For the first time, the use of the Grothendieck topos ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$ finds its application as the correct setting where to formulate universal properties of generalized numbers, functions and operators. This represents one of the very few applications of topos theory to mathematical analysis and could leads to further connections with topos theory and with the theory of diffeological spaces.
- Universal properties are able to describe spaces of GF in a particularly simple way, underscoring the differences with Sobolev-Schwartz distributions and highlighting the particular choices which the theory depends on. This can clarify the entire construction and could lead to further useful generalizations.
- The planned approach to QM using GSF and IBO (and where solutions of the Schrödinger equation for a GSF Hamiltonian follow from the general Picard-Lindelöf theorem for PDE) illustrates an important solution to non-linear operations used in physics with distributions. The possibility to use infinitesimal and infinite numbers in modeling physical systems is another important further feature of our approach.

Potential users can hence be foreseen both in pure and applied mathematics, in physics and engineering applications such as applications of the Galerkin theorem to finite elements methods for highly singular problems.

4.1 Importance for human resources

The new results achieved in the present proposal would allow us to write an fundamental monograph about GSF, with both very original new theoretical and applied results. This constitutes an important step to consolidate P. Giordano's research profile and might even open the possibility of acquiring an ERC grant. For the CV of all the members of the research group, see below in this proposal.

4.2 Ethical Issues

There are no ethical, security-related or regulatory aspects of the proposed research project.

4.3 Sex-specific and gender-related aspects

There are no sex-specific and gender-related aspects of the proposed research project.

5 Dissemination strategy and time planning

Work organization and human resources

Almost all the ideas of this project concerning IBO have been originally developed by P. Giordano. For this reason, he will actively contribute to their progress during the entire project. This contribution is essential to the success of this application and hence it justifies the 100% engagement of the PI into the project.

P. Giordano, M. Kunzinger and H. Vernaeve will supervise all the other members of the group; P. Giordano will work on all the WP; H. Vernaeve and M. Kunzinger will also work more specifically on WP 1 and WP 2 because of their interest and expertise in functional analysis and generalized functions. The first Ph.D. thesis will focus more on the first part of WP 1 and on some mathematical applications of WP 2. The second Ph.D. thesis will focus on the second part of WP 1 (spectral theory of basic IBO) and on applications to QM.

Moreover, we plan to organize several joint meetings in order to initiate and improve the collaboration, both in the solutions of problems and to get new ideas:

- Weekly seminar of the PI P. Giordano with both Ph.D. students (two separate meetings per week).
- Monthly seminar with all the member of the research group (and possibly interested external people), with presentation of present state, open problems and new results. This seminars will always be organized with an online video connection in case H. Vernaeve will not be present at the University of Vienna.
- One invitation per year of 5 days of H. Vernaeve at the University of Vienna to collaborate at the project.

Dissemination strategy

Our strategy for the dissemination of the results of this proposal is addressed both to an internal and an international audience:

- Regular seminars of the DIANA group of the University of Vienna addressed to all the interested colleagues are planned, with presentation of present state, open problems and new results.
- Contributions for two international conferences per year per person for the presentation of relevant results are planned. In particular, we are thinking of conferences such as the *ISAAC Conference* in 2021, the *International Conference on Generalized Functions* in 2020, the *International Congress of Mathematical Physics* 2021, the *International Conference on Applications of Geometric Methods of Functional Analysis* and the next *Operator Theory, Analysis and Mathematical Physics Conference.*
- Several articles for peer reviewed green open access journals and the related dissemination by means of preprint-servers is planned. We aim at journals such as: *Journal of Functional Analysis, Transaction of the American Mathematical Society, Communications in Mathematical Physics, AIP Journal of Mathematical Physics*, *Acta Applicandae Mathematicae*, *Advances in Nonlinear Analysis, Calculus of Variations and Partial Differential Equations.*

Time planning

The research project is designed for four co-workers: P. Giordano, M. Kunzinger and two Ph.D. candidates. To estimate the total amount of work to be dedicated into each one of the three parts of this project, we plan about six months to fully understand the background on GSF and hence 28 months for the development of each WP. A period of two months is planned to accomplish administrative duties such as theses writing and correction. The entire research project is hence planned to be concluded in 36 months and the time planning is represented in Fig. 1.

