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FERMAT REALS: INFINITESIMALS WITHOUT LOGIC

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Abstract. We review the theory of Fermat reals and Fermat extensions, a relatively new theory of nilpotent infinitesimals which does not need any background in Mathematical Logic. We focus on some differences from Nonstandard Analysis and Synthetic Differential Geometry using the viewpoint of intuitive interpretation and applicability in Physics. Finally, we state some open problems.

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1. Imagine a student...

Let me start with a short, fictitious, but hopefully meaningful story. Imagine a student taking first year courses in Physics and Calculus. In Calculus she would see something like

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x : \left|\frac{x}{c}\right| < \delta \quad \Longrightarrow \quad \left| \frac{\frac{1}{\sqrt{1-\frac{x^2}{c^2}}} - 1 - \frac{x^2}{2c^2}}{x^2} c^2 \right| < \varepsilon.$$

However, in Physics she would frequently work with formulas similar to Einstein's formulas

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{v^2}{2c^2}, \qquad \sqrt{1-h_{44}(x)} = 1 - \frac{1}{2}h_{44}(x) \tag{1.1}$$

$$f(x, t+\tau) = f(x,t) + \tau \cdot \frac{\partial f}{\partial t}(x,t), \tag{1.2}$$

that explicitly use infinitesimals like $v/c \ll 1$, $\tau \ll 1$ or $h_{44}(x) \ll 1$, such that, e. g., $h_{44}(x)^2 = 0$, see, e. g., [6] and [5]. Of course, our student asks herself whether the two lecturers use the same ring of scalars or not. She senses that there are inconsistencies, and she can hence try to search for a solution to this clash of methodologies.

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However, some students frequently adopt solutions that refer to the fact that different courses require different mentalities, and hence different answers according to different professors. So, let us assume that our student starts to search on the Internet (and also in the library...) for a method to address this *inconsistency between intuition and formal Mathematics*. After a few steps, she finds that probably the difficulties can be resolved studying Nonstandard Analysis (NSA, [1, 17, 23]) and Synthetic Differential Geometry (SDG, [2, 20–22]). Our student begins to study these wonderful and formally powerful theories, but since she has a great critical mind, she quickly finds that these theories are not as intuitively clear as expected:

NSA: Let $I \in {}^*\mathbb{R}$ be an infinite number, where ${}^*\mathbb{R} \supseteq \mathbb{R}$ is the totally ordered field of NSA. Therefore, we have $|I| > n$ for all $n \in \mathbb{N}$. In NSA it is very easy to prove that every finite number $r \in {}^*\mathbb{R}$, i. e. such that r is not infinite, is infinitely close to a standard real number ${}^\circ r \in \mathbb{R}$, called its *standard part*. The property of being infinitely close is denoted by $r \simeq {}^\circ r$ and indicates that $r - {}^\circ r$ is infinitesimal, i. e. $|r - {}^\circ r| < \frac{1}{n}$ for all $n \in \mathbb{N}_{\neq 0}$. It is also very easy to prove that $-1 \leq \sin(I) \leq 1$ is a finite hyperreal, so that we get $\sin(I) \simeq {}^\circ \sin(I)$. What is the intuitive meaning of this real number ${}^\circ \sin(I)$? It could be questioned whether this is an objection to NSA, or in fact to standard real analysis based as it is on infinitary hypotheses common in Zermelo-Fraenkel set theory. This is because the same question can be asked of a real number I which is so large as to be inexpressible by even a computer the size of the universe in the span of the total time allotted to our civilization. If one cannot, even in principle, express such a real number I, what could possibly be the intuitive meaning of the value of the sine function at I? Does the usual geometrical interpretation with the unit circle of $\sin(I) = \sin(J)$, $J = I \bmod 2\pi$, work in this case?

NSA: Similarly to the extension $\mathbb{R} \mapsto {}^*\mathbb{R}$, every subset $X \subseteq \mathbb{R}$ can be extended to ${}^*X \subseteq {}^*\mathbb{R}$. This extension operation has wonderful formal properties, since it preserves all the logical operations. Let $e \in {}^*\mathbb{R}_{\neq 0}$ be a non zero infinitesimal, and $[x] \in {}^*\mathbb{N}$ be the integer part of $x \in {}^*\mathbb{R}$. Then, it is not hard to prove (see, e. g., [4, 24] and references therein) that $\mathcal{U} := \left\{X \subseteq \mathbb{N} \mid \left[\frac{1}{e}\right] \in {}^*X\right\}$ is an ultrafilter on $\mathbb{N}$, and using $\mathcal{U}$ it is possible to construct a non-measurable set without using the axiom of choice. Connes [3] expressed the opinion that the possibility of easily producing a non-measurable set by every non zero infinitesimal shows that every example of infinitesimal in NSA is not "completely knowable". On the other hand, this possibility is used by Tao [25] to prove the existence of non-measurable sets. The meaning of Connes' criticism to NSA is hence largely debated; for a detailed analysis see the recent paper [19].

