Picard-Lindelöf for PDE

Paolo Giordano, Lorenzo Luperi Baglini

The idea:

Let's consider, e.g., the transport equation

$$\partial_t y(t,x) = \alpha(t,x)\cdot \partial_x y(t,x) \qquad (1)$$

where y, α are GSF. More generally, we can think at Cauchy problem of the form:

$$\begin{cases} \partial_t^k y(t,x) = G\left[t,x,\left(\partial_t^j \partial_x^\alpha y\right)_{j<k}\right] \\ \partial_t^j y(t_0,x) = y_{0j}(x) \qquad 0 \le j < k \end{cases} \qquad (2)$$

In (1), we set

$F(t,x,y) := \alpha(t,x)\cdot \partial_x y(t,x)$ Rem: $F(-,-,y)$ is a GSF $\forall y$ GSF

So (1) is equivalent to the integral equation

$$y(t,x) = y_0(x) + \int_{t_0}^{t} F(s,x,y)\,ds = y_0(x) + \int_{t_0}^{t} \alpha(s,x)\cdot \partial_x y(s,x)\,ds$$

If we try to test the Lipschitz property of F, we get something like $\left((t,x) \in K \Subset_f \widetilde{\mathbb{R}}^2\right)$

$$\left\| F(t,x,u) - F(t,x,v) \right\|_0 \le \max_{(t,x)\in K} \left|\alpha(t,x)\right| \cdot \max_{(t,x)\in K} \left|\partial_x (u-v)(t,x)\right|$$

$$\le M_0 \cdot \left\| u - v \right\|_1$$

there is an increase of norms, which is usually called a "loss of derivatives"

Analogously, e.g.

 2

$$\left\| F(t,x,u) - F(t,x,v) \right\|_1 = \max \Big[\| \cdot\cdot \|_0 \ , \ \max_K \partial_t \alpha \cdot \partial_x (u-v) + \alpha \cdot \partial_t \partial_x (u-v) \ , \ \max_K \partial_x \alpha \cdot \partial_x (u-v) + \alpha \, \partial_x^2 (u-v) \Big]$$

$$\leq M_1 \cdot \| u - v \|_2$$

Rem : In thm 1.7 we'll prove for (2).1 (for $K=1$) that for any PDE of the form

$$G\left[t, x, \left(\partial_x^a y(t,x)\right)_{|a| \leq L}\right] = \partial_t y(t,x)$$

we've

$$\left\| F(t,x,u) - F(t,x,v) \right\|_i \leq \Lambda_i \cdot \| u - v \|_{i+L} \qquad (1)$$

where $L \in \mathbb{N}_{>0}$ depends only on F (i.e. on the PDE), and where

$$F : (t,x,y) \in T \times S \times Y \longmapsto G\left[t, x, \left(\partial_x^a y(t,x)\right)_{|a| \leq L}\right] \in \widetilde{\mathbb{R}}^d$$

Banach FPT with loss of derivatives:

If we look at the classical proof of the BFPT together with the corresponding Picard–Lindelöf thm, and thinking at a generalization based on a property like (1).2 for PDE, the following ideas may come to mind:

1) In the BFPT the last step would become something like

$$\|T^m(u) - T^n(u)\|_i \leq \dots \leq \frac{\alpha^n_{i+Ln} - \alpha^m_{i+Lm}}{1-\alpha_{i+Ln}} \cdot \|T(u)-u\|_{i+Ln}$$

This still goes to zero if we have that $\|T(u)-u\|_{i+Ln}$ doesn't go to infinity too quickly w.r.t. α^n_{i+Ln}.
All this, at least for some u!

2) In the proof of P–L we don't need the full generality of BFPT ("for all u, $(T^n(u))_n$ converges...") because we've a particular u to start with: the initial condition y_0

3) In the proof of P–L, we estimate exactly a term of the form $\|T(y)-y_0\|$ (and hence also of the form $\|T(y_0)-y_0\| \dots$)

From these ideas, we obtain

Def:

Let $\emptyset \neq K \Subset_f \widetilde{\mathbb{R}}^n$ be a solid set

See Def 25 preprint ODE

$$y_0 \in X \subseteq \mathcal{G}\mathcal{C}^\infty(K, \widetilde{\mathbb{R}}^d) \quad , \quad L \in \mathbb{N}$$

then we say that

P is a finite contraction on X
with loss of derivatives L
starting from y_0

if

1) $P : X \longrightarrow X$ is a set-theoretical map

2) $\forall i \in \mathbb{N}\ \exists \alpha_i \in \widetilde{\mathbb{R}}_{>0}\ \forall u, v \in X$:

$$\| P(u) - P(v) \|_i \leq \alpha_i \cdot \| u - v \|_{i+L}$$

3) $\forall i \in \mathbb{N}$: $\displaystyle\lim_{\substack{n,m \to +\infty \\ n \leq m}} \alpha^m_{i+m\cdot L} \cdot \| P(y_0) - y_0 \|_{i+m\cdot L} = 0$ (sharp top.)

