UNIVERSAL PROPERTIES OF SPACES OF GENERALIZED FUNCTIONS

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Abstract.

1. Introduction

Mathematicians eventually try to solve a problem in the best possible way, e.g. considering a geometrically intrinsic solution, the best computational algorithm or the most general answer. Frequently motivated by the searching of beauty, [18], the required solution have to be the "simplest" one, i.e. it has to use the minimal amount of conventional constructions and data other than the given ones from which the problem must depend on. At a first sight, a non-trivial possible mathematical formalization of the idea of *simplest solution* should involve information theory (see e.g. [23] and references therein) or mathematical logic.

In this paper, using only a minimal amount of category theory, we see a common informal interpretation of *universal solution* as the simplest way to solve a given problem. It is well-known that universal constructions appears everywhere in mathematics, [19], and hence this interpretation justifies why this happens. We list several examples justifying this interpretation, in particular for spaces of generalized functions (GF) both in linear and nonlinear frameworks.

We will see that a universal solution not only candidates itself as the simplest way to solve a given problem, but its universal property is able to highlight what are the data of the problem and the conventional choices in any other possible construction. Frequently, this paves the way for generalizations, and it always directly yields an axiomatic characterization of these universal solutions. In the point of view of several mathematicians, universal properties are so important that they take them as the right starting point: "it is not important how you solve this problem (and we see that a possible solution exists); the key point is that you have a universal solution, which is unique up to isomorphisms".

Concerning spaces of GF, we clearly consider Schwartz's distributions as "the simplest way to have derivatives of continuous functions" (see [24]) and hence the corresponding not-well known universal property (see also [22]), see Sec. 3. Following the algebraic construction of Sebastiao e Silva ([25]) of distributions of finite order, we see how to obtain a similar universal construction for arbitrary manifold-valued distributions defined in another manifold or in infinite dimensional spaces such as $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)$. Any other solution of the same problem will reasonably satisfy

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the same minimal and meaningful universal property, and hence it will be isomorphic to our solution, see Sec. 3.0.1.

Among nonlinear settings for GF extending Schwartz's distributions, Colombeau's special algebra (see e.g. [2, 3, 4, 6]) is frequently perceived as the simplest one. We actually prove that it is the most simple quotient algebra one can consider, see Sec. 4. We will also consider the recent *generalized smooth functions* (see [12, 13] and references therein) because it is even more general than Colombeau's algebras, but with several improved properties such as more general domains, the closure with respect to composition, a better integration theory and Hadamard's well-posedness for every PDE, see Sec. 5. As a secondary result, we hence have an axiomatic description up to isomorphisms of Colombeau's special algebra and of generalized smooth functions. In particular, the ring of Colombeau's generalized numbers reveals to be the simplest quotient ring containing the infinite numbers $[\varepsilon^{-q}]_\sim$, $q \in \mathbb{N}$, and where every zero-net $[x_\varepsilon]_\sim = 0$ is generated by an infinitesimal function: $\lim_{\varepsilon\to 0^+} x_\varepsilon = 0$ (see Sec. 4.3; see also [27] for another axiomatic description in the framework of nonstandard analysis and where the latter property does not hold).

In the following, we will use the conventions:

- universal = terminal = limit = projective: the unique arrows arrives to the universal object.
- co-universal = initial = co-limit = injective: the unique arrows starts from the universal object.

The paper is self-contained, in the sense that it requires only the notions of category, functor, natural transformation and Schwartz's distributions.
We start by introducing the notion of universal solution using a simple and non-abstract language.

2. General definition of (co-) universal property

We start by defining in general terms what a universal property is. We will use only basic notions of category theory, and will give a definition near to the common use of universal properties.

Definition 1. Let $\mathbf{C}$ be a category. $\mathcal{P}(C)$ and $\mathcal{Q}(f, A, B)$ are two properties of A, B, C and f, where A, B, C are objects of $\mathbf{C}$ and f is an arrow of $\mathbf{C}$. Assume that:

$$\mathcal{Q}(f, A, B),\ \mathcal{Q}(g, B, C)\ \Rightarrow\ \mathcal{Q}(g \circ f, A, C), \tag{2.1}$$

$$Q(1_A, A, A). \tag{2.2}$$

Then we say that C *is a universal solution of* $\mathcal{P}$ *with respect to* $\mathcal{Q}$ if

(i) $\mathcal{P}(C)$,
(ii) $\forall D \in \mathbf{C} :\ \mathcal{P}(D)\ \Rightarrow\ \exists! \varphi : D \longrightarrow C :\ \mathcal{Q}(\varphi, D, C)$.

Similarly, we say that C *is a co-universal solution of* $\mathcal{P}$ *with respect to* $\mathcal{Q}$ if

(iii) $\mathcal{P}(C)$,
(iv) $\forall D \in \mathbf{C} :\ \mathcal{P}(D)\ \Rightarrow\ \exists! \varphi : C \longrightarrow D :\ \mathcal{Q}(\varphi, C, D)$.

The proof of the following theorem trivially generalizes the classical proofs concerning the uniqueness of universal objects up to isomorphisms:

Theorem 2. *Suppose that C_1 and C_2 are two (co-)universal solutions of $\mathcal{P}$ with respect to $\mathcal{Q}$. Then C_1 is isomorphic to C_2 in $\mathbf{C}$.*

Proof. Since C_1 is a universal solution of $\mathcal{P}$ and $\mathcal{Q}$, using (ii) Def. 1 for $D = C_2$, there exist a unique $\varphi_1 : C_2 \longrightarrow C_1$ such that the property $\mathcal{Q}(\varphi_1, C_2, C_1)$ holds. In a similar way, there exist a unique φ_2 such that $\varphi_2 : C_1 \longrightarrow C_2$ so we have $\mathcal{Q}(\varphi_2, C_1, C_2)$. By assumption (2.1) on $\mathcal{Q}$, the property $\mathcal{Q}(\varphi_2 \circ \varphi_1, C_2, C_2)$ holds. Using again Def. 1.(ii) with $D = C_2$, we get that only one arrow φ satisfies $\mathcal{Q}(\varphi, C_2, C_2)$. Since $\mathcal{Q}(1_{C_2}, C_2, C_2)$ also holds by (2.2), then $\varphi_2 \circ \varphi_1 = 1_{C_2}$. In a similar way, we have $\varphi_1 \circ \varphi_2 = 1_{C_1}$, which proves the theorem. □

Starting from the properties $\mathcal{P}$ and $\mathcal{Q}$, we can define a new category $\mathbf{C}(\mathcal{P}, \mathcal{Q})$. Its objects are the objects of the category $\mathbf{C}$ that verify the property $\mathcal{P}$ and its arrows are the arrows f of the category $\mathcal{C}$ for which $\mathcal{Q}(f, C, D)$ holds where C, $D \in \mathbf{C}(\mathcal{P}, \mathcal{Q})$, i.e.:

- $C \in \mathbf{C}(\mathcal{P}, \mathcal{Q}) :\iff \mathcal{P}(C)$,
- $D \xrightarrow{\varphi} C$ in $\mathbf{C}(\mathcal{P}, \mathcal{Q})$: $\iff \mathcal{Q}(\varphi, D, C)$, $D \xrightarrow{\varphi} C$ in $\mathbf{C}$,
- $\psi \circ \varphi$ in $\mathbf{C}(\mathcal{P}, \mathcal{Q}) :\iff \psi \circ \varphi$ in $\mathbf{C}$.

Then, we have that C is a universal solution of $\mathcal{P}$, $\mathcal{Q}$ if and only if C is terminal in $C(\mathcal{P}, \mathcal{Q})$ (i.e. for all $D \in C(\mathcal{P}, \mathcal{Q})$ there exists one and only one $\varphi : D \longrightarrow C$ in $C(\mathcal{P}, \mathcal{Q})$), and C is a co-universal solution of $\mathcal{P}$, $\mathcal{Q}$ if and only if C is initial in $C(\mathcal{P}, \mathcal{Q})$ (i.e. for all $D \in C(\mathcal{P}, \mathcal{Q})$ there exists one and only one $\varphi : C \longrightarrow D$ in $C(\mathcal{P}, \mathcal{Q})$).

As we mentioned above, a (co-)universal solution is considered as the (co-)simplest or (co-)most natural solution of that problem. Even considering only the following elementary examples, we can start to justify this interpretation:

Example 3.

(i) Let's consider the problem to put a topology on a set $X \in \mathbf{Set}$. The category $\mathbf{C}$ in this example is the category of all the topologies on X viewed as a poset, i.e. "$\subseteq$" is the unique arrow of $\mathbf{C}$, and we write $\tau \subseteq \sigma$ if the topology τ is coarser than the topology σ. The properties $\mathcal{P}$ and $\mathcal{Q}$ are defined as follow.

$$\mathcal{P}(\tau) :\iff \tau \text{ is a topology on } X,$$
$$\mathcal{Q}(i, \tau, \sigma) :\iff i = \subseteq,\ \tau \subseteq \sigma.$$

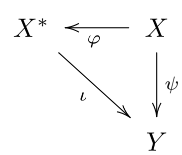
The trivial topology $(\{\emptyset\}, X)$ is the co-universal solution of the property $\mathcal{P}$ with respect to the property $\mathcal{Q}$ and the discrete topology is the universal solution. Clearly, these also appear as trivial solutions; on the other hand, note that they are also the simplest/non-conventional solutions starting from the unique data $X \in \mathbf{Set}$ and with respect to the problem "define a topology on X": any other solution would necessarily introduce (in the case of the trivial topology) or delete (in the case of the discrete topology) something which is not related to the problem or the data itself. This example also shows that the notion of *simplest solution* can be implemented in two ways: from "below" (co-universal) or from "above" (universal).

(ii) Let R be a ring and let $x \notin R$. What would be the smallest/simplest ring containing both x and R? Any ring that contains x and R must contains also sums of terms of the form $r \cdot x^n$ for any integer n and any element $r \in R$. Intuitively, the simplest solution is therefore $R[x]$. The co-universal property can be highlighted as follow: Let $S \in \mathbf{Ring}$, then we can consider the property $\mathcal{P}(S, s)$ whenever $x \in S$ and $s : R \longrightarrow S$ is a ring homomorphism, and the property $\mathcal{Q}(f,(S,s),(L,l))$ if $S \xrightarrow{f} L$ in **Ring** and $f \circ s = l$. $R[x]$ is the co-universal solution of $\mathcal{P}$ and $\mathcal{Q}$, i.e. the simplest way to extend the ring R by adding a new element $x \notin R$. Clearly, we have $\mathcal{P}(R[x], i)$, where $i : R \longrightarrow R[x]$ is the inclusion. Let $S \in \mathbf{Ring}$, and let $s : R \longrightarrow S$ be a ring homomorphism, i.e. $\mathcal{P}(S,s)$ holds, then the unique $\varphi : R[x] \longrightarrow S$ of Def. 1.(ii) is given by $\varphi\left(\sum_i r_i x^i\right) = \sum_i s(r_i)x^i$.