NSA: The hyperreal field ${}^*\mathbb{R}$ can be defined as a quotient field $\mathbb{R}^{\mathbb{N}}/\sim$, where two sequences $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ are intuitively said to be equivalent if $x_n =$

y_n on a large set of indexes $n \in \mathbb{N}$. To formalize this notion of "large subset of natural numbers", one can use the notion of ultrafilter $\mathcal{U}$ by saying that A *is large* if $A \in \mathcal{U}$, where $A \subseteq \mathbb{N}$. Let

$$P(A) := \lim_{n \to +\infty} \frac{\operatorname{card}(A_{\leq n})}{n+1}$$

be the density of A (in the case that this limit exists), see, e. g., [26]. The density $P(A)$ is usually interpreted as the finitely additive uniform probability of picking a number from A. It is not hard to prove that

$$\forall \varepsilon > 0 \, \exists S \in \mathcal{U} : \; P(S) < \varepsilon.$$

Using this set of indexes $S \subseteq \mathbb{N}$ and any hyperreal $x = [(x_n)_{n \in \mathbb{N}}]_\sim \in {}^*\mathbb{R}$, we can easily construct another sequence $(y_n)_{n \in \mathbb{N}}$ generating the same number, i. e. such that $x = [(y_n)_{n \in \mathbb{N}}]_\sim$, but such that $P\,[x_n = y_n] < 10^{-100}$. See [7, 10].

NSA: Even if we consider a hyperreal $x = [(x_n)_{n \in \mathbb{N}}]_\sim \in {}^*\mathbb{R}$ infinitely close to a standard real number $x \simeq r \in \mathbb{R}$, in general, it is not possible to prove that $\lim_{n \to +\infty} x_n = r$. Only if ${}^*\mathbb{R}$ is defined using a particular type of ultrafilter, whose existence follows, for example, by assuming the continuum hypothesis, we can prove that $x_n \to r$ along a subsequence $n \in S \in \mathcal{U}$. In any event, even with the better case we can define an infinitesimal hyperreal $x = [(x_n)_{n \in \mathbb{N}}]_\sim$ that goes to zero along $S \in \mathcal{U}$, but goes to $+\infty$ for $n \in \mathbb{N} \setminus S$ and such that $P(S) < 10^{-100}$ whereas $P(\mathbb{N} \setminus S) > 1 - 10^{-100}$. See [15].

SDG: In SDG we have nilpotent infinitesimals, i. e. numbers of a ring $h \in R$ such that $h^n = 0$ for some $n \in \mathbb{N}$. Therefore, we do not have a field, but this lacking is necessary for formulas such as (1.2). The order relation in the ring of scalars R is only a partial order and not a total order, so that one cannot deduce $x = y$ from $x \leq y$ and $y \leq x$. Moreover, in SDG for every infinitesimal h both $h \geq 0$ *and* $h \leq 0$ hold. Our student does not understand how to reconcile this order property with the tens of intuitive drawings of infinitesimal objects she must reproduce in the study of Physics, in particular since the length of every infinitesimal segment is positive and not negative.

SDG: SDG is so beautiful and powerful that it permits the development of several differential geometry topics in a cartesian closed framework, i. e., not only for ordinary smooth manifolds but also for infinite dimensional spaces such as the space of all smooth functions between two manifolds. On the other hand, it seems impossible to consider a physical theory dealing with a physical infinitesimal constant taken from the ring R of SDG (e. g., Planck's constant to study the relationships between Quantum and Classical Mechanics). Indeed, in this theory it is only possible to prove that

$$\neg\neg \exists h \in R : \; h \simeq 0,$$

and this does not imply that infinitesimal numbers exist in the ring R, because SDG is incompatible with classical logic and admits models only in intuitionistic logic, in which the logic law $\neg\neg\mathcal{A} \Rightarrow \mathcal{A}$ does not hold. These models must be constructed using a non trivial amount of Topos theory which has been judged by the best researchers in SDG as "more complicated" as compared to other cartesian closed theories of manifolds. See [22].

Our imaginary student is even more confused. These are surely the most beautiful theories of infinitesimals currently available in mathematics. Surely they are the most formally powerful ones. However, shouldn't it be possible to develop another construction, perhaps less powerful, but simpler and *always intuitively clear*? Wouldn't it be possible to define a ring of numbers with infinitesimals within "standard mathematics," without it being necessary to have a deep background in Mathematical Logic?

Since Einstein wrote (1.2) exactly using an equality sign, if we want to be faithful to his writings and intuition, we are forced to consider a ring with nilpotent infinitesimals (let $f(x,t) = t^2$ at $t = 0$ in (1.2) to deduce that, necessarily, $\tau^2 = 0$). Can we have a theory similar to SDG, but always intuitively clear, that is compatible with classical logic and deals with easier models? Can we start from an ordinary smooth manifold M and extend it with something like ${}^\bullet M \supseteq M$, similar to what we can do in NSA, but with the addition of "nilpotent infinitely close new points"? Our final aim would be to develop Differential Geometry also based on infinite dimensional spaces like in SDG, so we can obtain a theory with new results and not only useful and more elegant reformulations of well-known classical theorems. If our student's passion is still alive, after all these questions, the best we can hope for is that she try her own solution. Indeed, this is a natural step, since two different languages ($\varepsilon - \delta$ and informal infinitesimals) which describe a sufficiently large part of nature must be strongly related to each other. Using only elementary analysis, after a couple of decades and with the usual small amount of inspiration and a large amount of perspiration, she was successful in creating a new useful theory.

Our student called the new ring of scalars ${}^\bullet\mathbb{R}$ *the ring of Fermat reals*, because "Fermat would surely have liked it". For more formal motivations concerning this name, see [11, 12].