Thm: (BFPT with loss of derivatives)

Let K, X, y_0, L, P as above, and $\forall i \in \mathbb{N}$

X sharply Cauchy complete

$\alpha_i \leq \alpha_{i+1}$. Then:

0) P is sharply continuous

1) $\exists \displaystyle\lim_{n \to +\infty} P^n(y_0) =: y \in X$

2) $P(y) = y$

Proof:

0) V sharply open set of $\mathcal{G}^{\mathcal{F}}(K, \tilde{\mathbb{R}}^d)$
$P(u) \in V \cap X$ $\Big\}$ (Hp)

$\exists i \in \mathbb{N}\ \exists r \in \tilde{\mathbb{R}}_{>0} : B^i_r(Pu) \subseteq V$

$$\|Pv - Pu\|_i \le \alpha_i \|v - u\|_{i+L}$$

$$\forall v \in \underbrace{B^{i+L}_{\frac{r}{\alpha_i}}(u) \cap X}_{=:W} : \|v-u\|_{i+L} < \frac{r}{\alpha_i} \implies \|Pv - Pu\|_i < r$$

$$Pv \in B^i_r(Pu) \subseteq V \cap X$$

W sharply open in $\mathcal{G}^{\mathcal{F}}(K, \tilde{\mathbb{R}}^d)$

$\forall v \in W \cap X : Pv \in V \cap X$ ($)

1) By induction $\forall n \in \mathbb{N}$

$$\|P^{n+1}(y_0) - P^n(y_0)\|_i \le \alpha_i \cdot \alpha_{i+L} \cdot \ldots \cdot \alpha_{i+(n-1)L} \cdot \|P(y_0) - y_0\|_{i+nL}$$

But $\alpha_i \le \alpha_{i+1}$, so

$$\|P^{n+1}(y_0) - P^n(y_0)\|_i \le \alpha^n_{i+nL} \cdot \|P(y_0) - y_0\|_{i+nL}$$

For every $n, m \in \mathbb{N}$, $n < m$:

$$\|P^m(y_0) - P^n(y_0)\|_i \le \|P^m(y_0) - P^{m-1}(y_0)\|_i + \ldots + \|P^{n+1}(y_0) - P^n(y_0)\|_i \le$$

$$\le \alpha^{m-1}_{i+(m-1)L} \cdot \|P(y_0) - y_0\|_{i+(m-1)L} + \ldots + \alpha^n_{i+nL} \cdot \|P(y_0) - y_0\|_{i+nL}$$

$\alpha_i \le \alpha_{i+1}$
$\|-\|_i \le \|-\|_{i+1}$

$$\le \alpha^{m-1}_{i+(m-1)L} \cdot \|P(y_0) - y_0\|_{i+(m-1)L} + \ldots + \alpha^n_{i+(m-1)L} \cdot \|P(y_0) - y_0\|_{i+(m-1)L}$$

$$\leq \alpha_{i+(m-1)L}^{n} \cdot \|P(y_0) - y_0\|_{i+(m-1)L} \cdot \sum_{j=0}^{m-1-n} \alpha_{i+(m-1)L}^{j} =$$

$$= \alpha_{i+(m-1)L}^{n} \cdot \|P(y_0) - y_0\|_{i+(m-1)L} \cdot \frac{1 - \alpha_{i+(m-1)L}^{m-n}}{1 - \alpha_{i+(m-1)L}} =$$

$$= \frac{\alpha_{i+(m-1)L}^{n} - \alpha_{i+(m-1)L}^{m}}{1 - \alpha_{i+(m-1)L}} \cdot \|P(y_0) - y_0\|_{i+(m-1)L}$$

and this $\to 0$ by assumptions.

The rest of the proof is identical as in the ODE preprint:

http://www.mat.univie.ac.at/~giordap7/preprint_ODE.pdf

□

Picard - Lindelöf with loss of derivatives:

Def:

Let $\emptyset \neq T \subset\subset_f \widetilde{\mathbb{R}}$ be a solid set (time $t \in T$)

$\emptyset \neq S \subset\subset_f \subseteq \widetilde{\mathbb{R}}^m$ be a solid set (space $x \in S$)

$Y \subseteq \mathcal{G}\mathcal{C}^\infty(T \times S, \widetilde{\mathbb{R}}^d)$, $L \in \mathbb{N}$

then we say that

F is uniformly Lipschitz on Y

with constants $(\Lambda_i)_{i \in \mathbb{N}}$

and loss of derivatives L

if:

1) $F : T\times S\times Y \longrightarrow \widetilde{\mathbb{R}}^d$ is a set-theoretical map

2) $\forall y\in Y:\ F(-,-,y) =: F(y)(-,-)\in \mathcal{G}\mathcal{E}^\infty(T\times S,\widetilde{\mathbb{R}}^d)$

3) $\forall i\in\mathbb{N}\ \forall u,v\in Y:$

norms of $\mathcal{G}^{\mathcal{F}}(T\times S,\widetilde{\mathbb{R}}^d)$. See Def 34 ODE preprint.

$$\left\| F(t,x,u) - F(t,x,v)\right\|_i \le \Lambda_i\cdot \left\|u-v\right\|_{i+L}$$