(iii) Let (X,d) be a non-complete metric space. Then it can be completed in the following way (For more details and proofs, see); Let $\mathcal{C}(X)$ be the collection of all Cauchy sequences of X and let $\sim$ be an equivalence relation defined on $\mathcal{C}(X)$ by

$$(x_n)_n \sim (y_n)_n \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

We set $X^* := \mathcal{C}(X)/\sim$ and set $d^*([(x_n)],[(y_n)_n]) = \lim_{n\to\infty} d(x_n,y_n)$. Indeed, (X^*, d^*) is a complete metric space. Moreover, there exists an isometry $\varphi : X \longrightarrow X^*$ such that $\varphi(X)$ is dense in X^*. The triple (X^*, d^*, φ) is co-universal in the following sense: a triple (Y, δ, ψ) is such that (Y, δ) is a complete metric space and $\psi : X \longrightarrow Y$ is an isometry, then there exists a unique map $\iota : X^* \longrightarrow Y$ such that the following diagram commutes



ι is defined in the following way: Let $x^* \in X^*$. Since $\varphi(X)$ is dense in X^*, then there exists a sequence $(x_n)_n$ of X such that $(\varphi(x_n))_n$ converges to x^*. The sequence $(\varphi(x_n))_n$ is a Cauchy sequence and since φ and ψ are isometries, the sequence $(\psi(x_n))_n$ is also a Cauchy sequence in Y which converges because Y is complete. Thus we can set $\iota(x^*) = \lim_{n\to\infty}(\psi(x_n))$ and since φ and ψ are isometries then the ι is well defined, i.e. it does not depend on the sequence $(x_n)_n$.

(iv)

We can underscore that in all these universal solutions there are no conventional choices and they are the most natural solutions: any other solution would appear as less natural.

2.1. **Preliminary notions: presheaf and sheaf.** For the sake of completeness, in this section we briefly recall the notions of presheaf and sheaf, because they are used in our universal characterization of spaces of GF.

In the following, we denote by **Set**, the category of sets and functions, by $\mathbf{Mod}_R$ the category of R modules over the ring R, so that$\mathbf{Vect}_K := \mathbf{Mod}_K$ is the category of vector spaces over a given field K, $\mathcal{O}\mathbb{R}^\infty$ is the category of open sets of any dimension and smooth functions, **Ring** is the category of rings and ring-homomorphisms. If $\mathbb{T} = (|\mathbb{T}|, \tau)$ is a topological space, we use the same symbol to also denote the category induced by its open sets as a preorder, i.e. the category of open sets $A \in \tau$ of the given topology and only one arrow "$\subseteq$", i.e. we write $A \xrightarrow{\subseteq} B$ in $\mathbb{T}$ if $A \subseteq B$. We finally recall that for any category $\mathbf{C}$, we can define its opposite $\mathbf{C}^{\mathrm{op}}$ by reversing its arrows. That is, $\mathrm{ob}\left(\mathbf{C}^{\mathrm{op}}\right) := \mathrm{ob}(\mathbf{C})$ and $\mathbf{C}^{\mathrm{op}}(A,B) := \mathbf{C}(B,A)$, i.e. $f \in \mathbf{C}^{\mathrm{op}}(A, B)$ if and only if $B \xrightarrow{f} A$ holds in the original category $\mathbf{C}$. For example, we write $f \in (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}}(A, B)$ if f is a smooth function from $B \subseteq \mathbb{R}^b$ into $A \subseteq \mathbb{R}^a$, and $\mathbb{T}^{\mathrm{op}}(A, B)$ is non empty if and only if $B \subseteq A$.

Definition 4.

(i) Let R be a ring. A presheaf P of $\mathbf{Mod}_R$ is a functor $P : \mathbb{T}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_R$. We denote by $P(U) \in \mathbf{Mod}_R$ the evaluation on $U \in \mathbb{T}^{\mathrm{op}}$ and by $P_{U,V} := P(U \leq V) : P(U) \longrightarrow P(V)$ the evaluation on arrows (usually called restrictions from U to V).

(ii) If $(U_j)_{j \in J}$ is a covering in $\mathbb{T}$ of $U \in \mathbb{T}^{\mathrm{op}}$, then we say that $(f_j)_{j \in J}$ is a compatible family related to P if and only if
  (i) $\forall j \in J : f_j \in P(U_j)$.
  (ii) $\forall j, h \in J : P_{U_j, U_j \cap U_h}(f_j) = P_{U_h, U_h \cap U_j}(f_h)$.

(iii) Moreover, we say that $P : \mathbb{T}^{\mathrm{op}} \longrightarrow \mathbf{Mod}_R$ is a sheaf if it's a presheaf that satisfies the following conditions. For any $U \in \mathbb{T}^{\mathrm{op}}$, for any covering $(U_j)_{j \in J}$ of U in $\mathbb{T}$ and for any compatible family $(f_j)_{j \in J}$ related to P we have
  (i) $f, g \in P(U)$. Then $\forall j \in J : P_{U,U_j}(f) = P_{U,U_j}(g) \Longrightarrow f = g$. (This is called locality condition; if P satisfy only this condition, it's called a separated presheaf or a monopresheaf).
  (ii) $\exists f \in P(U)$ such that $\forall j \in J : P_{U,U_j}(f) = f_j$. (This called gluing condition).

Remark 5.

(i) (i), (ii) imply that $\exists! f \in P(U) : \forall j \in J : P_{UU_j}(f) = f_j$.

(ii) In the last definition, instead of $\mathbf{Mod}_R$,we could have written any category whatsoever and the definition would still work.In reality, we will be working only with presheaves and sheaves of $\mathbf{Mod}_R$. Hence, the definition above is largely sufficient.

3. Universal property of Schwartz distributions

In this section, we want to show a co-universal property of the space of Schwartz's distributions. More precisely, exactly as stated in [24], we will show that the space $\mathcal{D}'$ of Schwartz distributions is the most natural space in which we can take derivatives of continuous functions preserving derivatives of continuously differentiable functions.

In what follows, $\mathcal{C}^0$ and $\mathcal{C}^1$ denote the sheaves of continuous functions and continuously differentiable functions defined on open subsets of $\mathbb{R}^n$ and valued in $\mathbb{R}$. The notations $\mathcal{C}^1 \underset{\partial_i}{\overset{\iota}{\rightrightarrows}} \mathcal{C}^0$, $i = 1, \ldots, n$, are used to denote the inclusion and the partial derivatives of continuously differentiable functions.

Schwartz's solution leads to the following objects:

(i) $\mathcal{D}'$ is the sheaf of real valued distributions on $\mathbb{R}^n$.

(ii) $\mathcal{C}^0 \xrightarrow{\lambda} \mathcal{D}'$ is the inclusion of the space of continuous functions into the space of distributions. The map λ is a sheaf morphism, i.e. it is a natural transformation $\lambda_U : \mathcal{C}^0(U) \longrightarrow \mathcal{D}'(U)$ for all $U \in \mathrm{Op}(\mathbb{R}^n)$ and for all $V \in$

$\mathrm{Op}(\mathbb{R}^n)$ with $V \subseteq U$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}^0(U) & \xrightarrow{\lambda_U} & \mathcal{D}'(U) \\ {\scriptstyle \mathcal{C}^0_{U,V}}\downarrow & & \downarrow{\scriptstyle \mathcal{D}'_{U,V}} \\ \mathcal{C}^0(V) & \xrightarrow[\lambda_V]{} & \mathcal{D}'(V) \end{array}$$

(iii) $\mathcal{D}' \xrightarrow{D_i} \mathcal{D}'$, for $i = 1, \dots, n$, are the partial derivatives of distributions. Once again, each D_i is a sheaf morphism because $D_{iU} : \mathcal{D}'(U) \longrightarrow \mathcal{D}'(U)$ for all $U \in \mathrm{Op}(\mathbb{R}^n)$ and since they commutes with restrictions of distributions: $D_{iV}(\mathcal{D}'_{UV}(T)) = D_i(T|_V) = \mathcal{D}'_{UV}(D_{iU}(T)) = D_i(T)|_V$ if $V \in \mathrm{Op}(\mathbb{R}^n)$, $V \subseteq U$.

(iv) $D_i \circ D_j = D_j \circ D_i$ for all i, $j = 1, \dots, n$.

Theorem 6. *$(\mathcal{D}', \lambda, (D_i)_i)$ is a co-universal solution of the problem $\mathcal{P}(H, j, (\delta_i)_i)$ given by :*

(i) $H : (\mathbb{R}^n)^{op} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ *is a sheaf of real vector spaces.*

(ii) $j : C^0 \longrightarrow H$ *is a sheaf morphism.*

(iii) $\delta_i : H \longrightarrow H$, $i = 1, \dots, n$, *are compatible with partial derivatives of C^1 functions: $\delta_i \circ j \circ \iota = j \circ \partial_i$, i.e. the following diagram of sheaves morphisms commutes for all $i = 1, \dots, n$:*

$$\begin{array}{ccccc} C^1 & \overset{\iota}{\hookrightarrow} & C^0 & \xrightarrow{j} & H \\ & {\scriptstyle \partial_i}\searrow & & & \downarrow{\scriptstyle \delta_i} \\ & & C^0 & \xrightarrow{j} & H \end{array}$$

(iv) $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ *for all i, $j = 1, \dots, n$.*

The problem is solvable with respect to the property $\mathcal{Q}(\psi, H, j, (\delta_i)_i, \overline{H}, \overline{j}, (\overline{\delta}_i)_i)$ given by

$$\psi : H \longrightarrow \overline{H}, \qquad \psi \circ j = \overline{j}, \qquad \psi \circ \delta_i = \overline{\delta}_i \circ \psi \quad \forall i = 1, \dots, n,$$

i.e. when the following diagrams of sheaves morphisms commute

$$\begin{array}{ccc} H & \xrightarrow{\delta_i} & H \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \overline{H} & \xrightarrow[\overline{\delta}_i]{} & \overline{H} \end{array} \qquad \begin{array}{ccc} C^0 & \xrightarrow{j} & H \\ & {\scriptstyle \overline{j}}\searrow & \downarrow{\scriptstyle \psi} \\ & & \overline{H} \end{array}$$

Therefore, if $(\overline{\mathcal{D}}', \overline{\lambda}, (\overline{D}_i)_i)$ is another solution of (i)-(iv) then

$$\exists! \psi : \mathcal{D}' \longrightarrow \overline{\mathcal{D}}' : \ \overline{\lambda} = \psi \circ \lambda, \ \psi \circ D_i = \overline{D}_i \circ \psi \quad \forall i = 1, \dots, n.$$

Proof. The Key idea of the proof is to use the local structure of distributions to define ψ on distributions of finite order. Since any distribution is locally of finite order, ψ can be defined locally on any distribution. After that, we use the sheaf property to extend ψ to any distribution. The proof is divided into 6 steps:

(i) Definition of ψ for distributions of finite order: Let $T \in \mathcal{E}'(\Omega)$ be a distribution of finite order and Ω is an arbitrary open set. The local structure theorem of distributions of finite order states that

$$\exists k \in \mathbb{N} \exists f_\alpha \in C^0(\Omega), |\alpha| \leq k : T = \sum_{|\alpha| \leq k} D^\alpha \circ \lambda(f_\alpha) \tag{3.1}$$