2. The ring of Fermat reals

We start from the idea that a smooth ($\mathcal{C}^\infty$) function $f : {}^\bullet\mathbb{R} \to {}^\bullet\mathbb{R}$, where ${}^\bullet\mathbb{R} \supseteq \mathbb{R}$ is our new ring we have still to define, is actually *equal* to its tangent straight line in the first order neighborhood, e. g., of the point $x = 0$. Formally, we wish to write

$$\forall h \in D : \; f(h) = f(0) + h \cdot f'(0) \tag{2.1}$$

where D is the subset of ${}^\bullet\mathbb{R}$ which defines the above-mentioned neighborhood of $x = 0$. The equality (2.1) can be seen as a first-order Taylor's formula without remainder because, intuitively, we think that $h^2 = 0$ for any $h \in D$ (indeed the

property $h^2 = 0$ defines the first order neighborhood of $x = 0$ in ${}^\bullet\mathbb{R}$). These almost trivial considerations lead us to understand many things: ${}^\bullet\mathbb{R}$ must necessarily be a ring and not a field because in a field the equation $h^2 = 0$ implies $h = 0$; moreover we will surely have some limitation in the extension of some function from $\mathbb{R}$ to ${}^\bullet\mathbb{R}$. For example, we will necessarily have limitations in extending the square root because, using this function with the usual properties, the equation $h^2 = 0$ implies $|h| = 0$. On the other hand, we are also led to ask whether (2.1) uniquely determines the derivative $f'(0)$. Indeed, even if it is true that we cannot simplify by h, we know that the polynomial coefficients of a Taylor's formula are unique in classical analysis. In fact, we will prove that

$$\exists!\, m \in \mathbb{R}\ \forall h \in D:\ f(h) = f(0) + h \cdot m, \tag{2.2}$$

that is, the slope of the tangent is uniquely determined if it is an ordinary real number. We will call formulas like (2.2) *derivation formula*s.

If we try to construct a model for the formula (2.2), a natural idea is to think our new numbers in ${}^\bullet\mathbb{R}$ as equivalence classes $[h]$ of usual functions $h : \mathbb{R} \to \mathbb{R}$. In this way, we may hope both to include the real field using classes generated by constant functions, and that the class generated by $h(t) = t$ could be a first order infinitesimal number.

Remark 1. Sometimes, but not always, we will use a notation like $h_t := h(t)$ for real functions of the real variable t. This makes it possible to decrease the number of parenthesis used in formulas and to leave the classical notation $f(x)$ for functions of the form $f : {}^\bullet\mathbb{R} \to {}^\bullet\mathbb{R}$.

To understand how to define this equivalence relation, we have to think of (2.1) in the following sense:

$$f(h_t) \sim f(0) + h_t \cdot f'(0), \tag{2.3}$$

where the idea is that we are going to define $\sim$. If we think h_t "sufficiently similar to t", we can define $\sim$ so that (2.3) is equivalent to

$$\lim_{t \to 0^+} \frac{f(h_t) - f(0) - h_t \cdot f'(0)}{t} = 0,$$

that is,

$$x \sim y \ :\Longleftrightarrow\ \lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0. \tag{2.4}$$

In this way, (2.3) is very near to the definition of differentiability for f at 0.
It is important to note that, because of de L'Hôpital's theorem, we have the isomorphism

$$\mathcal{C}^1(\mathbb{R}, \mathbb{R})/\sim\ \simeq\ \mathbb{R}[x]/(x),$$

the left hand side is (isomorphic to) the usual tangent bundle of $\mathbb{R}$ and thus we obtain nothing new. It is not easy to understand what set of functions we have to choose for

x, y in (2.4) so as to obtain a non trivial structure. The first idea is to take continuous functions at $t = 0$, instead of more regular ones like $\mathcal{C}^1$-functions. In this way, we see that, e. g., $h_k(t) = |t|^{1/k}$ becomes a k-th order nilpotent infinitesimal because $h^{k+1} \sim 0$. For almost all the results presented in this paper, continuous functions at $t = 0$ work well. However, only in proving the non-trivial property

$$\big(\forall x \in {}^\bullet\mathbb{R} : \ x \cdot f(x) = 0\big) \quad \Longrightarrow \quad \forall x \in {}^\bullet\mathbb{R} : \ f(x) = 0 \tag{2.5}$$

we can see that it does not suffice to take continuous functions at $t = 0$. Another problem, necessarily connected with the basic idea (2.1), is that the use of nilpotent infinitesimals frequently leads to considering terms like $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n}$. For this type of products, the first problem is to know whether $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} \neq 0$ and what is the order k of this new infinitesimals, that is, for what k we have $(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n})^k \neq 0$ but $(h_1^{i_1} \cdot \ldots \cdot h_n^{i_n})^{k+1} = 0$. We will have a good frame if we are able to solve these problems starting from the order of each infinitesimal h_j and from the values of the powers $i_j \in \mathbb{N}$. On the other hand, almost all the examples of nilpotent infinitesimals are sums of terms of the form $h(t) = t^\alpha$, with $0 < \alpha < 1$. These functions also have very good properties in dealing with products of powers. It is for these reasons that we shall focus our attention on the following family of functions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}$ in the definition (2.4) of $\sim$.

Definition 2. We say that x *is a little-oh polynomial*, and we write $x \in \mathbb{R}_o[t]$ iff

(1) $x : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$
(2) We can write

$$x(t) = r + \sum_{i=1}^{k} \alpha_i \cdot t^{a_i} + o(t) \quad \text{as} \quad t \to 0^+$$

for suitable

$$k \in \mathbb{N}$$
$$r, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$$
$$a_1, \ldots, a_k \in \mathbb{R}_{\geq 0}.$$

Hence, a little-oh polynomial $x \in \mathbb{R}_o[t]$ is a polynomial function with real coefficients, in the real variable $t \geq 0$, with generic positive powers of t, and up to a little-oh function as $t \to 0^+$. Simple examples of little-oh polynomials are the following: $x(t) = 1 + t + t^{1/2} + t^{1/3} + o(t)$ and $x(t) = r + o(t)$.