We first prove that this definition is not too restrictive by proving that every GSF G in an arbitrary PDE of the form

$$G\left[t,x,\left(\partial_x^a y(t,x)\right)_{|a|\le L}\right] = \partial_t y(t,x)$$

is uniformly Lipschitz:

Thm:

Let $\emptyset\ne S\Subset_f\widetilde{\mathbb{R}}^n$ and $\emptyset\ne T\Subset_f\widetilde{\mathbb{R}}$ be solid sets

$L\in\mathbb{N}$, $\hat{L}:=\text{card}\left\{a\in\mathbb{N}^n \mid |a|\le L\right\}$

$G\in {}^\circ\mathcal{G}\mathcal{E}^\infty(T\times\widetilde{\mathbb{R}}^n\times\widetilde{\mathbb{R}}^{d\cdot\hat{L}},\widetilde{\mathbb{R}}^d)$

$y_0\in\mathcal{G}\mathcal{E}^\infty(S,\widetilde{\mathbb{R}}^d)$

$\|y_0\|_i\le s_i\in\widetilde{\mathbb{R}}_{>0}\quad\forall i\in\mathbb{N}\qquad (1)$

then for all $H\Subset_f\widetilde{\mathbb{R}}^d$, the function

$$(t,x,y)\in T\times S\times Y\longmapsto G\left[t,x,\left(\partial_x^a y(t,x)\right)_{|a|\le L}\right]\in\widetilde{\mathbb{R}}^d$$

is uniformly Lipschitz with loss of derivatives L on

$$Y:=\left\{y\in\mathcal{G}\mathcal{E}^\infty(T\times S,H)\ \middle|\ \|y-y_0\|_i\le r_i\ \forall i\in\mathbb{N}\right\}$$

Proof:

Note that

$$y_0 \in \mathcal{G}\mathcal{E}^\infty(S, \widetilde{\mathbb{R}}^d) \implies \partial^a y_0(S) \subset\subset_f \widetilde{\mathbb{R}}^d$$

so the assumption (1).7 is not restrictive.

$i = 0$:

We need to evaluate

$$\left\| G\left[t,x,\left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] - G\left[t,x,\left(\partial_x^a v(t,x)\right)_{|a|\le L}\right]\right\|_0 =: (1)$$

where $t \in T$, $x \in S$ and $u, v \in Y$.

But

$$\left|\partial_x^a u(t,x)\right| \le \|u\|_{|a|} \le \|u - y_0\|_{|a|} + \|y_0\|_{|a|} \le r_{|a|} + s_{|a|}$$

So, setting

$$D := \prod_{i=0}^{\hat{L}} \overline{B_{r_i+s_i}(0)} \subset\subset_f \widetilde{\mathbb{R}}^{d\cdot\hat{L}}$$

We can write

$$(1) \le \left\| G|_{T\times S\times D} \right\|_1 \cdot \|u - v\|_L$$

$i \ge 1$:

Now we need to evaluate $(|h| \le i)$:

$$\left|\partial^h G\left[t,x,\left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] - \partial^h G\left[t,x,\left(\partial_x^a v(t,x)\right)_{|a|\le L}\right]\right| =: (2)$$

using the Faà di Bruno formula (see Lem. 43 ODE preprint) with

$$f = G : T \times S \times \widetilde{\mathbb{R}}^{d\cdot\hat{L}} \longrightarrow \widetilde{\mathbb{R}}^d \qquad m = 1 + d + d\cdot\hat{L}$$

$$g(t,x) := \left(t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right) \qquad \forall (t,x) \in T\times S$$

we obtain

$$\partial^h G\Big[t,x,\big(\partial_x^a u(t,x)\big)_{|a|\le L}\Big] = \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\Big[t,x,\big(\partial_x^a u(t,x)\big)_{|a|\le L}\Big] \cdot \sum_{(k,m)\in P(h,\alpha)} h! \prod_{j=1}^{|h|} \frac{[\partial^{m_j} g(t,x)]^{k_j}}{k_j!\,(m_j!)^{|k_j|}}$$

We note that

$$\partial^{m_j} g(t,x) = \partial^{m_j}\Big[\big(t,x,\big(\partial_x^a u(t,x)\big)_{|a|\le L}\big)\Big]$$

so that

$$\sum_{(k,m)\in P(h,\alpha)} h! \prod_{j=1}^{|h|} \frac{[\partial^{m_j} g(t,x)]^{k_j}}{k_j!\,(m_j!)^{|k_j|}} =$$

$$= \sum_{(k,(1,0,\dots,0))\in P(h,\alpha)} h! \prod_{j=1}^{|h|} \frac{1}{k_j!} +$$

$$+ \sum_{(k,(0,1,0,\dots,0))} h! \prod_{j=1}^{|h|} \frac{1}{k_j!} +$$

$$+ \dots +$$

$$+ \sum_{(k,(0,\dots,0,1,0,\dots,0))} h! \prod_{j=1}^{|h|} \frac{1}{k_j!} +$$

$$+ \sum_{\substack{(k,m)\\ m_1=\dots=m_d=0}} h! \prod_{j=1}^{|h|} \frac{[\partial^{m_j} u(t,x)]^{k_j}}{k_j!\,(m_j!)^{|k_j|}} =: Q_{h\alpha}\big[u(t,x)\big] \qquad (1)$$

therefore

9

$$(2).\mathcal{P} = \Bigg| \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[u(t,x)\right] -$$

$$- \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\left[t, x, \left(\partial_x^a v(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] \Bigg| \le$$

$$\le \Bigg| \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[u(t,x)\right] -$$

$$- \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] +$$

$$+ \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] -$$

$$- \sum_{1\le|\alpha|\le|h|} \partial^\alpha G\left[t, x, \left(\partial_x^a v(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] \Bigg|$$