In this case we can set

$$\psi(T) = \sum_{|\alpha| \leq k} \overline{D}^\alpha \circ \overline{\lambda}(f_\alpha). \tag{3.2}$$

The right hand side of 3.2 is well defined as an element of $\overline{\mathcal{D}}'$. If $\exists k' \in \mathbb{N} \exists g_\alpha \in C^0(\Omega), |\alpha| \leq k' : T = \sum_{|\alpha| \leq k'} D^\alpha \circ \lambda(g_\alpha)$ then

$$\sum_{|\alpha| \leq k} \overline{D}^\alpha \circ \overline{\lambda}(f_\alpha) = \sum_{|\alpha| \leq k'} \overline{D}^\alpha \circ \overline{\lambda}(g_\alpha). \tag{3.3}$$

In order to prove 3.3 we set $\tilde{\alpha} = (\max(k, k')_i)$ and D_i^{-1} the (left) inverse of D_i. Thus we have

$$\begin{aligned} T &= \sum_{|\alpha| \leq k} D^\alpha \circ \lambda(f_\alpha) = \sum_{|\alpha| \leq k} D^{\tilde{\alpha}} \circ D^{\alpha - \tilde{\alpha}} \lambda(f_\alpha) \\ &= D^{\tilde{\alpha}} \sum_{|\alpha| \leq k} D^{\alpha - \tilde{\alpha}} \lambda(f_\alpha) := D^{\tilde{\alpha}} \circ \lambda(\hat{f}) \end{aligned}$$

where $\hat{f}$ is a continuous function. Similarly, we have

$$T = D^{\tilde{\alpha}} \circ \lambda(\hat{g})$$

and $\hat{g}$ is a continuous function. From the last two equations we have $D^{\tilde{\alpha}} \circ \lambda(\hat{f} - \hat{g}) = 0$. It follow that

$$\begin{aligned} \sum_{|\alpha| \leq k} \overline{D}^\alpha \circ \overline{\lambda}(f_\alpha) &= \overline{D}^{\tilde{\alpha}} \circ \overline{\lambda}(\hat{f}) \\ &= \overline{D}^{\tilde{\alpha}} \circ \overline{\lambda}(\hat{g}) + \overline{D}^{\tilde{\alpha}} \circ \overline{\lambda}(\hat{f} - \hat{g}) \\ &= \sum_{|\alpha| \leq k'} \overline{D}^\alpha \circ \overline{\lambda}(g_\alpha) + \psi\left(D^{\tilde{\alpha}} \circ \lambda(\hat{f} - \hat{g})\right) = \sum_{|\alpha| \leq k'} \overline{D}^\alpha \circ \overline{\lambda}(g_\alpha) \end{aligned}$$

which proves 3.3. Note that 3.2 implies that

$$\psi \circ \lambda(f) = \overline{\lambda}(f), \forall f \in C^0 \text{ and } \forall i : \psi \circ D_i(T) = \overline{D}_i \circ \psi(T) \, \forall T \in \mathcal{E}'. \tag{3.4}$$

(ii) Definition of ψ locally for any $T \in \mathcal{D}'(\Omega)$: Let $K \Subset \Omega$. The local structure theorem of distributions asserts that $T|_K \in \mathcal{E}'(\Omega)$. Thus we can define $\psi(T|_K)$ as in the first step as follow

$$\psi(T|_K) = \sum_{|\alpha| \leq k} \overline{D}^\alpha \circ \overline{\lambda}(f_\alpha|_K) \tag{3.5}$$

where $k \in \mathbb{N}$ and $\forall |\alpha| \leq k : f_\alpha \in C^0(\Omega)$.

(iii) For $T \in \mathcal{D}'(\Omega)$, we simply consider

$$B(T) := \{V \subseteq \Omega | T|_V \in \mathcal{E}'(\Omega)\}.$$

Since $\overline{D}^{\alpha} \circ \overline{\lambda}$ is a sheaf-morphism, then $\psi(T|_V)_{V \in B(T)}$ is a compatible family, i.e $\psi(T|_V)|_{U\cap V} = \psi(T|_U)|_{V\cap U}$ for all $U, V \in B(T)$. Using the sheaf property of $\overline{\mathcal{D}}'$ we set

$$\psi(T) = \overline{\mathcal{D}}'\text{-gluing}\left[\psi(T|_V)_{V\in B(T)}\right]. \tag{3.6}$$

(iv) Proving that ψ is a sheaf-morphism on $\mathcal{D}'$: Since $\overline{D}^{\alpha} \circ \overline{\lambda}$ is a sheaf-morphism, ψ is a sheaf-morphism on distribution of finite order. Let A, A' be two open sets such that $A \subseteq A'$ and $T \in \mathcal{D}'(A')$. We should prove that $\psi(T)|_A = \psi(T|_A)$. Using the last step We have

$$\begin{aligned}\psi(T|_A) &= \overline{\mathcal{D}'}\text{-gluing}\left[\psi((T|_A)|_V)|_{V\in B(T)}\right]\\ &= \overline{\mathcal{D}'}\text{-gluing}\left[(\psi((T)|_V)|_A)_{V\in B(T)}\right] = \psi(T)|_A.\end{aligned}$$

(v) Proving that for $i = 1, 2...$,the equality $\psi \circ D_i(T) = \overline{D}_i \circ \psi(T)$ holds for any $T \in \mathcal{D}'$: We have, using the last two step

$$\begin{aligned}\psi(D_i(T)) &= \overline{\mathcal{D}}'\text{-gluing}\left[\psi(D_i T|_V)_{V\in B(T)}\right] = \overline{\mathcal{D}}'\text{-gluing}\left[\overline{D}_i \circ \psi(T|_V)_{V\in B(T)}\right]\\ &= \overline{\mathcal{D}}'\text{-gluing}\left[(\overline{D}_i \circ \psi(T)|_V)_{V\in B(T)}\right] = \overline{D}_i \circ \psi(T).\end{aligned}$$

(vi) Uniqueness of ψ: Suppose that there exist two sheaf-morphisms ψ_1 and ψ_2 that verify the theorem. It follows that $\psi_1 \circ \lambda = \overline{\lambda} = \psi_2 \circ \lambda$. We have also $\psi_j \circ D_i = \overline{D}_i \circ \psi_j$ for $j = 1, 2$. It yields

$$\psi_1 \circ D^{\alpha} \circ \lambda(f) = \overline{D}^{\alpha} \circ \psi_1 \circ \lambda(f) = \overline{D}^{\alpha} \circ \psi_2 \circ \lambda(f) = \psi_2 \circ D^{\alpha} \circ \lambda(f) \tag{3.7}$$

which implies that ψ_1 and ψ_2 are equal on distribution of finite order. Assume that for some $T \in \mathcal{D}'(\Omega)$ and for some $V \subset \Omega$ we have

$$\psi_1(T)|_V \neq \psi_2(T)|_V.$$

Without loss of generality we can assume that $T|_V \in \mathcal{E}'(\Omega)$ otherwise we can find $V' \subseteq V$ such that $T|_{V'} \in \mathcal{E}'(\Omega)$. Since both ψ_1, ψ_2 are sheaf-morphisms, it follows that

$$\psi_1(T|_V) = \psi_2(T|_V)$$

which contradict 3.7. It follow that $\psi_1(T) = \psi_2(T)$. Hence, the sheaf-morphism ψ is unique.

□

Remark 7.

(i) Similarly we can prove that $\mathcal{D}'(\Omega)$ where Ω is any open set, satisfies a similar co-universal property. The proof is identically the same.

(ii) If we suppose only that $H : (\mathcal{O}\mathbb{R}^{\infty})^{\text{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ is a presheaf of $\mathbb{R}$-vector spaces in the lats theorem, then the co-universal solution would be $(\mathcal{D}'^{(F)}, \lambda, (D_i)_i)$, where $\mathcal{D}'^F$ is the space of distributions of finite order. The local structure theorem for distribution states that every distribution of finite order is a finite sum of derivative of continuous functions. Thus, the presheaf-morphism ψ in this case is given by 3.2.

(iii) Let $\mathcal{G}^s$ be a Colombeau AG algebras with some fixed gauge (see definition below), ι the embedding of continuous functions into $\mathcal{G}^s$ using convolution with Colombeau mollifier, and $(\overline{D}_i)_i$ are the derivatives of Colombeau generalized functions. Note that we have $\mathcal{P}(\mathcal{G}^s, \iota, \overline{D})$. (Reference) Thus if we take $(\overline{\mathcal{D}}, \overline{\lambda}, (\overline{D}_i)_i) = (\mathcal{G}^s, \iota, (\overline{D}_i)_i)$, then theorem 6 ensures the existence of a unique embedding of the space of Schwartz distributions into the space of Colombeau AG algebras. The same can be said for the space of generalized functions where the property $\mathcal{P}(\mathcal{GC}^\infty, \iota, \overline{D})$ is proved in [reference: theorem 25 of Topos 1].

3.0.1. *Application: manifold-valued distributions.* As we have mentioned in the last proof, the theorem on local structure of distributions states that any distribution is locally a derivative of continuous functions, i.e. $\forall T \in \mathcal{D}'(\Omega), \forall V \Subset \Omega : T|_V \in \mathcal{E}'(\Omega)$. In a certain sense, this theorem states that T is obtained by gluing together distributions of finite order. Therefor we would like to proceed as follows

(i) Define $\partial_V^\alpha f := [(\alpha, f)]_\sim$ using the ideas of Sebastião e Silva.
(ii) Obtain from (i) the separated presheaf P of distributions of finite order.
(iii) Consider the simplest sheaf $\overline{P}$ associated to P. Elements $T \in \overline{P}(U)$ will be called distributions on U.
(iv) Prove that $\overline{P}$ satisfies a universal property.

Let $T \in \mathcal{D}'(U)$ and consider

$$B := \{V \in Op(\mathbb{R}^n) | T|_V \in \mathcal{E}'(V)\} := B(T). \tag{3.8}$$

Then the local structure of distributions implies that $B(T)$ is a covering of U and T is obtained by gluing the family $(T|_V)_{V\in B}$ of $\mathcal{E}'$. This family is maximal one, in the sense that if $S \in \mathcal{E}'(W)$, $W \in Op(\mathbb{R}^n)$, and

$$S|_{W\cap V} = (T|_V)\,|_{W\cap V} = T|_{W\cap V} \qquad \forall V \in B \tag{3.9}$$

then gluing together $(T|_{W\cap V})_{V\in B}$, we get $T|_W$ and hence $S = T|_W$. Since $S \in \mathcal{E}'$, $W \in B$. Hence, every distribution of finite order S that locally belongs to the family $(T|_V)_{V\in B}$ (i.e it satisfies 3.8) is actually already an elements of this family. The distribution T can be identified with the maximal family $(T|_V)_{V\in B}$ of distribution of finite order. In fact gluing together the compatible family $(T|_V)_{V\in B}$ we return back to T; in other words

$$\mathcal{D}'(U) \simeq \{(T|_V)_{V\in B} | T \in \mathcal{D}'(U)\}. \tag{3.10}$$

where $T \longmapsto (T|_V)_{V\in B}$ and $(T|_V)_{V\in B} \overset{\text{gluing}}{\longmapsto} T$. Note that if $(U_j)_{j\in J}$ is another covering of U, we similarly have

$$\mathcal{D}'(U) \simeq \{(T|_{U_j})_{j\in J} | T \in \mathcal{D}'(U)\}$$

but we expect that 3.10 is the simplest (because it's maximal) way to associate to $T \in \mathcal{D}'(U)$ a compatible family of finite order distributions.