Definition 3. Let $x, y \in \mathbb{R}_o[t]$. Then we say that $x \sim y$ or that $x = y$ in ${}^\bullet\mathbb{R}$ iff $x_t = y_t + o(t)$ as $t \to 0^+$. Because it is easy to prove that $\sim$ is an equivalence relation, we can define the quotient ring ${}^\bullet\mathbb{R} := \mathbb{R}_o[t]/\sim$, where in $\mathbb{R}_o[t]$ we consider the pointwise ring operations. We will use the notation $x = [x_t] \in {}^\bullet\mathbb{R}$ for the equivalence class generated by the little-oh polynomial $t \in \mathbb{R}_{\geq 0} \mapsto x_t \in \mathbb{R}$. Moreover:

(1) We define the standard part map as ${}^\circ(-) : x \in {}^\bullet\mathbb{R} \mapsto {}^\circ x = x(0) \in \mathbb{R}$
(2) $\mathrm{d}t_a := [t^{\frac{1}{a}}] \in {}^\bullet\mathbb{R}$ for all $a \in \mathbb{R}_{\geq 0}$.
(3) Let $x = [x_t]$, $y = [y_t] \in {}^\bullet\mathbb{R}$. Then we say that $x < y$ iff $x \neq y$ and there exists $z \in \mathbb{R}_o[t]$ such that

$$\exists \delta \in \mathbb{R}_{>0}\, \forall t \in (0, \delta] : \; x_t \leq y_t + z_t$$

$$z_t = o(t) \quad \text{as } t \to 0^+.$$

(4) Let $A \subseteq \mathbb{R}^n$ be an open subset, $f \in \mathcal{C}^\infty(A, \mathbb{R})$ a smooth function. Define ${}^\bullet A$ like in (2.8) (or, equivalently, as ${}^\bullet A = A_o[t]/\sim$, where $A_o[t]$ is the set of little-oh polynomials taking values in A). For $x \in {}^\bullet A$ define

$${}^\bullet f(x) := [f(x_t)] \in {}^\bullet\mathbb{R}.$$

Using these definitions, we have all we need to characterize rings like ${}^\bullet\mathbb{R}$ using the following axioms.

Axiom 1 (decomposition)**.** *${}^\bullet\mathbb{R}$ is a commutative ring with unity.*

Every Fermat reals $x \in {}^\bullet\mathbb{R}$ can be written, in a unique way, as

$$x = {}^\circ x + \sum_{i=1}^{N} \alpha_i \cdot \mathrm{d}t_{a_i}, \tag{2.6}$$

where ${}^\circ x, \alpha_i, a_i \in \mathbb{R}$ are standard reals, $a_1 > a_2 > \cdots > a_N \geq 1$, $\alpha_i \neq 0$. The term ${}^\circ x \in \mathbb{R}$ is called *standard part* of x, and $\alpha_i =: {}^\circ x_i$ its i-th standard part. Vice versa, any writing of the type (2.6), which is called the *decomposition* of x, gives a Fermat real, so that e. g. ${}^\bullet\mathbb{R} \supseteq \mathbb{R}$ and ${}^\circ r = r$ for all $r \in \mathbb{R}$.

Axiom 2 (base infinitesimals)**.** *The terms* $\mathrm{d}t_a$ *verify the following properties*

$$\begin{aligned} \mathrm{d}t_a \cdot \mathrm{d}t_b &= \mathrm{d}t_{\frac{ab}{a+b}} \\ (\mathrm{d}t_a)^p &= \mathrm{d}t_{\frac{a}{p}} \quad \forall p \in \mathbb{R}_{\geq 1} \\ \mathrm{d}t_a &= 0 \quad \forall a \in \mathbb{R}_{<1}. \end{aligned} \tag{2.7}$$

Therefore, among Fermat reals we also have nilpotent infinitesimals, like $x = 3\,\mathrm{d}t_2$, since $x^3 = 27\,\mathrm{d}t_{\frac{2}{3}} = 0$. These are exactly the same type of infinitesimals used by Einstein in formulas like (1.1). We will simply use the symbol $\mathrm{d}t$ for $\mathrm{d}t_1$. Intuitively speaking, looking at (2.7), we can also intuitively say that the greater is a and the greater is the nilpotent infinitesimal $\mathrm{d}t_a$. Our axiom on the total order relation $<$ will confirm this intuition. This relation should not be confused with the following notion.

Axiom 3 (order of infinitesimals)**.** *The* order $\omega(x) =: a_1$ (*see* (2.6)) *can be interpreted as the leading term in the decomposition and hence it has the following*

expected properties

$$\omega(x+y) \leq \max\left[\omega(x), \omega(y)\right]$$
$$\frac{1}{\omega(x \cdot y)} = \frac{1}{\omega(x)} + \frac{1}{\omega(y)},$$

whenever x, y are infinitesimals such that $x + y \neq 0$, respectively $x \cdot y \neq 0$.

In the decomposition (2.6), the term $\alpha_i =: \omega_i(x)$ will be called the i-th order of x. Directly from (2.6) it is not hard to prove that if $k \in \mathbb{N}_{>1}$, then $x^k = 0$ if and only if $\omega(x) < k$. Nilpotent Fermat reals can be thought of as non zero numbers which are so small that a suitable power of them gives zero.