In this way we've to deal with two types of summands.
The 1^{st} one is:

$$\Bigg| \partial^\alpha G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[u(t,x)\right] -$$

$$\partial^\alpha G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] \Bigg| \le$$

$$\le \left\| G\big|_{T\times S\times D} \right\|_{i+1+L} \cdot L_{i\alpha} \cdot \left\| u - v \right\|_{i+L}$$

where

$$\Big| Q_{h\alpha}\left[u(t,x)\right] - Q_{h\alpha}\left[v(t,x)\right] \Big| \le L_{i\alpha} \cdot \left\| u - v \right\|_{i+L}$$

The 2^{n} type of summand is

$$\left| \partial^{\alpha} G\left[t, x, \left(\partial_x^a u(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] - \right.$$

$$\left. - \partial^{\alpha} G\left[t, x, \left(\partial_x^a v(t,x)\right)_{|a|\le L}\right] \cdot Q_{h\alpha}\left[v(t,x)\right] \right| \le$$

$$\le \left\| G|_{T\times S\times D} \right\|_{i+1+L} \cdot \left\| u - v \right\|_{i+L} \cdot \left| Q_{h\alpha}\left[v(t,x)\right]\right|$$

But

$$\left| Q_{h\alpha}\left[v(t,x)\right]\right| \le \sum_{(k,(1,0,\dots,0))\in P(h,\alpha)} h! \prod_{j=1}^{|h|} \frac{1}{k_j!} +$$

$$+ \sum_{(k,(0,1,0,\dots,0))} h! \prod_{j=1}^{|h|} \frac{1}{k_j!} +$$

$$+ \dots +$$

$$+ \sum_{(k,(0,\dots,0,1,0,\dots,0))} h! \prod_{j=1}^{|h|} \frac{1}{k_j!} +$$

$$+ \sum_{\substack{(k,m)\\ m_1=\dots=m_d=0}} i! \prod_{j=1}^{|h|} \frac{\left|\partial^{m_j} u(t,x)\right|^{k_j}}{k_j!\,(m_j!)^{|k_j|}} =$$

and $\left|\partial^{m_j} u(t,x)\right| \le \|u\|_{i+L} \le r_{i+L}$ because $u \in Y$.

Therefore

$$\left| Q_{h\alpha}\left[v(t,x)\right]\right| \le B_{h\alpha}\left(1, \dots, 1, r_{i+L}, \dots, r_{i+L}\right)$$

We finally give an estimate of $C_{i\alpha}$:

$$\left| Q_{h\alpha}\left[u(t,x)\right] - Q_{h\alpha}\left[v(t,x)\right]\right| \le$$

$$\leq \sum_{\substack{(k,m) \\ m_1=\ldots=m_d=0}} h! \prod_{j=1}^{|h|} \frac{\left| \left[\partial^{m_j} u(t,x)\right]^{k_j} - \left[\partial^{m_j} v(t,x)\right]^{k_j} \right|}{k_j! \,(m_j!)^{|k_j|}} \leq$$

$$\leq \sum_{\substack{(k,m) \\ m_1=\ldots=m_d=0}} i! \prod_{j=1}^{i} \frac{r_{i+L}^{k_j-1}}{(k_j-1)! \,(m_j!)^{|k_j|}} \cdot \| u - v \|_{i+L} .$$

The obtained (not necessarily optimal) Lipschitz constants are hence:

$$\Lambda_0(T\times S) := \left\| G|_{T\times S\times D} \right\|_1 , \qquad D := \prod_{i=0}^{\hat{L}} \overline{B_{r_i+\delta_i}(0)}$$

$$\Lambda_i(T\times S) := \max_{|h|\leq i} \sum_{1\leq|\alpha|\leq|h|} \left[L_{i\alpha} + B_{h\alpha}\left(1,\ldots,1, r_{|i|+L},\ldots, r_{|i|+L}\right)\right] \cdot$$

$$\cdot \left\| G|_{T\times S\times D} \right\|_{i+1+L}$$

We also note that

$$\left.\begin{array}{l} T \subseteq T' \subset\subset_f \widetilde{\mathbb{R}} \\ S \subseteq S' \subset\subset_f \widetilde{\mathbb{R}}^d \\ T\times S \subseteq T'\times S' \end{array}\right\} \Longrightarrow \Lambda_i(T\times S) \leq \Lambda_i(T'\times S') \qquad (2)$$

and

$$\Lambda_i(T\times S) \leq \Lambda_{i+1}(T\times S) \quad \forall i\in\mathbb{N} \qquad (1)$$

because

$$L_{i\alpha} \geq 0 , \quad B_{h\alpha}\left(1,\ldots,1, r_{i+L},\ldots, r_{i+L}\right) \geq 0 \qquad (\text{see } (1).3)$$

$$\left\| G|_{T\times S\times D} \right\|_i \leq \left\| G|_{T\times S\times D} \right\|_{i+1}$$