Actually, this is a general and universal way to associate a sheaf to a separated presheaf.

Definition 8. Let U, $W \in \mathbb{T}^{\text{op}}$, B is a covering of U, $S \in P(W)$, $(T_V)_{V\in B}$ a compatible family related to P. Then we say that $S \in_W (T_V)_{V\in B}$ and we say "S locally belongs to $(T_V)_{V\in B}$ on W" if and only if

$$\forall V \in B : P_{W,W\cap V}(S) = P_{V,W\cap V}(T_V).$$

Moreover, we say that $(T_V)_{V\in B}$ is a maximal family on U if and only if

(i) $(T_V)_{V\in B}$ is a compatible family

(ii) $\forall W\in \mathbb{T}^{\text{op}}\forall S\in P(W): S\in_W (T_V)_{V\in B}\Longrightarrow W\in B, S=T_W$.

The separateness of P is used in the following result, that allows us to consider the maximal family generated by a given compatible family. The idea is to consider all the section $S\in P(W)$ of the presheaf that locally belongs to the given family.

Theorem 9. *Let B be a covering of $U\in\mathbb{T}^{op}$ and let $(T_V)_{V\in B}$ be a compatible family. Set*

$$\mathcal{R}:=\{W\in\mathbb{T}^{op}|\exists S\in P(W): S\in_W (T_V)_{V\in B}\} \tag{3.11}$$

then, we have

(i) $\forall W\in\mathcal{R}\exists! S\in P(W): S\in_W (T_V)_{V\in B}$. *We denote by S_W this unique S.*

(ii) $\forall V\in B: V\in\mathcal{R},\ S_V=T_V$.

(iii) $(S_V)_{V\in\mathcal{R}}$ *is a maximal family on U.*

We'll use the notation $\max[(T_V)_{V\in B}]=(S_V)_{V\in\mathcal{R}}$.

We can now define the sheaf $\overline{P}$ on objects:

Definition 10. If $U\in\mathbb{T}^{\text{op}}$, we set $(T_V)_{V\in B}\in\overline{P}(U)$ if and only if

(i) B is a covering of U

(ii) $(T_V)_{V\in B}$ is a maximal family on U.

To eventually get an R-Mod (case of real-valued distribution), we also have to define module operations:

Definition 11. Let $U\in\mathbb{T}^{\text{op}}$, $r\in R$ and let $(T_V)_{V\in B}$, $(S_W)_{W\in C}\in\overline{P}(U)$. Then

(i) $(T_V)_{V\in B}+(S_W)_{W\in C}:=\max[(T_A+S_A)_{A\in B\cap C}]$

(ii) $r\cdot(T_V)_{V\in B}:=\max[(r\cdot T_V)_{V\in B}]$.

Using these operations, it's possible to prove that $(\overline{P}(U),+,\cdot)\in\mathbf{Mod}_R$.

We still use the symbol $\overline{P}(U)$ to denote this R-module. We finally define $\overline{P}$ on arrows.

Definition 12. Let U, $V\in\mathbb{T}^{\text{op}}$, $V\subseteq U$. Then

(i) $\mathcal{E}:=\{W\cap V|W\in\mathcal{C}\}$ where $\mathcal{C}$ is a covering of U.

(ii) $\overline{P}_{UV}:(T_W)_{W\in\mathcal{C}}\in\overline{P}(U)\longmapsto(P_{W,V\cap W}(T_W))_{W\in\mathcal{E}}\in\overline{P}(V)$.

The link between P and $\overline{P}$ is given by the following natural transformation

$$\eta_U: T\in P(U)\mapsto\max\left[(P_{UV}(T))_{V\in Op(U)}\right]\in\overline{P}(U)$$

where $Op(U)$ is the induced topology on U.

With these definitions, we have the following universal property:

Theorem 13. *If $P:\mathbb{T}^{op}\longrightarrow R$-**Mod** is separated then*

(i) $\overline{P}:\mathbb{T}^{op}\longrightarrow R$-**Mod** *is a sheaf*

(ii) $\eta: P\longrightarrow\overline{P}$ *is a natural transformation*

(iii) $(\overline{P},\eta)$ *is co-universal among all $(\overline{P},\eta)$ that satisfy (i), (ii).*

Example 14. [The idea of Sebastião e Silva] Let r, $s\in\mathbb{N}$, J is an open set of $\mathbb{R}$ and f, $g\in C^0(J)$ two real-valued continuous functions on J. The space $C^k(J)$ is the space of k times continuously differentiable functions on J. For every $(r,f)\in\mathbb{N}\times C^0(J)$, we denote by F_r the map that belongs to $C^r(J)$ and verifies $D^rF_r=f$.

Let $(s,g) \in \mathbb{N} \times C^0(J)$. We write $(r,f) \sim (s,g)$ (intuitively $D^r f = D^s g$ in the sense of distributions) if setting $m = \max(r,s)$ we have $F_{m-r} - G_{m-s} \in \mathbb{R}_{m-1}[x]$, i.e. is a polynomial of degree less or equal to $m-1$.

We check now that $\sim$ is an equivalent relation; The Reflexivity and the symmetry are easy to check and we will prove only the transitivity. Let (r,f), (s,g) $(t,h) \in \mathbb{N} \times C^0(J)$ such that $(r,f) \sim (s,g)$ and $(s,g) \sim (t,h)$. Setting $m = \max(r,s)$ and $m' = \max(s,t)$ then we have

$$(r,f) \sim (s,g) \Longrightarrow F_{m-r} - G_{m-s} \in \mathbb{R}_{m-1}[x]$$

and

$$(s,g) \sim (t,h) \Longrightarrow G_{m'-s} - H_{m'-t} \in \mathbb{R}_{m'-1}[x].$$

Setting now $n = \max(m,m')$ then

$$F_{m-r} - G_{m-s} \in \mathbb{R}_{m-1}[x] \Longrightarrow F_{n-r} - G_{n-s} \in \mathbb{R}_{n-1}[x],$$

and

$$G_{m'-s} - H_{m'-t} \in \mathbb{R}_{m'-1}[x] \Longrightarrow G_{n-s} - H_{n-t} \in \mathbb{R}_{n-1}[x].$$

It follows that $F_{n-s} - H_{n-t} \in \mathbb{R}_{n-1}[x]$ which implies that $(r,f) \sim (t,h)$.

We consider now $P(J) := (\mathbb{N} \times C^0(J))/\sim$ and the equivalence class of $(r,f) \in \mathbb{N} \times C^0(J)$ will be denoted by $[(r,f)]$.

The space $P(J)$ can be equipped by a structure of an $\mathbb{R}$-module. For $[(r,f)]$, $[(s,g)] \in P(J)$, we define the sum $[(r,f)] + [(s,g)]$ in the following way:

$$\begin{aligned}
[(r,f)] + [(s,g)] = [(\max(r,s), F_{m-r})] &+ [(\max(r,s), G_{m-s})] \\
&= [(\max(r,s), F_{m-r} + G_{m-s})].
\end{aligned}$$

The multiplication by scalar $l \in \mathbb{R}$ is given by $l \times [(r,f)] = [(r, lf)]$.

We define now the restriction operator: If V is another open set of $\mathbb{R}$ such that $V \subseteq J$ then the restriction $P_{JV} : P(J) \longrightarrow P(V)$ from J to V is given by $P_{JV}([(r,f)] = [(r, C^0_{JV}(f))]$ where $C^0_{JV} : C^0(J) \longrightarrow C^0(V)$ is the restriction on V of continuous function on J. Note that P_{JV} does not depend on r. This is because the derivative is a morphism of presheaf. More precisely, let $(s,g) \in \mathbb{N} \times C^0(J)$ be such that $[(r,f)] = [(s,g)]$ and we will prove that $[(r, C^0_{JV}(f))] = [(s, C^0_{JV}(g))]$. The equality $[(r,f)] = [(s,g)]$ implies that $F_{m-r} - G_{m-s} \in \mathbb{R}[x]$. In the other hand, one can check that

$$D^{m-r} C^0_{JV}(F_{m-r}) = C^0_{JV} D^{m-r} F_{m-r} = C^0_{JV}(f) \text{ and } D^{m-s} C^0_{JV}(G_{m-s}) = C^0_{JV}(g).$$

Then

$$\begin{aligned}
F_{m-r} - G_{m-s} \in \mathbb{R}[x] &\Longrightarrow C^0_{JV}(F_{m-r}) - C^0_{JV}(G_{m-s}) \in \mathbb{R}[x] \\
&\Longrightarrow [(r, C^0_{JV}(f))] = [(s, C^0_{JV}(g))].
\end{aligned}$$

It follows that $P : Op(\mathbb{R}) \longrightarrow \mathbb{R}\text{-}\mathbf{Mod}$ is a natural transformation.

We claim now that $P(J)$ is separated. Indeed, let $(U_i)_i$ be a covering of J and let $T = [(r,f)]$, $S = [(s,g)] \in P(J)$ be such that $P_{JU_i}(T) = P_{JU_i}(S)$ for every i, which implies that $[(r, C^0_{JU_i}(f)] = [(s, C^0_{JU_i}(g))]$ for every i. Then $C^0_{JU_i}(F_{m-r}) - C^0_{JU_i}(G_{m-s}) \in \mathbb{R}_{m-1}[x] \forall i$. Thus $F_{m-r} - G_{m-s} \in \mathbb{R}_{m-1}[x]$ which implies that $T = S$ and thus $P(J)$ is separated.

According to theorem 13, we can obtain a sheaf $\overline{P}(J)$ from the separated presheaf $P(J)$. Elements of $\overline{P}(J)$ are elements of the form $(T_V)_{V \in B} = [(r_V, f|_V)]_{V \in B}$ where B is a covering of J and $(T_V)_V$ is a maximal family on J.

If we consider the triple $(\overline{P}, \overline{\lambda}, \overline{D})$ where $\overline{\lambda} : C^0 \longrightarrow \overline{P}$ is a sheaf-morphism defined for every $f \in C^0$ by $\overline{\lambda}(f) = \eta([(0, f)]) \in \overline{P}$ and $\overline{D} : \overline{P} \longrightarrow \overline{P}$ defined by $\overline{D} \circ \eta([(r, f)]) = \eta_U([r+1, f)])$. If we denote by $\iota : C^1 \longrightarrow C^0$ then we can easily check that for every $f \in C^1$ we have

$$\overline{D} \circ \overline{\lambda} \circ \iota(f) = \overline{D} \circ \eta([(0, f)]) = \eta([(1, f)]) = \eta([(0, \partial f)]) = \overline{\lambda} \circ \partial f$$

which proves that $\overline{D}$ is compatible with the derivative of C^1 functions. Therefore, by theorem 6 there exists a unique sheaf morphism $\psi : \mathcal{D}' \longrightarrow \overline{P}$ such that $\overline{\lambda} = \psi \circ \lambda$ and $\overline{D} \circ \psi = \psi \circ D$.