6 Scientific environment

The ideal environment for realizing this project is the DIANA research group at Vienna University. This will enable us to collaborate with some of the leading scientists in the field, as well as several PostDocs and Ph.D. students. As an added value, we also intend to initiate a collaboration with the highly active research group of Prof. M. Ruzhansky of the University of Ghent, Belgium, about the theory of IBO and its applications.

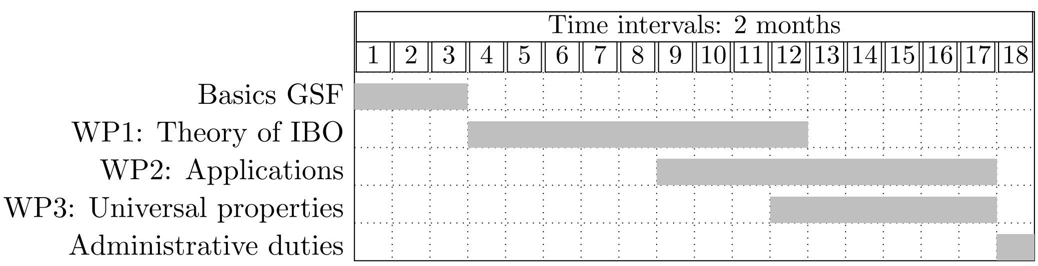

Figure 1: Basic time organization

Nomenclature

CGF	Colombeau generalized function(s)
DE	Differential equation(s)
GF	Generalized function(s)
GSF	Generalized smooth function(s)
IBO	Infinite bounded operator(s)
NSA	Nonstandard analysis
ODE	Ordinary differential equation(s)
PDE	Partial differential equation(s)
QM	Quantum mechanics
WP	work package(s)

Annex 1: financial aspects

Available personnel and infrastructure

The University of Vienna (AT) is the planned research institution to host the present research project. The only available personnel is Prof. M. Kunzinger, who is already employed at the University of Vienna. Prof. M. Kunzinger is one of the head of the DIANA research group of the University of Vienna, which regularly held weekly seminars (see Dissemination strategy in Sec. 5 of the proposal).

Personnel costs

In our view, work on the project goals can be pursued by funding one senior post-doc position for Dr. P. Giordano, and two Ph.D. candidate positions for 2 years. As is quite common at the University of Vienna, the remaining funding for the completion of these Ph.D. studies will be requested in separate specific project proposals. Therefore, on the basis of the 2019 FWF salary rates, we have the following:

- 1 senior post-doc position for 36 months: 74'380 €/y * 3 y = 223'140 €.
- 2 Ph.D. candidate positions for 2 years = 38'550 €/y * 4 y = 154'200 €.

This amounts to a total of 377'340 €.

Equipment and material costs

These costs concern two graphic tablets for the two Ph.D. students (which should not be confused with a tablet computer; see e.g. https://www.youtube.com/watch?v=eEJtMFdkzzU) for exchange of mathematical handwritten notes and PDF annotations. A classical graphic tablet could be the Wacom Pth-860-S, size L (413 € Amazon.com price as of 17 October 2019). Note that this kind of IT tool is not available as standard equipment of the University of Vienna.

- 1 graphic tablet * 2 researchers = 413 € * 2 = 826 €.

Travel costs

We plan to pay the three invitations to Prof. H. Vernaeve (yearly invitation for the entire duration of the project). We plan 900 € per invitation. We note here that it is cheaper that Prof. H. Vernaeve visits Vienna than all project members visit him.

- 3 invitations * 900 € = 2'700 €.