$\square$

We can now prove the Picard - Lindelöf thm for PDE. In it we can always assume that F is given by

$$F : (t,x,y) \in T\times S\times Y \longmapsto G\left[t, x, \left(\partial_x^a y(t,x)\right)_{|a|\leq L}\right] \in \widetilde{\mathbb{R}}^d$$

13

Thm:

Let $t_0 \in \widetilde{\mathbb{R}}$, $\alpha, r_i \in \widetilde{\mathbb{R}}_{>0}$ $\forall i \in \mathbb{N}$, $T_\alpha := [t_0 - \alpha, t_0 + \alpha]$

$\widetilde{\mathbb{R}}^d \supseteq H$ sharply closed set, $S \Subset_f \widetilde{\mathbb{R}}^n$

$y_0 \in \mathcal{G}\mathcal{E}^\infty(S, H)$, $\overline{B_{r_0}(y_0(x))} \subseteq H$ $\forall x \in S$

Set $Y_\alpha := \left\{ y \in \mathcal{G}\mathcal{E}^\infty(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\}$

and assume that F is unif. Lip. on Y_α with const. $(\Lambda_i)_{i \in \mathbb{N}}$ and loss of derivatives L. Finally assume that

$\Lambda_i \le \Lambda_{i+1}$ $\forall i \in \mathbb{N}$

$\|F(-,-,y)\|_i \le M_i(y)$, $\alpha \cdot M_i(y) \le r_i$ $\forall y \in Y_\alpha$

$$\lim_{\substack{n,m \to +\infty \\ n \le m}} \alpha^{n+1} \cdot \Lambda^n_{i+mL} \cdot \|F(-,-,y_0)\|_{i+mL} = 0 \qquad \forall i \in \mathbb{N}$$

Then there exist a solution $y \in Y_\alpha$ of the Cauchy problem:

$$\begin{cases} \partial_t y(t,x) = F(t,x,y) \\ y(t_0, x) = y_0(x) \end{cases} \qquad (1)$$

Proof:

That Y_α is sharply closed in $\mathcal{G}\mathcal{F}(T_\alpha \times S, H)$ can be proved as in the ODE preprint.

The function $y_0 : (t,x) \in T_\alpha \times S \mapsto y_0(x) \in H$ is in $Y_\alpha \neq \emptyset$.

 14

Set

$$P : Y_\alpha \longrightarrow \mathcal{GE}^\infty(T_\alpha \times S, \widetilde{\mathbb{R}}^d)$$

here y is correct, not $y(s,x)$!

$$P(y)(t,x) := y_0(x) + \int_{t_0}^{t} F(s,x,y)\, ds \qquad \forall (t,x) \in T_\alpha \times S \quad \forall y \in Y_\alpha$$

We've: $P(y) \in \mathcal{GE}^\infty(T_\alpha \times S, \widetilde{\mathbb{R}}^d)$ because of 2).7 in Def 1.6.

For all $y \in Y_\alpha$:

$$\| P(y) - y_0 \|_i \le \max_{\substack{t \in T_\alpha \\ x \in S}} \int_{t_0}^{t} \| F(s,x,y) \|_i \, ds \le \alpha \cdot M_i(y) \le r_i$$

Moreover

$$\forall (t,x) \in T_\alpha \times S : \quad | P(y)(t,x) - y_0(x) | \le \| P(y) - y_0 \|_0 \le r_0$$

so that $P(y)(t,x) \in \overline{B_{r_0}(y_0(x))} \subseteq H$.

Now, we've to prove that P is a finite contraction on Y_α with loss of derivatives L starting from y_0.

We have:

$$\| P(y_0) - y_0 \|_{i+mL} \le \max_{\substack{t \in T_\alpha \\ x \in S}} \int_{t_0}^{t} \| F(s,x,y_0) \|_{i+mL} \, ds$$

$$\le \alpha \cdot \| F(-,-,y_0) \|_{i+mL} \qquad (1)$$

and

$$\| P(u) - P(v) \|_i \le \max_{\substack{t \in T_\alpha \\ x \in S}} \int_{t_0}^{t} \| F(s,x,u) - F(s,x,v) \|_i \, ds \le$$

$$\le \alpha \cdot \Lambda_i \cdot \| u - v \|_{i+L}$$

We finally need to evaluate

15

$$\lim_{\substack{m,n\to+\infty \\ n\le m}} \alpha^n \cdot \Lambda_{i+mL} \cdot \left\| P(y_0) - y_0 \right\|_{i+mL} \overset{(1).14}{\le}$$

$$\lim_{\substack{m,n\to+\infty \\ n\le m}} \alpha^n \cdot \Lambda_{i+mL} \cdot \alpha \cdot \left\| F(-,-,y_0) \right\|_{i+mL} \simeq 0$$

and

$$\alpha \cdot \Lambda_i \le \alpha \cdot \Lambda_{i+1} \qquad \forall i \in \mathbb{N}$$

Thm 1.4 yields $\exists!$ of the solution on Y_α.