In the other hand, we consider the natural transformation $\overline{\eta} : P \longrightarrow \mathcal{D}'$ defined as follow: Let $(r_V, f|_V)_{V \in B} \in P(U)$ where B is a covering of U. We set $\overline{\eta}\left((r_V, f|_V)_{V \in B}\right) := \mathcal{D}'\text{-gluing}\left((D^{r_V} f|_V)_{V \in B}\right)$. Thus the couple $(\mathcal{D}', \overline{\eta})$ satisfies (i)-(ii)of theorem 13 and since $(\eta, \overline{P})$ is co-universal, then there exists $\overline{\psi} : \overline{P} \longrightarrow \mathcal{D}'$ a sheaf morphism. Therefore, $\mathcal{D}'$ and $\overline{P}$ are isomorphic.

The first generalization of this idea is hence with respect to the domain and the co-domain.

Let $(E, ||\cdot||)$ be a normed space over $\mathbb{R}$ and let U be an open set. Recall that a continuous map $f : U \longrightarrow E$ is said to be Fréchet differentiable at $x \in U$ if there exists a continuous linear map $d_x f : E \longrightarrow E$, $d_x f : h \longrightarrow d_x f(h)$ such that

$$\lim_{||h|| \to 0} \frac{||f(x+h) - f(x) - d_x f(h)||}{||h||} = 0.$$

The operator $d_x f$ when it exists, is unique and it's called the Fréchet derivative of f at x. Note that if E is an algebra and $f : E \longrightarrow E$, $f(x) = x^n$ then it is easy to check that $d_x f = n x^{n-1}$.

Let's now generalize the construction of Sebastião e Silva on an open set U of a Banach space E. In the case of continuous functions on an open set of $\mathbb{R}^d$, the functions F_r always exists since every continuous function has a primitive. In the case of $C(U)$, we don't know if this property holds. So, instead of $C(U)$ we will work on $D(U)$ the subspace of $C(U)$ such that if $f \in D(U)$ then for every $r \in \mathbb{N}$, there exists $F_r \in C^r(U) \cap D(U)$ such that $d^r F_r = f$.

Given r, $s \in \mathbb{N}$ and f, $g \in D(U)$. We write $(r, f) \sim (s, g)$ if setting $m = \max(r, s)$ we have $F_{m-r} - G_{m-s} \in E_{m-1}[x]$. We set $P(U) := (\mathbb{N} \times D(U))/\sim$. Indeed, $(P(U), +, \times)$ is an $\mathbb{R}$-module where the sum and the product are defined exactly as in the previous case. We have also that P is a separated presheaf of R-**Mod** over the topological space E where the restriction $P_{UV} : P(V) \longrightarrow P(V)$, $(V \subseteq U)$ is given by $P_{UV}[(r, f)] = [(r, D_{UV}(f)]$ where $D_{UV} : D(U) \longrightarrow D(V)$ is restriction operator, i.e. $D_{UV}(f) = f|_V$ and the separateness property can be proved exactly as in the previous case.

According to theorem 13, for every open set U of E, we can obtain a sheaf $\overline{P}(U)$ from $P(U)$.

We can repeat the same construction as above and change the derivative operator by an arbitrary operator. Let's fix the idea: Let R be a ring and let E be an R-module. Let $T : E \longrightarrow E$ be a R-linear operator. For an open set $U \subseteq E$, $D(U)$ is a space that has the following property: $f \in D(U)$ if and only if $\forall n \in \mathbb{N}$ there exists $F_n : U \longrightarrow E$ such that $T^n F_n = f$ and if $f = 0$ then $F_n = 0$ for every n.

Let now (r, f), $(s, g) \in \mathbb{N} \times D(U)$. Similarly, we write $(r, f) \sim (s, g)$ (intuitively $T^r f = T^s g$) if and only if $F_{m-r} = G_{m-s}$ where $m = \max(r, s)$. We consider now

$P(U) := (\mathbb{N} \times D(U)) / \sim$. The equivalence class of (r, f) will be denoted by $[r, f]$. We can equip $P(U)$ by a structure of a module where for every $[r, f]$, $[s, g] \in P(U)$ and for every $l \in R$ we set

$$[r, f] + [s, g] = [m, F_{m-r}] + [m, G_{m-s}] = [m, F_{m-r} + G_{m-s}]$$

and

$$l \times [r, f] = [r, lf].$$

Thus $(P(U), +, \times)$ is a R-module. Assume now that for every $r \in \mathbb{N}$ and for every $f \in D(U)$, we have

$$T^r(F_r|_V) = T^r F_r|_V.$$

Then $(P(U), +, \times)$ is a separated presheaf. Theorem 13 implies that we can obtain a sheaf $\overline{P}(U)$ from $P(U)$.

Now we generalize the idea to a Banach manifold. We start with some basic definitions.

[from wikipedia: Slightly reformulated] Similarly to a n-dimensional manifold in which any point has an neighborhood homeomorphic to an open set of $\mathbb{R}^n$, a Banach manifold is a manifold modeled on Banach space. Thus it is a topological space in which each point has a neighborhood homeomorphic to an open set in a Banach space. Banach manifolds are one possibility of extending manifolds to infinite dimensions.

A further generalization is to Fréchet manifolds, replacing Banach spaces by Fréchet spaces. On the other hand, a Hilbert manifold is a special case of a Banach manifold in which the manifold is locally modeled on Hilbert spaces.

Given two Banach spaces E and F and $r \in \mathbb{N}_{\geq 1}$. The space $\text{lin}(E^r, F)$ will denote the space of linear maps from E^r into F.

Definition 15. Let X be a set. An atlas of class C^r, $r \in \mathbb{N} \cup \{\infty\}$, on X is a collection of pairs called charts (U_i, φ_i), $i \in I$, such that

(i) each U_i is a subset of X and the union of U_i is the whole of X;

(ii) each φ_i is a bijection from U_i to an open set $\varphi_i(U_i)$ of some Banach space E_i, and for any indices i and j, $\varphi_i(U_i \cap U_j)$ is open of E_i;

(iii) The crossover map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is an r-times continuously differentiable function for every i, $j \in I$; that is, the rth Fréchet derivative

$$d^r(\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \longrightarrow \text{Lin}(E_i^r, E_j)$$

exists and it's continuous with respect to the E_i-norm topology on subsets of E_i and the operator norm topology on $\text{lin}(E_i^r, F_j)$.

If all the Banach spaces E_i are equal to the same space E, the atlas is called an E-atlas. X is then called a C^r E-manifold

Indeed, (on can show that) there exists a unique topology on X such that each U_i is open and each φ_i is a homeomorphism. For the sequel, we assume that this topology is Hausdorff.

Definition 16. Let E and F be two Banach spaces and let M be a $C^\infty E$-manifold and let N be a C^∞ F-manifold. A map $f : M \longrightarrow N$ is said to be in $C^r(M, N)$

if and only if for every φ_i of the atlas $\{\varphi_i, U_i\}_i$ of M, for every ψ_j of the atlas $\{\psi_j, V_j\}_j$ of N the map

$$\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \psi_j(V_j)$$

is in $C^r(E, F)$ with respect to E-norm topology and the F-norm topology.

[I will finish this later] Let M be a C^∞ E-manifold and let U be an open set of M. Let $\mathcal{A} = \{\varphi_i, V_i\}$ be an atlas of M. We denote by $\tilde{U}$ the open set of E given by

$$\tilde{U} := \cup_i \varphi_i(V_i \cap U).$$

In the following remarks I will state some difficulties to apply the idea of Sebasito

Remark. I start by recalling how do we obtain a distribution on an open set U of a manifold M of finite dimension. Assume that, $\varphi : U \longrightarrow V \subset \mathbb{R}^n$ is a diffeomorphism of class C^∞. Let $f \in \mathcal{D}'(V)$ and let $(f_n)_n \in C^0(V)$ be a sequence of continuous functions that converges to f (in the sense of distribution. Then $\varphi^* f = \lim \varphi^* f_n$ where $\varphi^* f_n = f_n \circ \varphi$ which is well defined (as a distribution on U) and the limit exists in the sens of $\mathcal{D}'(U)$.

In our case we define distributions using the sheafification of a separated presheaf. So if we assume that M is C^∞ E-manifold, U is an open set of M and $\varphi : U \longrightarrow V \subset E$ is a diffeomorphism of class C^∞. We denote by $P(V)$ the separated presheaf as defined above. So, an element $f \in P(V)$ is a compatible family $[(r, f)]$.

The first problem is how define the pullback of a class $[(r, f)]$ by φ. If $r = 0$ then we can set

$$\varphi^*[(0, f)] = [(0, f \circ \varphi)].$$

If for example $r = 1$ and $f \in C^1(V)$ then $(1, f)$ refers to the continuous functions $d^1 f$ and $\varphi^*(r, f) = (d^r f) \circ \varphi$ ad this is different from $(1, f \circ \varphi)$.

4. Colombeau AG algebras

4.1. **Co-universal property as quotient of moderate nets.** The first idea to find a co-universal property for Colombeau AG algebras is to formulate the classical co-universal property of every quotient, only at a "higher level", i.e. talking of functors of $\mathbb{R}$-algebras and natural transformations instead of algebras and their morphisms. In the following, any net $\rho = (\rho_\varepsilon)_\varepsilon \in \mathbb{R}^I_{>0}$ such that $\rho_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ will be called a gauge and the set $Ag(\rho^{-1}) := \{(\rho_\varepsilon^{-a})_\varepsilon \in \mathbb{R}^{(0,1]} \mid a \in \mathbb{R}_{>0}\}$ will be called the asymptotic gauge generated by ρ.

Definition 17. Let Ω be an open set of any dimension. The Colombeau AG algebras is defined by the quotient $\mathcal{G}^s(AG(\rho^{-1}), \Omega) := \mathcal{E}_M(AG(\rho^{-1}), \Omega)/\mathcal{N}(AG(\rho^{-1}), \Omega)$ where

$$\mathcal{E}_M(AG(\rho^{-1}), \Omega) := \left\{ (u_\varepsilon) \in \mathcal{C}^{\infty I}(\Omega) | \forall K \Subset \Omega \forall \alpha \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O\left(\rho_\varepsilon^{-N}\right) \right\}$$

$$\mathcal{N}(AG(\rho^{-1}), \Omega) := \left\{ (u_\varepsilon) \in \mathcal{C}^{\infty I}(\Omega) | \forall K \Subset \Omega \forall \alpha \forall N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O\left(\rho_\varepsilon^{N}\right) \right\}.$$

The equivalence class of $(u_\varepsilon)_\varepsilon \in M(\Omega)$ in $\mathcal{G}^s(AG(\rho^{-1}), \Omega)$ is denoted by $[u_\varepsilon]_\rho$.

Definition 18. Let $I = (0, 1]$. We denote by $\mathbf{Col}_I$ the category of Colombeau and we write $(G, \pi) \in \mathbf{Col}_I$ if

(i) $G : (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}} \longrightarrow \mathbf{ALG}_\mathbb{R}$ is a functor, where $\mathbf{ALG}_\mathbb{R}$ denote the category of $\mathbb{R}$-Algebras
(ii) $\pi : \mathcal{E}_M(AG(\rho^{-1}), -) \longrightarrow G$ is a natural transformation such that $\mathcal{N}(AG(\rho^{-1}), \Omega) \subseteq Ker(\pi_\Omega)$, $\forall \Omega \in \mathcal{O}\mathbb{R}^\infty$.