Therefore, the total amount requested for the present proposal (considering 5% of general costs) is 399'909.3 €.

Annex 2: List of references

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Curriculum vitae of P. Giordano

Name:	Giordano, Paolo
ORCID:	0000-0001-7653-1017
Nationality:	Italian
Date of birth:	27 March 1966
Web site:	www.mat.univie.ac.at/~giordap7/
Affiliation:	Wolfgang Pauli Institute, Oskar-Morgenstern-Platz 1 1090 Wien
Email:	paolo.giordano@univie.ac.at

Present research interests

- Non-Archimedean geometry and analysis
- Nonlinear theories of generalized functions
- Foundation of differential geometry
- Mathematical theories of complex systems

Education

- *Habilitation*, University of Vienna, Austria, 2019.
- *Rheinischen Friedrich-Wilhelms-Universität Bonn* (DE), Ph.D. in Mathematics, 2009.
- *Università degli Studi di Milano* (IT), M.Sc. in Mathematics, 1997.

Academic experiences

Selected research activities as principal investigator or co-director (selection)

- August 2017 - present: project leader of FWF stand alone research project *Hyperfinite methods for generalized smooth functions*, Wolfgang Pauli Institute, Vienna. Co-applicant and collaborator of the project is Prof. M. Kunzinger; 397'000 Euro.
- December 2012 - May 2017: project leader of FWF stand alone research project *Analysis and Geometry based on generalized numbers*, Dep. of Mathematics, University of Vienna. Co-applicant and collaborator of the project is Prof. M. Kunzinger; 321'000 Euro.
- June 2013 - May 2016: project leader of FWF stand alone research project *Non-Archimedean Geometry and Analysis*, Dep. of Mathematics, University of Vienna (AT). Co-applicants of the project are Prof. M. Kunzinger and Prof. V. Benci; 349'000 Euro.

- October 2010 - September 2012: project leader of the research project *Nilpotent Infinitesimals and Generalized Functions*, Dep. of Mathematics, University of Vienna, supported by an FWF Lise Meitner grant. Co-applicant of the project: Prof. M. Kunzinger; 115'200 Euro.

- March 2002 - February 2004: Marie Curie individual fellowship of the European Commission, *A new approach to differential geometry of spaces of mappings and its applications,* Institute of Applied Mathematics, University of Bonn, 140'200 Euro.

5 selected invited lectures

1. Invited talk at Institute for Scientific Interchange (ISI), "MaTryCS - A mathematical theory of complex systems", 2016.

2. Invited plenary lecture at the conference "Algebra, Geometry and Mathematical Physics", Brno, Czech Republic - "Infinitesimal without Logic", 2012.

3. Invited talk at the University of Pisa - "Generalized smooth functions", 2015.

4. Invited opening talk at the workshop "Workshop on diffeologies etc", Aix en Provence, France - "Theory of infinitely near points in smooth manifolds: the Fermat functor", 2014.

5. Invited speaker at the Interdiziplinäre Zentrum für Komplexe Systeme (IZKS, Bonn, Germany) - "Dynamics of cities: A mathematical planning tool for shopping malls", 2009.

Supervision of Ph.D. students and postdoctoral fellows

- 2018 - current: A. Mukhammadiev, D. Tiwari, Ph.D. students, University of Vienna, AT.

- 2012 - 2016: L. Luperi Baglini and E. Wu, post-docs, University of Vienna, AT.

- 2006 - 2009: G.L. Ciampaglia, M. Esmaeili, Ph.D. students, University of Italian Switzerland, CH.

Reviewing activities

I am reviewer for: Acta Mathematica, American Mathematical Monthly, Advances in Complex Systems, Environmental modelling and software, Physics Letters A, Topology proceedings, Commentationes Mathematicae Universitatis Carolinae, Arabian Journal of Mathematics.