$\square$

Rem:

1) We cannot repeat the proof of uniqueness as we did for ODE. In fact, if:

$$z \in \mathcal{G}\mathcal{C}^\infty(T_\alpha \times S, \widetilde{\mathbb{R}}^d)$$

$$\begin{aligned} \partial_t z(t,x) &= F(t,x,z) \\ z(t_0,x) &= y_0(x) \end{aligned} \qquad \forall (t,x) \in T_\alpha \times S$$

then, these imply

$$\left| y(t,x) - z(t,x) \right| \le \max_{\substack{t\in T_\alpha \\ x \in S}} \int_{t_0}^{t} \left\| F(-,-,y) - F(-,-,z) \right\|_0 \le \alpha \cdot \Lambda_0 \left\| y - z \right\|_L$$

$$\left\| y - z \right\|_0 \le \alpha \cdot \Lambda_0 \left\| y - z \right\|_L \quad \ldots ?$$

2) Note that the assumptions

$$\widetilde{\mathbb{R}}^d \supseteq H \text{ sharply closed set}$$

$$\overline{B_{r_0}(y_0(x))} \subseteq H \quad \forall x \in S$$

hold e.g. for $H = \overline{B_s(0)}$, where $s = r_0 + \|y_0\|_0$.

In fact $\forall x \in S$:

 (16)

$$\forall y \in \overline{B_{r_0}(y_0(x))} \quad : \quad |y - y_0(x)| < r_0$$

$$|y| \leq |y - y_0(x)| + |y_0(x)| < r_0 + \|y_0\|_0$$

$$y \in H$$

We finally want to connect Thm. 1.13 (PLT for PDE) to Thm. 1.7 (existence of Lipschitz constants for PDE with arbitrary GSF) in order to give clearer conditions for the existence of $\alpha \in \widetilde{\mathbb{R}}_{>0}$.

Cor:

Let $t_0 \in \widetilde{\mathbb{R}}$, $\beta, r_i \in \widetilde{\mathbb{R}}_{>0}$ $\forall i \in \mathbb{N}$, $T := [t_0 - \beta, t_0 + \beta]$

$L \in \mathbb{N}$, $\hat{L} := \operatorname{card}\{a \in \mathbb{N}^n \mid |a| \leq L\}$

$G \in {}^{\rho}\mathcal{GC}^{\infty}(T \times \widetilde{\mathbb{R}}^n \times \widetilde{\mathbb{R}}^{d \cdot \hat{L}}, \widetilde{\mathbb{R}}^d)$

$\emptyset \neq S \Subset_f \widetilde{\mathbb{R}}^n$, $y_0 \in \mathcal{GC}^{\infty}(S, \widetilde{\mathbb{R}}^d)$

$\|y_0\|_i \leq s_i \in \widetilde{\mathbb{R}}_{>0}$ $\forall i = 1, \dots, \hat{L}$

Set $H := \overline{B_{r_0 + s_0}(0)} \subseteq \widetilde{\mathbb{R}}^d$, $D := \prod_{i=0}^{\hat{L}} \overline{B_{r_i + s_i}(0)}$, $M_i := \left\| G|_{T \times S \times D} \right\|_i$.

Let $(\Lambda_i)_{i \in \mathbb{N}}$ be the Lipschitz constants for G as stated in Thm. 1.7 (and hence depending on $H, T, S, (r_i)_i, (s_i)_i$).

Finally, assume that $\alpha \in (0, \beta]$ and

$$\exists R \in \widetilde{\mathbb{R}}_{>0}\ \forall i \in \mathbb{N} : \Lambda_i \leq R \qquad (1)$$

$$\exists a \in \widetilde{\mathbb{R}}_{>0} : \alpha \leq \min\left(\frac{d\rho^a}{R}, \frac{r_i}{M_i}\right) \qquad (2)$$

$$\lim_{n,j \to +\infty} d\rho^{na} \cdot \left\| G\left(-,-,(\partial_x^a y_0)_{|a| \leq L}\right) \right\|_j = 0 \qquad (3)$$

Then there exist a solution $y \in Y_\alpha$ of (1).13.

Proof:

Set $T_\alpha := [t_0-\alpha, t_0+\alpha]$, $M_{i\alpha}(y) := M_{i\alpha} := \left\| G\big|_{T_\alpha\times S\times D} \right\|_i \quad \forall y\in Y_\alpha$.

Let $(\Lambda_{i\alpha})_{i\in\mathbb{N}}$ be the Lipschitz constants of

$$(t,x,y)\in T\times S\times Y_\alpha \longmapsto G\left[t,x,\left(\partial_x^a y(t,x)\right)_{|a|\le L}\right]\in\widetilde{\mathbb{R}}^d$$

on Y_α as in Thm 1.7 (see pag. 12).