Moreover, we write $(G, \pi) \xrightarrow{\tau} (F, \alpha)$ in $\mathbf{Col}_I$ if and only if the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_M(AG(\rho^{-1}), -) & \xrightarrow{\alpha} & F \\ \big\downarrow \pi & \nearrow_{\tau} & \\ G & & \end{array}$$

Theorem 19. *For every $(G, \pi) \in \mathbf{Col}_I$, there exist a unique $\tau : (\mathcal{G}^s(AG(\rho^{-1}), -), [-]_\rho) \longrightarrow (G, \pi)$ in $\mathbf{Col}_I$, i.e. $(\mathcal{G}^s(AG(\rho^{-1}), -), [-]_\rho)$ is co-universal in $\mathbf{Col}_I$.*

Proof. We should find τ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_M(AG(\rho^{-1}), -) & \xrightarrow{\pi} & G \\ \big\downarrow [-]_\rho & \nearrow_{\tau} & \\ \mathcal{G}^s(AG(\rho^{-1}), -) & & \end{array}$$

The only way τ can be defined is by setting $\tau_\Omega([u_\varepsilon]_\rho) = \pi_\Omega((u_\varepsilon)), \forall \Omega \in \mathcal{O}\mathbb{R}^\infty$. In order to prove that τ_Ω is well defined, i.e. it doesn't depend on the representative (u_ε) of $[u_\varepsilon]_\rho$, take two moderate nets $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ such that $[u_\varepsilon]_\rho = [v_\varepsilon]_\rho$, then we have $\tau_\Omega([u_\varepsilon]_\rho) = \pi_\Omega((u_\varepsilon)) = \pi_\Omega((v_\varepsilon + (u_\varepsilon - v_\varepsilon))) = \pi_\Omega((v_\varepsilon)) + \pi_\Omega((u_\varepsilon - v_\varepsilon))$ because for every Ω, π_Ω is an algebra-homomorphism. Since we have $\mathcal{N}(AG(\rho^{-1}), \Omega) \subseteq Ker(\pi_\Omega)$, we have $\tau_\Omega([u_\varepsilon]_\rho) = \pi_\Omega((u_\varepsilon)) = \pi_\Omega((v_\varepsilon)) = \tau_\Omega([v_\varepsilon]_\rho)$. ☐

This simple co-universal property highlights the following data:

4.2. **Co-universal properties as quotient algebras.** We want to show another co-universal property of Colombeau AG algebras by completing the idea that *a Colombeau algebra is a quotient of $\mathcal{C}^\infty(-)^I$, and we are forced to consider moderate and negligible nets because...*

Definition 20. . We say that (G, π) is a *quotient algebra of* $\mathcal{C}^\infty(-)^I$, and we write $(G, \pi) \in \mathrm{QALG}(\mathcal{C}^{\infty I})$, if

(i) $G : (\mathcal{O}\mathbb{R}^\infty)^{\mathrm{op}} \longrightarrow \mathbf{ALG}_\mathbb{R}$ is a functor
(ii) $\pi : M \longrightarrow G$ is a natural transformation such that $\mathcal{C}^\infty(\Omega)^I \supseteq M(\Omega)$ and $\pi_\Omega : M(\Omega) \longrightarrow G(\Omega)$ is an epimorphism of $\mathbb{R}$-algebras for all $\Omega \in \mathcal{O}\mathbb{R}^\infty$.

Let's discover the properties we need. Since for every $\Omega \in \mathcal{O}\mathbb{R}^\infty$, π_Ω is an algebra homomorphism, then we have for any $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in M(\Omega)$ and for any $r \in \mathbb{R}$

(i) $\pi_\Omega(u_\varepsilon) + \pi_\Omega(v_\varepsilon) = \pi_\Omega(u_\varepsilon + v_\varepsilon)$
(ii) $\pi_\Omega(u_\varepsilon) \cdot \pi_\Omega(v_\varepsilon) = \pi_\Omega(u_\varepsilon \cdot v_\varepsilon)$
(iii) $r \cdot \pi_\Omega(u_\varepsilon) = \pi_\Omega(r \cdot u_\varepsilon)$.

Moreover, the condition "π_Ω is an epimorphism" means that for every $g \in G(\Omega)$, there exist $(u_\varepsilon)_\varepsilon \in M(\Omega)$ such that $\pi_\Omega(u_\varepsilon) = g$. This implies

$$G(\Omega) \simeq M(\Omega)/Ker(\pi_\Omega) \text{ in } \mathbf{ALG}_{\mathbb{R}}.$$

The isomorphic associates to any class $[u_\varepsilon] \in M(\Omega)/Ker(\pi_\Omega)$, the element $\pi_\Omega(u_\varepsilon)$. In order to prove the injectivity, suppose that for $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in M(\Omega)$, we have $\pi_\Omega(u_\varepsilon) = \pi_\Omega(v_\varepsilon)$. Using properties (i) and (ii) we obtain $\pi_\Omega(u_\varepsilon - v_\varepsilon) = 0$, which implies that $[u_\varepsilon] = [v_\varepsilon]$.

Why we are forced to consider moderate nets? Let $(z_\varepsilon)_\varepsilon \in M(\Omega)$ be such that $\pi_\Omega(z_\varepsilon) = 0$ and consider $J_\varepsilon \in M(\Omega) \cap \mathbb{R}^I$ such that $\lim_{\varepsilon \longrightarrow 0^+} |J_\varepsilon| = +\infty$, then we have

$$\pi_\Omega(z_\varepsilon) \cdot \pi_\Omega(J_\varepsilon) = 0 \cdot \pi_\Omega(J_\varepsilon) = \pi_\Omega(z_\varepsilon \cdot J_\varepsilon). \tag{4.1}$$

Let's assume that: If $(w_\varepsilon)_\varepsilon \in M(\Omega)$ such that $\pi_\Omega(w_\varepsilon) = 0$, then for every $K \Subset \Omega$ and for every multi-index α we have $\sup_{x \in K} |\partial^\alpha w_\varepsilon(x)| := p_{K,\alpha}(w_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$. We say then that every zero-net in (G, π) is a potential infinitesimal (with respect to $p_{K,\alpha}$). Equation 4.1 implies that for every $K \Subset \Omega$ and multi-index α we have $p_{K,\alpha}(z_\varepsilon \cdot J_\varepsilon) = p_{K,\alpha}(z_\varepsilon) \cdot |J_\varepsilon| \longrightarrow 0$ which implies that $\forall^0 \varepsilon : p_{K,\alpha}(z_\varepsilon) \le |J_\varepsilon|^{-1}$. Set

$$\text{Infinite}(M(\Omega)) := \left\{ (|J_\varepsilon|) \mid (J_\varepsilon) \in M(\Omega) \cap \mathbb{R}^I,\ \lim_{\varepsilon \longrightarrow 0^+} |J_\varepsilon| = +\infty \right\}. \tag{4.2}$$

We have two possibilities:

(i) $M(\Omega)$ contains all the infinite nets. This implies that $\forall K \Subset \Omega$ and for all multi-index α we have $p_{K,\alpha}(z_\varepsilon) = 0$ for every ε small. In this case, the quotient must be trivial and this situation corresponds to the Schmieden-Laugwitz-Egorov model.

(ii) $M(\Omega)$ doesn't contain all the infinite nets.

The following theorem shows that if a non empty subalgebra of $\mathbb{R}^I$ is bounded by a gauge then it has a maximal gauge.

Theorem 21. *Let $\mathcal{R} \subseteq \mathbb{R}^I$ be a subring such that inf($\mathcal{R}$), the set of all the infinite nets of $\mathcal{R}$, is non empty. Assume that there exist a gauge σ such that Infinite($\mathcal{R}$) $\le AG(\sigma^{-1})$. Then there exist a gauge ρ such that $\rho^{-1} \in \mathcal{R}$ and inf($\mathcal{R}$) $\le AG(\rho^{-1})$. We say that $\mathcal{R}$ has a maximal asymptotic gauge.*

By Infinite($\mathcal{R}$) $\le Ag(\sigma^{-1})$ we mean that for every $(v_\varepsilon) \in$ Infinite($\mathcal{R}$), there exist $a \in \mathbb{R}_{>0}$ such that $\forall^0 \varepsilon : v_\varepsilon \le \sigma_\varepsilon^{-a}$.

Proof. If $\sigma^{-1} \in \mathcal{R}$, then we have nothing to do. We suppose now that $\sigma^{-1} \notin_s \mathcal{R}$. First, note that

$$\forall m \in \mathbb{R},\ \forall k \in \mathbb{R}_{>0}, \forall a \in \mathbb{R}_{>0} : AG(\rho^{-1}) = AG(k\rho^{-1}) = AG((\rho^{-1}+m)^{-1}) = AG(\rho^{-a}). \tag{4.3}$$

The inclusions $AG(\rho^{-1}) \subseteq AG(k\rho^{-1}) \subseteq AG((\rho^{-1} + m)^{-1})) \subseteq AG(\rho^{-a})$ are a consequence of

$$\forall^0 \varepsilon : \rho_\varepsilon^{-1} \le (\rho_\varepsilon^{-1} + m)^2 \le (k\rho_\varepsilon)^{-3} \le (\rho_\varepsilon)^{-4a},$$

whereas the opposite inclusions follow from

$$\forall^0 \varepsilon : \rho_\varepsilon^{-a} \le (k\rho_\varepsilon)^{-a-1} \le (\rho_\varepsilon^{-1} + m)^{a+2} \le (\rho_\varepsilon)^{-a-3}.$$

4.3 leads us to consider the infinite nets ρ^{-1}, $(k\rho)^{-1}$, $\rho^{-1}+m$, ρ^{-a} as the same "object" in $\inf(\mathcal{R})$. To do so, we define the equivalence relation $\sim$ on $\mathbb{R}^I$ by

$$x := (x_\varepsilon) \sim (y_\varepsilon) =: y \iff AG(x) = AG(y).$$

We denote by $\mathcal{K} = \inf(\mathcal{R})/\sim$. $\mathcal{K}$ is non empty ordered set. We claim that is inductive. Indeed, let $\mathcal{Q} = \{AG(\rho_i^{-1})\}_i \subset \mathcal{K}$ be totally ordered subset. Assume that $AG(\rho_i^{-1}) \le AG(\rho_j^{-1})$ when $i \le j$. Using 4.3 we can assume that

$$\forall i, j \text{ if } i \le j \text{ then } \forall^0 \varepsilon : \rho_{i\varepsilon}^{-1} \le \rho_{j\varepsilon}^{-1}.$$