International research partner (selection)

Vieri Benci, University of Pisa, Italy; Sergio Albeverio, University of Bonn, Germany; Hans Vernaeve, University of Ghent, Belgium; Alberto Vancheri, SUPSI, Switzerland; Maryam Esmaeili, University of Massachusetts Amherst.

Main areas of research and selected results

- *Theory of Fermat reals*: we developed a new theory of nilpotent infinitesimals and its applications to infinite-dimensional spaces.

- *Theory of Generalized Smooth Functions*: a new theory of generalized functions resulting in the closure with respect to composition, a better behavior on unbounded sets and new general existence results.
- *Theory of Interaction Spaces*: a new unifying theory of complex systems which includes several types of complex systems models.
- *Colombeau theory*: we unified several different Colombeau-like algebras into a single general abstract notion having the same simplicity of the special algebra.

Most important publications

For the links to these publications and the complete list, see:
https://www.mat.univie.ac.at/~giordap7/#publications

1. Giordano P., Kunzinger M., A convenient notion of compact set for generalized functions. Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92. DOI: 10.1017/S0013091516000559
2. Lecke A., Luperi Baglini L., Giordano P., The classical theory of calculus of variations for generalized functions. Advances in Nonlinear Analysis, Vol. 8, issue 1, 2017. DOI: 10.1515/anona-2017-0150
3. Giordano P., Kunzinger M., Inverse Function Theorems for Generalized Smooth Functions. Chapter in "Generalized Functions and Fourier Analysis", Volume 260 of the series Operator Theory: Advances and Applications pp 95-114. DOI: 10.1007/978-3-319-51911-1_7
4. Giordano P., Wu E., Calculus in the ring of Fermat reals. Part I: Integral calculus. Advances in Mathematics 289 (2016) 888–927. DOI: 10.1016/j.aim.2015.11.021
5. Giordano P., Kunzinger M., Vernaeve H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56-71. DOI: 10.1016/j.jmaa.2014.08.03
6. Vancheri A., Giordano P., Andrey D., Fuzzy logic based modeling of traffic flows induced by regional shopping malls. Advances in Complex Systems Vol. 17, N. 3 & 4, 2014, (39 pages). DOI: 10.1142/S0219525914500179
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9. Giordano P., Infinitesimals without logic, Russian Journal of Mathematical Physics, 17(2), pp.159-191, 2010. DOI: 10.1134/S1061920810020032
10. Giordano P., Fermat-Reyes method in the ring of Fermat reals. Advances in Mathematics 228, pp. 862-893, 2011. DOI: 10.1016/j.aim.2011.06.008

Michael Kunzinger

Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
☎ *+43 1 4277 50681*
✉ *michael.kunzinger@univie.ac.at*
http://www.mat.univie.ac.at/∼mike/, ORCID: 0000-0002-7113-0588

Curriculum Vitæ

Main Areas of Research

Functional analysis, generalized functions, semi-Riemannian geometry, mathematical physics, Lie group analysis of partial differential equations.

Education

2001 **Habilitation**, *University of Vienna*, Austria.

1996 **Ph.D**, *University of Vienna*, Austria.

1993 **Mag.rer.nat. (Mathematics)**, *University of Vienna*, Austria.

2002 **Mag.rer.nat. (Physics)**, *University of Vienna*, Austria.

Employment

Since 2019 **Full Professor**, *University of Vienna*, Austria.

2001–2019 **Associate Professor**, *University of Vienna*, Austria.

06/01–09/01 **Visiting Scientist**, *University of Southampton*, GB.

1996–2001 **Assistant Professor**, *University of Vienna*, Austria.

1995–1999 **Research Assistant**, *University of Vienna*, Austria, FWF research grant P-12023MAT, 'Distributional Methods in Einstein's Theory of Gravitation'.

1994–1995 **Assistant Professor**, *University of Vienna*, Austria.

Supervised PhD Students

2018 **Melanie Graf**, *Singularity theorems and rigidity in Lorentzian geometry*.