We've:

- $\forall x\in S$: $|y_0(x)| \le \|y_0\|_0 \le s_0 < r_0+s_0 \implies y_0\in\mathcal{G}\mathcal{C}^\infty(S,H)$
- $\overline{B_{r_0}(y_0(x))}\subseteq H \quad \forall x\in S$ by Rem. 2.15)
- $\Lambda_{i\alpha}\le\Lambda_{i+1,\alpha} \quad \forall i\in\mathbb{N}$ by (1).12
- Setting $F(t,x,y) := G\left[t,x,\left(\partial_x^a y(t,x)\right)_{|a|\le L}\right] \quad \forall y\in Y_\alpha$, we've

$$\|F(-,-,y)\|_i \le \left\| G\big|_{T_\alpha\times S\times D} \right\|_i = M_{i\alpha}(y) \quad \forall y\in Y_\alpha$$

- $\alpha\cdot M_{i\alpha}(y) = \alpha\cdot M_{i\alpha} \underset{T_\alpha\in T}{\le} \alpha\cdot M_i \le r_i$ by (2).16

- $$\alpha^{n+1}\cdot\Lambda^n_{i+nL,\alpha}\cdot\|F(-,-,y_0)\|_{i+nL} \overset{(2).12}{\le}$$

$$\le \alpha^{n+1}\cdot\Lambda^n_{i+nL}\cdot\|F(-,-,y_0)\|_{i+nL} \overset{(2).16}{\le}$$

$$\le \alpha^{n+1}\cdot R^n\cdot\|F(-,-,y_0)\|_{i+nL} \overset{(2).16}{\le}$$

$$\le \alpha\cdot \mathrm{d}\rho^{na}\cdot\|F(-,-,y_0)\|_{i+nL} \longrightarrow 0 \quad \text{by (3).16}$$

We can hence apply thm. 1.13 to get the conclusion. $\square$

Cor:

Let us assume all the hypotheses of Cor. 1.16 but (2).16, (3).16.
If we assume that

$$M, r \in \widetilde{\mathbb{R}}_{>0}$$

$$\left.\begin{array}{l} 0 < M_i \le M \\ r \le r_i \end{array}\right. \quad \forall i \in \mathbb{N} \qquad (1)$$

$$\exists a \in \widetilde{\mathbb{R}}_{>0} : \quad \alpha \le \min\left(\frac{d\rho^a}{R}, \frac{r}{M}\right)$$

Then there exist a solution $y \in Y_\alpha$ of (1).13.

Proof:

We've:

$$\alpha \le \frac{r}{M} \le \frac{r_i}{M} \le \frac{r_i}{M_i} \qquad \forall i$$

and

$$\left\| G(-,-,(\partial_x^a y_0)_{|a| \le L} \right\|_j \le \left\| G|_{T \times S \times D} \right\|_j = M_j \le M$$

Therefore

$$\lim_{n,j \to +\infty} d\rho^{na} \cdot \left\| G(-,-,(\partial_x^a y_0)_{|a| \le L} \right\|_j \le \lim_{n,j \to +\infty} d\rho^{na} \cdot M = 0$$

$\square$

Rem:

1) The existence of $\alpha \in \widetilde{\mathbb{R}}_{>0}$ in these corollaries depends on the following facts

- The Lipschitz constants must be bounded by some $R \in \widetilde{\mathbb{R}}$ uniformly w.r.t. $i \in \mathbb{N}$:

$$\forall i \in \mathbb{N} : \Lambda_i \le R$$

E.g. we cannot solve

$$y' = e^{\delta(0) \cdot y} \qquad \text{or} \qquad \begin{cases} y' = \delta(y) \\ y(0) = 0 \end{cases}$$

- The radii r_i have not to decrease to zero:

$$0 < r \le r_i \quad \forall i \in \mathbb{N}$$

- The norms $M_i = \left\| G|_{T\times S\times D} \right\|_i$ must be uniformly bounded:

$$0 < M_i \le M \qquad \forall i \in \mathbb{N}$$

- The product $\alpha^n \cdot \Delta_j^n$ must go to zero as $n \to +\infty$ more quickly than $\left\| G(-,-,(\partial_x^a y_0)_{|a|\le L} \right\|_j$ as $j \to +\infty$.

  E.g. $\alpha^n \cdot \Delta_j^n = O(d\rho^{na})$ and

$$\left\| G(-,-,(\partial_x^a y_0)_{|a|\le L} \right\|_j = O\left(\log d\rho^{jb}\right)$$

  or M_i uniformly bounded as above.

2) We deal with PDE as ODE because the proof of P-L is essentially identical, but with "$\forall x \in S$".
Only uniqueness is conceptually different.
The functionally compact solid set $S \subseteq \widehat{\mathbb{R}}^n$ seems to be completely arbitrary, even if assumptions (2). Cor. 1.16 and (1). Cor. 1.18 depend on S.