Indeed, for $i \le j$, the inequality $AG(\rho_i^{-1}) \le AG(\rho_j^{-1})$ implies the existence of $a > 0$ such that $\forall^0 \varepsilon : \rho_{i\varepsilon}^{-1} \le (\rho_{j\varepsilon}^a)^{-1}$ and from 4.3 we have $AG(\rho_j^{-1}) \le AG(\rho_j^{-a})$. Therefore we have

$$\exists \eta \in (0,1] \forall \varepsilon \in (0,\eta] \forall i, j \in J, i \le j \Longrightarrow \rho_{i\varepsilon}^{-1} \le \rho_{j\varepsilon}^{-1} \le \sigma_\varepsilon^{-1}.$$

Thus, for every ε small, the sequence $(\rho_{i\varepsilon}^{-1})_{i\in J}$ is increasing and bounded, hence it converge. Setting $\rho^{-1} = [\rho_\varepsilon^{-1}] = [\sup_j \rho_{j\varepsilon}^{-1}] \in \mathcal{R}$. We may therefore apply Zorn's lemma, and so we have a maximal asymptotic gauge $Ag(\rho^{-1})$ in $\mathcal{K}$ and thus in $Inf(\mathcal{R})$. □

Definition 22. Let (G,π), $(H,\eta) \in \mathrm{QALG}(\mathcal{C}^\infty(-)^I)$ and assume that π and η have the same domain $M(\Omega)$ which is a subalgebra of $\mathcal{C}^\infty(-)^I$. Then a *morphism of quotient algebras* is given by :

$$j : \mathrm{Ker}(\pi_\Omega) \hookleftarrow \mathrm{Ker}(\eta_\Omega), \quad \forall \Omega \in \mathcal{O}\mathbb{R}^\infty. \tag{4.4}$$

Theorem 23. *Quotient algebras of $\mathcal{C}^\infty(-)^I$ and their morphisms form a category.*

Proof. $\mathrm{QALG}(\mathcal{C}^\infty(-)^I)$ and its morphisms is an ordered set. Therefore it's a category. □

Therefore, a co-universal quotient algebra has the largest kernel.

Remark 24. Given (G,π), $(H,\eta) \in \mathrm{QALG}(\mathcal{C}^\infty(-)^I)$. We can give a more general definition of morphism of quotient by assuming that π and η have a two different domains. A *morphism of quotient algebras* (i,j) is given by :

$$i : Inf(M_1(\Omega)) \hookrightarrow Inf(M_2(\Omega)), \quad j : \mathrm{Ker}(\pi_\Omega) \hookleftarrow \mathrm{Ker}(\eta_\Omega), \quad \forall \Omega \in \mathcal{O}\mathbb{R}^\infty.$$

where $M_1(\Omega)$,$M_2(\Omega)$ are two different subalgebras of $\mathcal{C}^\infty(-)^I$ and we have $G \simeq M_1(\Omega)/ker(\pi_\Omega)$ and $H \simeq M_2(\Omega)/ker(\eta_\Omega)$. In this case, a co-universal quotient algebra has the smallest class of infinities and the largest kernel. But for any given subalgebra $M(\Omega)$ we can always find another subalgebra that have a smaller class of infinities. Thus, the co-universal quotient algebra could (does) not exist.

Theorem 25. *Let $M(\Omega) \subseteq \mathcal{C}^\infty(-)^I$ be a subalgebra. Assume that*

(i) *Infinite($M(\Omega) \cap \mathbb{R}^I$) is non-empty and bounded by an asymptotic gauge $Ag(\sigma^{-1})$*
(ii) $\forall (u_\varepsilon) \in M(\Omega), \forall K \Subset \Omega, \forall \alpha : p_{K,\alpha}(u_\varepsilon) \in M(\Omega) \cap \mathbb{R}^I$
(iii) *Let $(u_\varepsilon) \in \mathcal{C}^\infty(-)^I$. If $\forall K \Subset \Omega, \forall \alpha$, there exist $(v_\varepsilon) \in$ Infinite($M(\Omega) \cap \mathbb{R}^I$) such that $\forall^0 \varepsilon : p_{K,\alpha}(u_\varepsilon) \le v_\varepsilon$, then $(u_\varepsilon) \in M(\Omega)$*
(iv) *If $(u_\varepsilon) \in M(\Omega) \cap \mathbb{R}^I$, then $\exists (v_\varepsilon) \in$ Infinite($M(\Omega) \cap \mathbb{R}^I$), $\forall^0 \varepsilon : u_\varepsilon \le v_\varepsilon$.*

Then there exists a gauge ρ such that

$$M(\Omega) = \mathcal{E}_M(Ag(\rho^{-1}), \Omega). \tag{4.5}$$

Proof. First, note that we have

$$M(\Omega) = \mathcal{E}_M(\text{Infinite}(M(\Omega) \cap \mathbb{R}^I), \Omega).$$

In fact, assumptions (ii) and (iv) imply that $\forall (u_\varepsilon) \in M(\Omega), \forall K \Subset \Omega, \forall \alpha, \exists (v_\varepsilon) \in \text{Infinite}(M(\Omega) \cap \mathbb{R}^I)$ such that $\forall^0 \varepsilon : p_{K,\alpha}(u_\varepsilon) \le v_\varepsilon$. It follows that $M(\Omega) \subseteq \mathcal{E}_M(\text{Infinite}(M(\Omega) \cap \mathbb{R}^I), \Omega)$. The opposite inclusion is given by assumption (iii). Since M(Ω) is a subalgebra and $M(\Omega) \cap \mathbb{R}^I$ is non-empty and bounded by an asymptotic gauge $Ag(\sigma^{-1})$, theorem 21 applied for $\mathcal{R} = M(\Omega) \cap \mathbb{R}^I$ ensures the existence of a gauge ρ such that $\rho^{-1} \in M(\Omega) \cap \mathbb{R}^I$ and $\text{Infinite}(M(\Omega) \cap \mathbb{R}^I) \le Ag(\rho^{-1})$. This gives $\mathcal{E}_M(\inf(M(\Omega) \cap \mathbb{R}^I), \Omega) = \mathcal{E}_M(Ag(\rho^{-1}), \Omega)$ and thus we have 4.5. □

Theorem 26. *Let $M(\Omega) = \mathcal{E}_M(Ag(\rho^{-1}), \Omega)$. Then $(\mathcal{G}^s(AG(\rho^{-1}), \Omega), [-]_\rho)$ is co-universal among all the quotient differential algebras of $M(\Omega)$ such that every zero-net is a potential infinitesimal.*

By "differential" we mean the existence of a suitable operators $(D_i)_i$ verifying for all i, $D_i : G \longrightarrow G$ and $D_i \circ \pi = \pi \circ \partial_i$ where $(\partial_i)_i$ are the usual partial derivatives.

Proof. It's clear that $(\mathcal{G}^s(AG(\rho^{-1}), \Omega), [-]_\rho)$ is a quotient differential algebras where $D^\alpha [u_\varepsilon]_\rho = [\partial^\alpha u_\varepsilon]_\rho$ for every $(u_\varepsilon) \in M(\Omega)$. The fact that every zero-net is a potential infinitesimal follows directly from the definition of $\mathcal{N}(AG(\rho^{-1}), \Omega)$. We claim now that $[-]_\rho$ has the largest kernel. Indeed, let (G, π) be another quotient differential algebra of $M(\Omega)$ such that every zero-net is a potential infinitesimal and suppose that $Ker(\pi) \backslash Ker[-]_\rho \neq \emptyset$. Thus

$$\exists (u_\varepsilon) \in M(\Omega), \exists K \Subset \Omega, \exists \alpha, \exists n \in \mathbb{N}, \exists L \subseteq_0 I, \forall \varepsilon \in L : p_{K,\alpha}(u_\varepsilon) > \rho_\varepsilon^n \text{ and } \pi(u_\varepsilon) = 0. \tag{4.6}$$

Inequality 4.6 implies

$$\forall \varepsilon \in L \exists x_\varepsilon \in K : \rho_\varepsilon^{-n} |\partial^\alpha (u_\varepsilon(x_\varepsilon))| > 1 \Longrightarrow \rho_\varepsilon^{-n} |e_L(\varepsilon) \partial^\alpha (u_\varepsilon(x_\varepsilon))| > e_L(\varepsilon)$$

π is a sheaf morphism because it's a natural transformation between two sheaves and it's an order-preserving map because $M(\Omega)$ and $G(\Omega)$ are partially ordered sets (See Ashfaque 4.6.2). Thus it follows from the last inequality that

$$\pi(\rho_\varepsilon^{-n} |e_L(\varepsilon) \partial^\alpha u_\varepsilon(x_\varepsilon)|) = |\pi(\rho_\varepsilon^{-n} e_L(\varepsilon) \partial^\alpha u_\varepsilon(x_\varepsilon)))| \ge \pi(e_L(\varepsilon)). \tag{4.7}$$

Since (G, π) is a quotient differential algebra, there exist $(\delta_i)_i$ such that for every i, $\delta_i : G \longrightarrow G$ and $\delta_i \circ \pi = \pi \circ \partial_i$. This property together with 4.7 implies

$$|\delta^\alpha \pi(\rho_\varepsilon^{-n} e_L(\varepsilon) u_\varepsilon(x_\varepsilon)| \ge \pi(e_L(\varepsilon)) \tag{4.8}$$

which implies that $\pi(e_L(\varepsilon)) = 0$ because $\pi(u_\varepsilon) = 0$. This is a contradiction because $e_L(\varepsilon)$ is not a potential infinitesimal. So we have $Ker(\pi) \subseteq Ker[-]_\rho$ which proves the theorem. □

4.3. **A particular case: co-universal property of Colombeau generalized numbers.**

Definition 27. The Robinson-Colombeau ring of generalized numbers is defined by ${}^\rho\widetilde{\mathbb{R}} = \mathbb{R}_\rho / \sim_\rho$ where

(i) $\mathbb{R}_\rho \subset \mathbb{R}^I$ and $(x_\varepsilon)_\varepsilon \in \mathbb{R}_\rho \iff (x_\varepsilon)_\varepsilon \in \mathbb{R}^I | \exists n \in \mathbb{N} :\ x_\varepsilon = O(\rho_\varepsilon^{-n})$ as $\varepsilon \longrightarrow 0^+$. $\mathbb{R}_\rho$ is called the set of ρ-moderate nets.

(ii) $(x_\varepsilon)_\varepsilon \sim_\rho (y_\varepsilon)_\varepsilon \iff (x_\varepsilon)_\varepsilon, (y_\varepsilon)_\varepsilon \in \mathbb{R}_\rho$ and $\forall n :\ x_\varepsilon - y_\varepsilon = O(\rho_\varepsilon^{-n})$ as $\varepsilon \longrightarrow 0^+$.

$[x_\varepsilon]_\rho$ denote the equivalence class of $(x_\varepsilon)_\varepsilon \in \mathbb{R}_\rho$ in ${}^\rho\widetilde{\mathbb{R}}$. The ring of Robinson-Colombeau of generalized numbers can be seen as a subset of the Colombeau AG algebra.

Theorem 28. *Let ρ be a gauge. Then*

$$ {}^\rho\widetilde{\mathbb{R}} = \left\{ [u_\varepsilon]_\rho \in \mathcal{G}^s(AG(\rho^{-1}), \mathbb{R}) : \frac{d}{dx}[u_\varepsilon]_\rho = 0 \right\}. \tag{4.9} $$

Proof. $\Rightarrow$: This is trivial: Let $c = [c_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ and take $u_\varepsilon(x) = c_\varepsilon$. We have immediately $\frac{d}{dx}[u_\varepsilon(x)]_\rho = \left[\frac{d}{dx}c_\varepsilon\right]_\rho = [0]_\rho = 0$.