2016 **Alexander Lecke**, *Non-smooth Lorentzian Geometry and Causality Theory*.

2015 **Milena Stojković**, *Causality Theory for $C^{1,1}$-metrics*.

2010 **Eduard Nigsch**, *A Nonlinear Theory of Tensor Distributions on Riemannian Manifolds*.

2009 **Jasmin Sahbegović**, *Short-Time Fourier Transform and Modulation Spaces in Algebras of Generalized Functions*.

2008 **Sanja Konjik**, *Group Analysis and Variational Symmetries for Non-Smooth Problems*.

2006 **Eberhard Mayerhofer**, *The wave equation on singular space-times*.

5 Selected Invited Lectures

Jul 26, 2017 **Singularity theorems in regularity $C^{1,1}$**, *Geometry and Relativity*, Erwin Schrödinger Institue, Vienna.

Sep 9, 2016 **International Conference on Generalized Functions GF2016**, *Generalized functions as set-theoretical maps*, University of Dubrovnik.

Dez 6, 2015 **Workshop on generalized functions**, *Low-regularity semi-Riemannian geometry and the singularity theorems of General Relativity*, University of Zagreb.

Sep 6, 2012 **PDEMTA, Topics in PDE, Microlocal and Time-frequency Analysis**, *Abstract regularity theory*, University of Novi Sad.

Jan 6, 2006 **1st International Workshop on Mathematical Sciences: Mathematical Analysis and Applications.**, *Nonlinear distributional geometry*, Sogang University, Seoul.

Academic Prizes/Awards

2004 **START-prize**, *Austrian Science Fund FWF*.

2003 **ÖMG-Prize**, *Austrian Mathematical Society*.

Most important peer review activities, editorships and/or memberships in academic organisations

President, *International Association for Generalized Functions*.

Editor, *Publications de l'Institut Mathematique*, Belgrade.

2008-2016 **Member of the Austrian Academy of Sciences**.

Selected Research Projects

2017–2021 **P30233**, *Regularity Theory in Algebras of Generalized Functions*, Austrian Science Fund.

2016–2021 **P28770**, *Singularity Theorems and Comparison Geometry*, Austrian Science Fund.

2008-2011 **P20525**, *Global Analysis in Algebras of Generalized Functions*, Austrian Science Fund.

2005-2011 **START-Project Y237**, *Nonlinear Distributional Geometry*, Austrian Science Fund.

International Cooperation Partners (in the last 5 years)

James D. E. Grant, *University of Surrey*, GB.

James A. Vickers, *University of Southampton*, GB.

Hans Vernaeve, *University of Ghent*, Belgium.

Darko Mitrovic, *University of Montenegro*, Montenegro.

Clemens Sämann, *University of Toronto*, Canada.

Main areas of research and selected results

Symmetry Analysis of Differential Equations: New classification methods (conditional equivalence groups, normalized classes), study of conservation laws, characteristics and potential symmetries, formal compatibility, generalized conditional symmetries, singular reduction modules.

Algebras of Generalized Functions: Geometrization of the Theory, diffeomorphism invariant embeddings of distributions, applications to singular spacetimes in General Relativity, microlocal analysis, development of a theory of generalized smooth functions.

Mathematical General Relativity: Study of highly singular (distributional) spacetimes (pp-waves), extension of the singularity theorems of Hawking and Penrose to spacetimes of regularity $C^{1,1}$, study of singular spacetimes via new methods of metric geometry.

Differential Geometry: Nonlinear distributional geometry, low regularity pseudo-Riemannian geometry: new comparison methods, development of a new metric theory of Lorentzian geometry in metric spaces: Lorentzian length spaces, study of synthetic curvature bounds in this setting and applications to extendability of spacetimes.

Partial Differential Equations: Study of highly singular PDEs in the framework of algebras of generalized functions, kinetic theory (Vlasov-Klein-Gordon), degenerate parabolic equations on compact Riemannian manifolds.