Axiom 4 (ideals of infinitesimals)**.** *For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the set*

$$D_a := \left\{x \in {}^\bullet\mathbb{R} \mid {}^\circ x = 0,\ \omega(x) < a + 1\right\}$$

is an ideal. Moreover, for $k \in \mathbb{N}_{\geq 1}$, we have $D_k = \{x \in {}^\bullet\mathbb{R} \mid x^{k+1} = 0\}$.

We will simply use D for D_1. The ideal D_k is, therefore, a perfect candidate for the k-th order infinitesimal neighborhood of zero, with no k-th order Taylor formula having a remainder, since $x^{k+1} = 0$. This is indeed the subject of the next

Axiom 5 (Taylor formulas)**.** *Set ${}^\bullet\mathbb{R}^d := {}^\bullet\mathbb{R} \times \overset{d}{\ldots} \times {}^\bullet\mathbb{R}$, then every ordinary smooth function $f \in \mathcal{C}^\infty(A, \mathbb{R})$ defined on an open set $A \subseteq \mathbb{R}^d$ can be extended to the set*

$$^\bullet A := \left\{x \in {}^\bullet\mathbb{R}^d \mid {}^\circ x \in A\right\}, \tag{2.8}$$
$$^\bullet f : {}^\bullet A \to {}^\bullet\mathbb{R},$$

obviously obtaining a true extension, i. e., ${}^\bullet f(x) = f(x)$ if $x \in A$. Moreover, the following Taylor formula

$$\forall h \in D_k^d : {}^\bullet f(x+h) = \sum_{\substack{j \in \mathbb{N}^d \\ |j| \leq k}} \frac{h^j}{j!} \cdot \frac{\partial^{|j|} f}{\partial x^j}(x) \tag{2.9}$$

holds, where $x \in A$ is a standard point, and $D_k^d = D_k \times \overset{d}{\ldots} \times D_k$.

Therefore, smooth functions become exactly equal to polynomials of degree k in the infinitesimal k-th order neighborhood $x + D_k$. In particular, $f(x+k) = f(x) + h \cdot f(x)$ for $h \in D$, i. e. every smooth function is equal to its tangent line in a first order infinitesimal neighborhood. Einstein's formulas (1.1) are particular cases of this infinitesimal Taylor formula.

Axiom 6 (cancellation laws)**.** *Let $h_1, \ldots, h_n \in D_\infty$, $i_1, \ldots, i_n \in \mathbb{N}$, $x \in {}^\bullet\mathbb{R}$, then we have*

(1) $h_1^{i_1} \cdot \ldots \cdot h_n^{i_n} = 0$ *if and only if* $\sum_{k=1}^n \frac{i_k}{\omega(h_k)} > 1$.

(2) x is invertible if and only if ${}^\circ x \neq 0$.
(3) If $x \cdot r = x \cdot s$ in ${}^\bullet\mathbb{R}$, where $r, s \in \mathbb{R}$ and $x \neq 0$, then $r = s$.

If you are scared of working in a ring instead of a field, these laws allow for efficient work with this type of infinitesimals. If x is invertible, and proceeding like in the case of formal power series, it is not hard to prove that

$$\frac{1}{x} = \frac{1}{{}^\circ x} \cdot \sum_{j=0}^{+\infty} (-1)^j \cdot \left(\sum_{i=1}^{N} \frac{\alpha_i}{{}^\circ x} \cdot \mathrm{d}t_{a_i} \right),$$

where the series is really a finite sum due to nilpotency.

Axiom 7 (total order relation)**.** *The ring of Fermat reals is totally ordered by the relation $<$. This relation verifies the following properties: let $x, y \in {}^\bullet\mathbb{R}$, if ${}^\circ x \neq {}^\circ y$, then*

$$x < y \iff {}^\circ x < {}^\circ y.$$

Vice versa, if ${}^\circ x = {}^\circ y$, then

(1) If $\omega(x) > \omega(y)$, then $x > y$ if and only if ${}^\circ x_1 > 0$.
(2) If $\omega(x) = \omega(y)$, then

$$\begin{aligned} {}^\circ x_1 > {}^\circ y_1 \quad &\Longrightarrow \quad x > y \\ {}^\circ x_1 < {}^\circ y_1 \quad &\Longrightarrow \quad x < y. \end{aligned}$$

The axiom gives an effective criterion to decide whether $x < y$ or not. Indeed:

(1) first of all $x < y$ is equivalent to $0 < y - x$, so we can describe the algorithm for the case $0 < x$, $x \in {}^\bullet\mathbb{R} \setminus \mathbb{R}$ only (from the first part of the previous axiom it follows that $<$ extends the usual order relation on $\mathbb{R}$). If the standard part ${}^\circ x \neq 0$, then the order relation can be decided on the basis of this standard part only. e. g. $2 + \mathrm{d}t_2 > 0$ and $1 + \mathrm{d}t_2 < 3 + \mathrm{d}t$.
(2) Otherwise, if the standard part ${}^\circ x = 0$, we look at the order $\omega(x)$ and at the first standard part ${}^\circ x_1$, which is the coefficient of the biggest infinitesimals in the decompositions of x: because $\omega(x) > \omega(0) = 0$, we have $x > 0$ iff ${}^\circ x_1 > 0$. e. g. $3\,\mathrm{d}t_2 > 0$; $\mathrm{d}t_2 > a\,\mathrm{d}t$ for every $a \in \mathbb{R}$; $0 < \mathrm{d}t < \mathrm{d}t_2 < \mathrm{d}t_3 < \ldots < \mathrm{d}t_k$ for every k.