3) In a similar way, we can deal with k-th order (in t) PDE

$$\begin{cases} \partial_t^k y(t,x) = G\left[t,x,(\partial_t^j \partial_x^a y)_{\substack{j<k \\ |a|+j\le L}}\right] \\[2ex] \partial_t^j y(t_0,x) = y_{0j}(x) \qquad 0 \le j < k \end{cases}$$

Note that, surprisingly, there is no limitation on the maximum order $a \in \mathbb{N}^n$ of derivatives $\partial_x^a y$. This doesn't imply that to solve, e.g.

$$\partial_x^2 y = \frac{1}{\kappa}\partial_t y$$

we need only 1 initial condition. It simply says that, in general, the two Cauchy problems:

$$\begin{cases} \partial_t y = \kappa\,\partial_x^2 y \\ y(0,x) = y_0(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_x^2 \bar{y} = \frac{1}{\kappa}\partial_t \bar{y} \\ \partial_x \bar{y}(0,x) = \bar{y}_1(x) \\ \bar{y}(0,x) = \bar{y}_0(x) \end{cases}$$

have, in general, different solutions $y \ne \bar{y}$ unless $\bar{y}_1(x) = \partial_x y(0,x)$ and $\bar{y}_0(x) = y(0,x)$ or $y_0(x) = \bar{y}(0,x)$.

4) We recall that the embedding of Schwartz distributions depends on the gauge ρ (if we want to have good properties of the embedding, see http://www.mat.univie.ac.at/~giordap7/GenFunMaps.pdf). To underscore this fact, we write δ_ρ for the ρ-embedding of the Dirac delta.

Therefore the fundamental solution of the heat equation:

$$\begin{cases} \partial_t u = \kappa\,\partial_x^2 u & x \in [-d\rho^{-1}, d\rho^{-1}] \\ u(x,0) = \delta_\rho(x) & t \in [0, d\rho^{-1}] \end{cases}$$

is the ρ-embedding of the classical heat kernel:

$$\langle K(t), \varphi \rangle := \begin{cases} \int_{\mathbb{R}} e^{-s^2/4t} / \sqrt{4\pi t}\,\varphi(s)\,ds & t \in \mathbb{R}_{>0} \\ \varphi(0) & t = 0 \end{cases} \qquad \forall \varphi \in \mathcal{D}_{\mathbb{R}}$$

i.e. it's $i_\rho(K(t))$, where $i_\rho : \mathcal{D}' \longrightarrow \mathcal{GC}^\infty$ is the sheaf embedding. It's an open problem to show that the solution obtained from the previous Picard-Lindelöf Thm is at least infinitely closed (associated) to $i_\rho(K(t))$.

 21

5) Let's consider e.g. the Cauchy problem

$$\begin{cases} \partial_t y(t,x) = G\left[t, x, y(t,x), \partial_x y(t,x), \partial_x^2 y(t,x)\right] \\ y(0,x) = \delta_\rho(x) = \frac{1}{d\rho} \cdot \mu\left(\frac{x}{d\rho}\right) \end{cases} \qquad (1)$$

where $G \in \widetilde{\mathbb{R}}[t, x, d_0, d_1, d_2]$ is a polynomial with coeff. in $\widetilde{\mathbb{R}}$ and μ is a Colombeau mollifier (http://www.mat.univie.ac.at/~giordap7/GenFunMaps.pdf).

In Cor. 1.18 we set

$\beta, \gamma \in \widetilde{\mathbb{R}}_{>0}$ arbitrary, $T := [-\beta, \beta]$, $S := [-\gamma, \gamma]$

$s_i := d\rho^{-i-1} \implies \|y_0\|_i \le s_i \quad \forall i = 0,1,2$

$0 < r \le r_i \le \bar{r} \in \widetilde{\mathbb{R}}_{>0}$ arbitrary, so $D = \prod_{i=0}^{2} \overline{B_{r_i + s_i}(0)}$

Since G is a polynomial, we've:

$\exists M \in \widetilde{\mathbb{R}}_{>0}\, \forall i \in \mathbb{N}: \; M_i := \|G|_{T \times S \times D}\|_i \le M$

Set Λ_i as at pag. 7, and from $M_i \le M$, $r_i \le \bar{r}$ we get

$\exists R \in \widetilde{\mathbb{R}}_{>0}\, \forall i \in \mathbb{N}: \; \Lambda_i \le R$

From Cor 1.18 we obtain $\exists \alpha \in \widetilde{\mathbb{R}}_{>0}$ and $\exists y \in Y_\alpha$ solution of (1).

Note that:

1) If R is not infinitesimal $\left(\exists k \in \mathbb{R}_{>0} : k < R\right)$ then $\alpha \simeq 0$
   If R is infinitesimal (i.e. all the coefficients of the polyn. G are infinitesimal) and $r, M \in \mathbb{R}_{>0}$ (e.g. $\beta, \gamma \in \mathbb{R}$), then we can take $\alpha \not\simeq 0$.

2) If we take all $r_i \simeq 0$, then $\alpha \simeq 0$ and the solution y is infinitely close to the initial condition δ_ρ because

$$\|y - \delta_\rho\|_i \le r_i \quad \forall i \in \mathbb{N}$$

The larger we take r_i, the larger is α and the solution y can depart from the initial condition.

3) To obtain a more powerful tool to get sufficient condition for $\alpha \in \mathbb{R}_{>0}$, we need hyperfinite iterations of Picard contractions.

4) Analogously, we can deal with PDE of greater order in t.

Acknowledgement: We would like to thank

- M. Grosser
- M. Kunzinger
- E. Nigsch
- C. Sämann
- R. Steinbauer

for helpful discussions and several suggestions.