Locally convex spaces: Study of weak barrelledness properties, applications of the theory of vector valued distributions to Cauchy-Dirichlet problems.

10 Most Important Publications

For the complete list of publications, see `https://www.mat.univie.ac.at/~mike/publications.php`

Books

1. **Barrelledness, Baire-like- and (LF)-Spaces**
   Pitman Research Notes in Mathematics Vol. 298,
   Longman, Harlow 1993.
2. with E. Farkas, M. Grosser und R. Steinbauer,
   On the Foundations of Nonlinear Generalized Functions I,II
   Mem. Amer. Math. Soc. 153, No. 729, 2001.
3. with M. Grosser, M. Oberguggenberger und R. Steinbauer
   Geometric Theory of Generalized Functions
   with Applications to General Relativity,
   Mathematics and its Applications, Kluwer 2001.

Journal Publications

4. with M. Oberguggenberger,
   Characterization of Colombeau generalized functions by their pointvalues
   Math. Nachr. 203, 147–157, 1999.
5. with M. Grosser, R. Steinbauer und J. Vickers
   A global theory of algebras of generalized functions
   Advances in Math. 166, No. 1, 50–72, 2002.
   http://arxiv.org/abs/math.FA/9912216
6. with G. Rein, R. Steinbauer, and G. Teschl
   Global weak solutions of the relativistic Vlasov-Klein-Gordon System
   Comm. Math. Phys., 238 (1-2), 367-378, 2003.
   http://arxiv.org/abs/math.AP/0209303
7. with P. Giordano
   Topological and algebraic structures on the ring of Fermat reals
   Israel J. Math., 193 no. 1, 459–505, 2013.
   http://arxiv.org/abs/1104.1492
8. with S. Dave,
   Singularity structures for noncommutative spaces
   Trans. Amer. Math. Soc., 367, no. 1, 251-273, 2015.
   http://arxiv.org/abs/1111.6570
9. with V. M. Boyko and R. Popovych
   Singular reduction modules of differential equations
   J. Math. Phys. 57, 101503 (2016)
   http://arxiv.org/abs/1201.3223
10. with J.D.E. Grant, M. Graf, and R. Steinbauer
    The Hawking-Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics
    Comm. Math. Phys., 360 (2018), no. 3, 1009-1042.
    https://arxiv.org/abs/1706.08426

Curriculum Vitae

Name: Hans Vernaeve
ORCID: 0000-0002-0976-2250

Date and Place of Birth: 12/10/1976, Aalst, Belgium

Address: Ghent University, Krijgslaan 281, B-9000 Gent, Belgium

Website: `http://cage.ugent.be/~hvernaev/research.html`

Main research areas: Functional analysis, nonstandard analysis, algebras of generalized functions

Education and employment

- October 1994–July 1998: studies in mathematics at Ghent University.
- October 1998–May 2002: Ph.D. in pure mathematics at Ghent University. Ph.D. Thesis: Nonstandard contributions to the theory of generalized functions.
- October 2002–August 2006: scientific employee in the department *Research and Development* of the software company ICORDA (Ghent, Belgium).
- October 2006–September 2008: postdoctoral fellowship (Lise Meitner grant) at the University of Innsbruck, Austria.
- October 2008–now: lecturer in the department of Mathematics at Ghent University.
- September 2009: member of the international research group *Differential Algebras and Nonlinear Analysis* (DIANA).

Scientific activities

Invited speaker at international conferences (selection)

- Generalized Functions 2009 (Vienna, Austria, September 2009).

- Summer School *Generalized functions in PDE, geometry, stochastics and microlocal analysis* (Novi Sad, Serbia, September 2010).
- Generalized Functions 2011 (Martinique, April 2011).
- Topics in PDE, Microlocal and Time-frequency Analysis (Novi Sad, Serbia, September 2012).
- WING Workshop (Innsbruck, Austria, July 2016).
- Riemannian Geometry and Generalised Functions (Paris, France, October 2018).