For a proof that these axioms characterize the structure $({}^\bullet\mathbb{R}, +, \cdot, <, {}^\circ(-), \mathrm{d}t_{(-)})$ up to isomorphisms of ordered rings, see [8]. For a proof that indeed in the ring ${}^\bullet\mathbb{R}$ these axioms hold, see [11, 12].

More advanced axioms are needed to deal with (quasi-standard) smooth functions that are more general than extension ${}^\bullet f$ of standard smooth functions f, e. g. like the very simple $g(x) = x + \mathrm{d}t$; see [10, 12] for more details. Clearly, we can define the absolute value, powers and logarithms of invertible Fermat reals and generalize their usual properties, [12]. We can also define meaningful metrics on ${}^\bullet\mathbb{R}$ and roots of (nilpotent!) infinitesimals, and prove applications to fractional derivatives, [16].

3. Geometrical representation

In our introductory fictitious story, our student was in search of a new theory which, on the one hand, can resolve the inconsistency between the intuition of the Physics' course and the formal Mathematics of the Calculus course. On the other hand, she was searching for a theory which is always intuitively clear. We can say that our student is looking for a theory which can keep a good dialectic between provable formal properties and their intuitive meanings. In this direction, we can see the possibility to find a geometrical representation of Fermat reals.

The idea is that, to any Fermat real $x \in {}^{\bullet}\mathbb{R}$, we can associate the function

$$t \in \mathbb{R}_{\geq 0} \mapsto {}^{\circ}x + \sum_{i=1}^{N} {}^{\circ}x_i \cdot t^{1/\omega_i(x)} \in \mathbb{R} \tag{3.1}$$

where N is, of course, the number of addends in the decomposition of x. Therefore, a geometric representation of this function is also a geometric representation of the number x because different Fermat reals have different decompositions, see Axiom 1 (decomposition). equality in ${}^{\bullet}\mathbb{R}$ depends only on the germ generated by each little-oh polynomial (see Definition 3), we can represent each $x \in {}^{\bullet}\mathbb{R}$ using only the first small part of the function (3.1).

Definition 4. If $x \in {}^{\bullet}\mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$, then

$$\mathrm{graph}_{\delta}(x) := \left\{ \left({}^{\circ}x + \sum_{i=1}^{N} {}^{\circ}x_i \cdot t^{1/\omega_i(x)}, t \right) \mid 0 \leq t < \delta \right\}$$

where N is the number of addends in the decomposition of x.

Note that the value of the function is placed in the abscissa position so that the correct representation of $\mathrm{graph}_{\delta}(x)$ is given by Figure 3.1.

This interchange of abscissa and ordinate in the $\mathrm{graph}_{\delta}(x)$ makes it possible to represent this graph as a line tangent to the classical straight line $\mathbb{R}$ and hence to have a better graphical picture. Finally, note that if $x \in \mathbb{R}$ is a standard real, then $N = 0$ and the $\mathrm{graph}_{\delta}(x)$ is a vertical line passing through ${}^{\circ}x = x$.

The following theorem makes it possible to represent the Fermat reals geometrically.

Theorem 5. *If $\delta \in \mathbb{R}_{>0}$, then the function*

$$x \in {}^{\bullet}\mathbb{R} \longmapsto \mathrm{graph}_{\delta}(x) \subset \mathbb{R}^2$$

is injective. Moreover if $x, y \in {}^{\bullet}\mathbb{R}$, then we can find $\delta \in \mathbb{R}_{>0}$ (depending on x and y) such that $x < y$ if and only if

$$\forall p, q, t : \quad (p, t) \in \mathrm{graph}_{\delta}(x)\ ,\ (q, t) \in \mathrm{graph}_{\delta}(y) \quad \Longrightarrow \quad p < q. \tag{3.2}$$

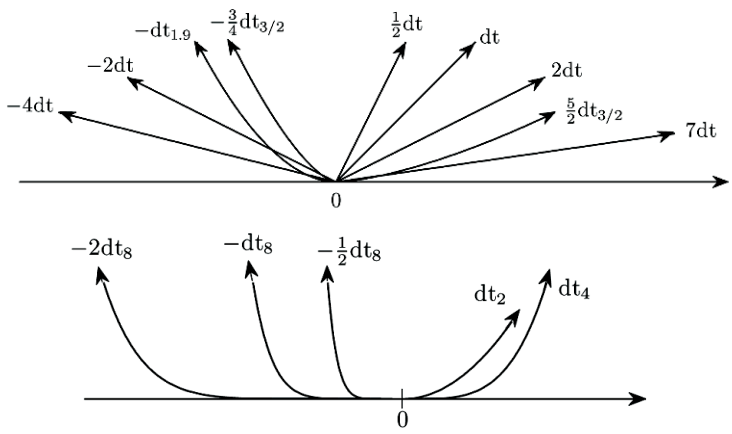

FIGURE 3.1. Some first and higher order infinitesimals

For a proof of this theorem, see [11]. See Figure 3.2 for the meaning of condition (3.2).

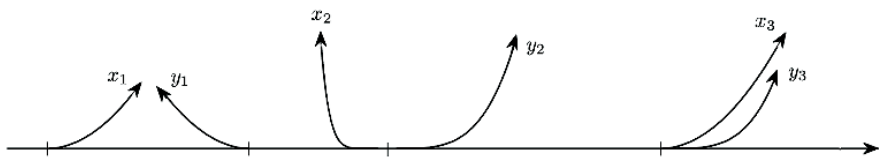

FIGURE 3.2. Different cases in which $x_i < y_i$

4. COMPUTER IMPLEMENTATION

The definition of the ring of Fermat reals is highly constructive. Therefore, using object oriented programming, it is not hard to write a computer code corresponding to ${}^{\bullet}\mathbb{R}$. We implemented a first version of this software using Matlab R2010b.

The constructor of a Fermat real is `x=FermatReal(s,w,r)`, where `s` is the $n+1$ double vector of standard parts (`s(1)` is the standard part ${}^{\circ}x$) and `w` is the double vector of orders (`w(1)` is the order $\omega(x)$ if $x \in {}^{\bullet}\mathbb{R} \setminus \mathbb{R}$, otherwise `w=[]` is the empty vector). The last input `r` is a logical variable and assumes the value `true` if we want that the display of the number `x` is implemented using the Matlab `rats` function for both its standard parts and orders. In this way, the number will be displayed using continued fraction approximations and, therefore, in many cases, the calculations will be exact. These inputs are the basic methods of every Fermat real,

and can be accessed using the `subsref`, and `subsasgn`, notations `x.stdParts`, `x.orders`, `x.rats`. The function `w=orders(x)` gives exactly the double vector `x.orders` if $x \in {}^{\bullet}\mathbb{R} \setminus \mathbb{R}$ and `0` otherwise.

The function `dt(a)`, where `a` is a double, constructs the Fermat real $\mathrm{d}t_a$. Because we have overloaded all the algebraic operations, like `x+y`, `x*y`, `x-y`, `-x`, `x==y`, `x~=y`, `x<y`, `x<=y`, `x^y`, we can define a Fermat real, e. g., using an expression of the form `x=2+3*dt(2)-1/3*dt(1)`, which corresponds to

```
x=FermatReal([2 3 -1/3],[2 1],true).
```

We have also implemented the function `y=decomposition(x)`, which gives the decomposition of the Fermat real `x`, `abs(x)`, `log(x)`, `exp(x)`, `isreal(x)`, `isinfinitesimal(x)`, `isinvertible(x)`.

The ratio `x/y` has been implemented for `x` and `y` infinitesimals and `y~=0`, or in case `y` is invertible. Finally, the function `y=ext(f,x)`, corresponds to ${}^{\bullet}f(x)$ and has been implemented using the evaluation of the symbolic Taylor formula of the inline function `f`.

The functions `x^p`, `sqrt(x)` and `nthroot(x,n)` have been implemented both for `x` infinitesimal or invertible using the formulas derived in [16].

Using these tools, we can easily find, e. g., that

$$\frac{\sin(\sqrt{\mathrm{d}t_3 + 2\,\mathrm{d}t_2})}{\cos(\sqrt[3]{-4\,\mathrm{d}t})} = \mathrm{d}t_6 + \mathrm{d}t_3 - \frac{2}{3}\,\mathrm{d}t_2 + \frac{1096}{2787}\,\mathrm{d}t_{\frac{6}{5}} + \frac{1234}{913}\,\mathrm{d}t.$$

The Matlab source code is freely available under open-source license, and can be requested from the author of the present paper.

5. Applications and developments

The simplicity of the ring of Fermat reals permits an easy extension of this approach along different directions.

We can see many informal uses of infinitesimals in Physics with this new point of view. Frequently, these informal calculations can be faithfully repeated using ${}^{\bullet}\mathbb{R}$, without any changes. This can be thought of as a partial proof that this theory provides a meaningful way to help us solve our modeling problems without forcing us to completely change our methodological approach. Formalizing these physical models is a good way to learn the properties of Fermat reals. On the other hand, working in a rigorous mathematical theory of infinitesimals allows us to gain a great formal power and to discover the physical meaning of several informal approximations. For example, in [11] it is proven that the wave equation is *equivalent* to a suitable condition of infinitesimal oscillations of a string. Other elementary examples have also been developed in [11]: the deduction of the heat equation, a study of the electric dipole, the Newtonian limit in relativity, the curvature of a smooth curve, the area of the circle and volumes of revolution, the stretching of a spring and others.

The theory of Fermat reals is strongly inspired by SDG and, indeed, it can be deeply developed to consider infinitesimal methods that are valid in both finite and infinite dimensional spaces. Every smooth manifold M can be extended using this type of infinitesimals analogously to what we have previously shown for the extension ${}^\bullet\mathbb{R} \supseteq \mathbb{R}$, and obtaining a smooth embedding ${}^\bullet M \supseteq M$. More generally, this extension is applicable to every diffeological space [18] obtaining a functor with very good preservation properties. The category of diffeological spaces is cartesian closed and embeds the category of smooth manifolds, so that these Fermat extensions can also be applied to infinite dimensional function spaces. In this framework, we can define tangent vectors of the space M as smooth maps of the form $t : D \to {}^\bullet M$, i. e. as infinitesimal linear curves traced on ${}^\bullet M$. We can also intrinsically define the sum of two tangent vectors as the diagonal of the infinitesimal parallelogram generated by these tangent vectors. We can prove that a vector field can be defined as an infinitesimal smooth transformation of the space ${}^\bullet M$ into itself, i. e. as $V : D \to {}^\bullet M^{{}^\bullet M}$ such that $V(0) = 1_{{}^\bullet M}$. We can define infinitesimal integral curves and prove their existence even in infinite dimensional spaces of functions such as $\mathcal{C}^\infty({}^\bullet M, {}^\bullet N)$. These are only a few examples from a very long ongoing project. For more details, see [10, 13, 14].