Membership in academic organisations

- Member of the European Mathematical Society (E.M.S.)
- Member of the Belgian Mathematical Society (B.M.S.)
- Member of the International Society for Analysis, its Applications and Computation (I.S.A.A.C.)

Research grants

- F.W.F., Lise Meitner project M949 *Topological Algebras in the Theory of Generalized Functions*, 2006–2008, EUR 116 000.
- Research Foundation Flanders (F.W.O.) individual project 1.5.138.13N *Microlocal analysis in generalized function algebras* 2013–2015, EUR 7 000
- B.O.F. project, Ghent University, *Spaces of analytic functions in linear and nonlinear generalized function theory* 2014–2018, 1 full time Ph.D. researcher + EUR 14 900
- B.O.F. project, Ghent University, *Nonstandard analysis in special models* 2018–2019, 1 full time Ph.D. researcher
- Research Foundation Flanders (F.W.O.) individual project 1.5.129.18N *Nonstandard analysis and generalized function theory* 2018–2020, EUR 6 000

Key international cooperation partners

Dr. P. Giordano, University of Vienna
Prof. M. Kunzinger, University of Vienna
Prof. S. Pilipović, University of Novi Sad
Prof. D. Scarpalezos, University Paris 7
Prof. T. Todorov, California Polytechnic State University

Selected results

- Nonstandard analysis: representations of distributions, point values of distributions
- Algebras of generalized functions: development and application of nonstandard techniques, pointwise properties, regularity and microlocal analysis, embeddings of distributions and ultradistributions, topological properties, algebraic properties, group invariance, homogeneity.

10 most important publications

For the complete list of publications, see: `https://cage.ugent.be/~hvernaev/research.html`

1. H. Vernaeve, *Optimal Embeddings of Distributions into Algebras*, Proc. Edinburgh Math. Soc. (2003) 46: 373–378. DOI:10.1017/S0013091500001188

2. H. Vernaeve, *Pointwise characterizations in generalized function algebras*, Monatsh. Math. (2009) 158: 195–213. DOI:10.1007/s00605-008-0032-8

3. H. Vernaeve, *Ideals in the ring of Colombeau generalized numbers*, Comm. Alg. (2010) 38: 2199–2228. DOI:10.1080/00927870903055222

4. H. Vernaeve, *Isomorphisms of algebras of Colombeau generalized functions*, Monatsh. Math. (2011) 162: 225–237. DOI:10.1007/s00605-009-0152-9

5. C. Garetto, H. Vernaeve, *Hilbert $\widetilde{\mathbb{C}}$-modules: structural properties and applications to variational problems.* Trans. Amer. Math. Soc. (2011) 363: 2047–2090. DOI: 10.1090/S0002-9947-2010-05143-8

6. H. Vernaeve, *Nonstandard principles for generalized functions*, J. Math. Anal. Appl. (2011) 384: 536–548. DOI:10.1016/j.jmaa.2011.06.002

7. H. Vernaeve, J. Vindas, *Characterization of distributions having a value at a point in the sense of Robinson*, J. Math. Anal. Appl. (2012) 396(1): 371–374. DOI:10.1016/j.jmaa.2012.06.023

8. P. Giordano, M. Kunzinger, H. Vernaeve, *Strongly internal sets and generalized smooth functions*, J. Math. Anal. Appl. (2015) 422: 56–71. DOI:10.1016/j.jmaa.2014.08.036

9. H. Vernaeve, *Microlocal analysis in generalized function algebras based upon generalized points and generalized directions*, Monatsh. Math. (2016) 181: 205–215. DOI:10.1007/s00605-015-0831-7

10. A. Debrouwere, H. Vernaeve, J. Vindas, *A non-linear theory of infrahyperfunctions.* To appear in Kyoto J. Math. See ArXiv: 1701.06996