6. Open problems and ideas

The following describes some open problems and ideas for the interested reader to undertake. Please feel free to contact the author to obtain more feedback, considering that some of these ideas are being investigated within the ongoing projects P25116-N25 and P25311-N25 of the Austrian Science Fund FWF.

6.1. *Infinities and nilpotent infinitesimals*

Almost every theory of actual infinitesimals is usually well coupled with a corresponding theory of infinite numbers. In the case of a field, this coupling is naturally provided by the reciprocal function. These infinitely large numbers can be fruitfully applied to the formulation of integrals as infinite sums, limits at infinite points, hyperfinite subdivisions of the real line, studies of singularities, and generalized functions, to cite only a few. Of course, in the case of nilpotent infinitesimals, the trivial possibility that infinities are reciprocals of infinitesimals does not hold because, in that case, $h^2 = 0$ would imply $h = 0$. Is there some meaningful way to bypass the impossibility to have infinities as reciprocals of nilpotent infinitesimals? What property of multiplication of numbers should be weakened to allow a coexistence between these infinities and nilpotent infinitesimals? Is the corresponding formalism sufficiently easy to use and intuitively meaningful? Can these infinities be used to define a suitable class of generalized functions? Hint: the reciprocal of $h = [t] \in {}^\bullet\mathbb{R}$ is necessarily generated by $t \in \mathbb{R}_{>0} \mapsto t^{-1} \in \mathbb{R}$.

6.2. *Weil functors and infinitesimals whose product is not zero*

In the ring ${}^\bullet\mathbb{R}$, the product of any two first order infinitesimals $h, k \in D$ is zero: $h \cdot k = 0$. As proved in [11, Theorem 24] this is a general consequence of the total order property, but having infinitesimals h, k such that $h^2 = 0 = k^2$ and $h \cdot k \neq 0$ would be useful for studying the double tangent bundle (see [20]). An idea to explore, inspired by rings like $\mathbb{R}[t, s]/\langle t^2 = 0, s^2 = 0\rangle$, can be roughly stated as "two first order infinitesimals $(h_t)_t$ and $(k_s)_s$ have a non zero product $(h_t \cdot k_s)_{t,s}$ if they depend on two independent variables t and s". A possible formalization of this idea can be sketched in the following way. Instead of little-oh polynomials, let us consider maps of the form $x : \mathbb{R}^v_{\geq 0} \to \mathbb{R}$ (v depending on x) such that

$$x(t_1, \ldots, t_v) = r + \sum_{j=1}^{k} \alpha_j \cdot t_1^{a_{1j}} \cdot \ldots \cdot t_v^{a_{vj}} + o(t_1) + \ldots + o(t_v), \tag{6.1}$$

Now the analogue of the equality in ${}^\bullet\mathbb{R}$ is that $x \sim y$ if and only if x and y are both defined on the same domain $\mathbb{R}^v_{\geq 0}$ and $x(t_1, \ldots, t_v) = y(t_1, \ldots, t_v) + o(t_1) + \ldots + o(t_v)$ as $t_k \to 0^+$ for all k. This idea seems positive for two reasons: first, if we define a new ring in this way, considering only the subring of all the maps $\mathbb{R}_o[t_i]$ which only depend on one variable t_i, we obtain a ring ${}^\bullet\mathbb{R}[t_i]$ isomorphic to the present ${}^\bullet\mathbb{R}$. Second, if we consider $h(t_1, t_2) := t_1$ and $k(t_1, t_2) := t_2$, then we have $h^2 \sim 0$ and $k^2 \sim 0$, but not $h \cdot k \sim 0$. Of course, from [11, Theorem 24] it follows that every subring ${}^\bullet\mathbb{R}[t_i]$ is totally ordered, but the entire ring cannot be totally ordered. Assuming that this new class of little-oh polynomials works as a sufficiently good theory, is there a good representation of every Weil functor using this new ring? For a representation of a subclass of Weil functors using ${}^\bullet\mathbb{R}$, see [9].

6.3. *Perturbation theory*

Several classical methods used in perturbation theory seem to simplify if we take the small perturbation parameter ε as a non zero nilpotent infinitesimal in the ring of Fermat reals, i.e. if $\varepsilon \in {}^\bullet\mathbb{R}_{\neq 0}$, $\varepsilon^n = 0$. For example, we can more easily use the nilpotency property of ε instead of using big-oh asymptotic estimates; we can also take advantage of exact Taylor formulas in ${}^\bullet\mathbb{R}$, i.e. with nilpotent infinitesimal increments and no remainder. More generally, we can use finite sums in the ring ${}^\bullet\mathbb{R}$ instead of a (convergent or formal) power series, and we can also take advantage of the existence of arbitrary roots of nilpotent infinitesimals (see [16]). Do these properties permit an easier calculus of classical perturbation methods?

6.4. *Example in (elementary) Physics*

If you have an example, or even only an idea, of how to use the ring of Fermat in (elementary) Physics, please feel free to contact the author. We are planning to write a textbook with an introduction to this ring and its first properties and applications.

Even in the case of a new example in elementary Physics, your name will be clearly cited as the author of the example. For examples in this direction, see [11].

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