$\Leftarrow$: Let $[u_\varepsilon]_\rho \in \mathcal{G}^s(AG(\rho^{-1}), \mathbb{R}) : \frac{d}{dx}[u_\varepsilon(x)]_\rho = 0 = \left[\frac{d}{dx}u_\varepsilon(x)\right]_\rho$. It follows $\forall\varepsilon : \frac{d}{dx}u_\varepsilon(x) = z_\varepsilon$ where $(z_\varepsilon)_\varepsilon$ is negligible net of $\mathbb{R}_\rho$. By taking the primitive we obtain $\forall\varepsilon : u_\varepsilon(x) = c_\varepsilon + z_\varepsilon x$ which implies that $[u_\varepsilon(x)]_\rho = [c_\varepsilon]_\rho \in {}^\rho\widetilde{\mathbb{R}}$ because $[u_\varepsilon]_\rho \in \mathcal{G}^s(AG(\rho^{-1}), \mathbb{R})$. □

The Robinson-Colombeau ring of generalized numbers verify a co-universal property similar to the one verified by $\mathcal{G}^s$.

Definition 29. We say that (G, π) is a *quotient ring of* $\mathbb{R}^I$, and we write $(G, \pi) \in$ QRing$(\mathbb{R}^I)$, if

(i) $G : \mathbb{R}^I \longrightarrow \mathbf{Ring}_{\mathbb{R}}$ is a functor
(ii) $\pi : \mathcal{R} \longrightarrow G$ is natural transformation such that $\mathbb{R}^I \supseteq \mathcal{R}$ and $\pi : \mathcal{R} \longrightarrow G$ is an epimorphism of rings.

Let $(H, \eta) \in$ QRing$(\mathbb{R}^I)$ such that π and η have the same domain $\mathcal{R}$. Then a *morphism of quotient ring* $j : (G, \pi) \longrightarrow (H, \eta)$ is given by

$$ j : \mathrm{Ker}(\pi) \hookleftarrow \mathrm{Ker}(\eta). \tag{4.10} $$

Similarly to the quotient algebra's case, a co-universal quotient ring of some fixed subring has the largest kernel.

Remark 30. If π and η have two different domain $\mathcal{R}_1$ and $\mathcal{R}_2$, then a *morphism of quotient ring* is a couple (i, j): $(G, \pi) \longrightarrow (H, \eta)$ given by

$$ i : \inf(\mathcal{R}_1) \hookrightarrow \inf(\mathcal{R}_2), \quad j : \mathrm{Ker}(\pi) \hookleftarrow \mathrm{Ker}(\eta). $$

In this case, a co-universal quotient ring has the smallest class of infinities and the largest kernel. Since the smallest class of infinities does not exist because, by taking the logarithmic we obtain a smaller class of infinities, the co-universal quotient ring do not exists in this case.

Theorem 31. $\mathcal{R} \subseteq \mathbb{R}^I$ *a non-empty subring. Assume that*

(i) *Infinite*$(\mathcal{R}) \le Ag(\sigma^{-1})$ *where* σ^{-1} *does not necessary belong to* $\mathcal{R}$.
(ii) *Let* $(x_\varepsilon)_\varepsilon \in \mathbb{R}^I$. *If there exist* $(v_\varepsilon)_\varepsilon \in$ *Infinite*$(\mathcal{R})$ *such that* $\forall^0\varepsilon : |u_\varepsilon| \le v_\varepsilon$, *then* $(u_\varepsilon)_\varepsilon \in \mathcal{R}$,
(iii) *If* $(x_\varepsilon)_\varepsilon \in \mathcal{R}$, *then* $\exists (v_\varepsilon)_\varepsilon \in$ *inf*$(\mathcal{R})$, $\forall^0\varepsilon : |x_\varepsilon| \le v_\varepsilon$.

Then $\mathcal{R} = \mathbb{R}_\rho$.

The proof is similar to the proof of theorem 26

The co-universal property of ${}^\rho\widetilde{\mathbb{R}}$ is given in the following theorem.

Theorem 32. *Let ρ be a gauge and suppose that $\mathcal{R} = \mathbb{R}_\rho$. Then ${}^\rho\widetilde{\mathbb{R}}$ is co-universal among all the quotient ring (G, π) such that every zero-net is a potential infinitesimal.*

The proof is identical to the proof of theorem 25.

Remark 33. Let σ be another gauge such that $\rho \leq \sigma$. Note that $Ker([-]_\rho) \subseteq Ker([-]_\sigma)$. The quotient ring $(\mathbb{R}_\rho/\sim_\sigma, [-]_\sigma)$ does not have the property that every zero-net of $(\mathbb{R}_\rho/\sim_\sigma, [-]_\sigma)$ is a potential infinitesimal. Indeed, take $(x_\varepsilon)_\varepsilon \in Ker([-]_\sigma)/Ker([-]_\rho)$. Then

$$\exists L \subseteq_0 I, \exists n \in \mathbb{N}_{>0}, \forall \varepsilon \in L : \rho_\varepsilon^n < |x_\varepsilon| \qquad \text{and} \qquad \forall m \in \mathbb{N}, \forall^0 \varepsilon : |x_\varepsilon| \leq \sigma_\varepsilon^m.$$

It follows that

$$\forall m \in \mathbb{N}, \forall \varepsilon^0 \in L : \rho_\varepsilon^n < \sigma_\varepsilon^m. \tag{4.11}$$

This implies that $[\rho_\varepsilon^n e_L(\varepsilon)]_\sigma = 0$. Since $d\rho^{-n} \in \mathbb{R}_\rho/\sim_\sigma$ we have $[e_L(\varepsilon)]_\sigma = [\rho_\varepsilon^{-n} e_L(\varepsilon)]_\sigma \times [\rho_\varepsilon^n e_L(\varepsilon)]_\sigma = [0]$. Therefore, e_L is a zero-net but it is not a potential infinitesimal.

5. Co-universal property of spaces of generalized smooth functions

Introduction to GSF and their main differences w.r.t. CGF

In this section we would like to formalize the idea that generalized smooth functions are the simplest way to have a class of arbitrarily differentiable functions on subsets of ${}^\rho\widetilde{\mathbb{R}}^n$ defined by nets of ordinary smooth function.

The next definition is Def 3.1 of "Generalized Analytic Functions on Generalized. Hans Vernaeve"

Definition 34. Let $A \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. We define ${}^\rho\mathcal{GC}^\infty(A) = \tilde{\mathcal{E}}_M(A)/\tilde{\mathcal{N}}(A)$,

$$\tilde{\mathcal{E}}_M(A) := \left\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)^I : \forall \alpha \in \mathbb{N}^d, \forall [x_\varepsilon] \in A, \exists N \in \mathbb{N}, \partial^\alpha u_\varepsilon(x) = O(\rho_\varepsilon^{-N})\right\}$$

$$\tilde{\mathcal{N}}(A) := \left\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)^I : \forall \alpha \in \mathbb{N}^d, \forall [x_\varepsilon] \in A, \forall N \in \mathbb{N}, \partial^\alpha u_\varepsilon(x) = O(\rho_\varepsilon^{N})\right\}.$$

Here $\forall [x_\varepsilon] \in A$ means: for each representative $(x_\varepsilon)_\varepsilon$ of an element of A. Since $\mathcal{E}_M(A)$ is a differential algebra (for the ε-wise operations) and $\tilde{\mathcal{N}}(A)$ is a differential ideal of $\tilde{\mathcal{E}}_M(A)$, ${}^\rho\mathcal{GC}^\infty(A)$ is a differential algebra. [This remark is from the paper Generalized Analytic Functions on Generalized. Hans Vernaeve]

Here also, $[u_\varepsilon]_\rho$ denotes the equivalence class of $(u_\varepsilon)_\varepsilon \in \tilde{\mathcal{E}}_M(A)$ in ${}^\rho\mathcal{GC}^\infty(A)$.

Definition 35. We denote by **GSF** the category of generalized smooth functions. We write $(G, \pi, (D_i)_i) \in$ **GSF** if

(i) $G : \mathcal{P}({}^\rho\widetilde{\mathbb{R}}^\infty)^{op} \longrightarrow \mathbf{ALG}_{{}^\rho\widetilde{\mathbb{R}}}$ is a functor, where $\mathcal{P}({}^\rho\widetilde{\mathbb{R}}^\infty)$ is the category of subsets of ${}^\rho\widetilde{\mathbb{R}}^d$ for any d, and the inclusions.

(ii) $\pi : \tilde{\mathcal{E}}_M(A) \longrightarrow G$ is a natural transformation such that $\tilde{\mathcal{N}}(A) \subseteq Ker(\pi_A)$, $\forall A \in \mathcal{P}({}^\rho\widetilde{\mathbb{R}}^\infty)$.

(iii) $(D_i)_i : G \longrightarrow G$ is compatible with the usual partial derivatives $(\partial_i)_i$, i.e. $D_i \circ \pi = \pi \circ \partial_i$, $\forall i$.

Moreover, we write $(G, \pi, (D_i)_i) \xrightarrow{\tau} (F, \lambda, (\delta_i)_i)$ in **GSF** if and only if the following diagrams commute for all i

$$\begin{array}{ccc} \tilde{\mathcal{E}}_M & \xrightarrow{\lambda} & F \\ {\scriptstyle\pi}\downarrow & \nearrow{\scriptstyle\tau} & \\ G & & \end{array} \quad , \quad \begin{array}{ccc} G & \xrightarrow{D_i} & G \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ F & \xrightarrow[\delta_i]{} & F \end{array} \quad . \tag{5.1}$$

Theorem 36. *For every* $(F, \lambda, (\delta_i)_i) \in \mathbf{GSF}$*, there exist a unique* $\tau : ({}^\rho\mathcal{GC}^\infty, [-]_\rho, (D_i)_i) \longrightarrow (F, \lambda, (\delta_i)_i)$ *in* $\mathbf{GSF}$ *i.e.* $({}^\rho\mathcal{GC}^\infty, [-]_\rho, (D_i)_i)$ *is co-universal in* $\mathbf{GSF}$.

Proof. For every $[u_\varepsilon]_\rho \in {}^\rho\mathcal{GC}^\infty$, set $\tau([u_\varepsilon]_\rho) = \lambda(u_\varepsilon)$. We claim that τ doesn't depend on the representative $(u_\varepsilon)_\varepsilon$ of $[u_\varepsilon]_\rho$. Indeed, let $(v_\varepsilon)_\varepsilon$ be another representative of $[u_\varepsilon]_\rho$. Since $\lambda_{(-)}$ is an algebra-morphism (because it's a natural transformation between two algebras) and $\tilde{\mathcal{N}}(-) \subseteq Ker\left(\lambda_{(-)}\right)$, we have

$$\tau([u_\varepsilon]_\rho) = \lambda(u_\varepsilon) = \lambda(v_\varepsilon) + \lambda(u_\varepsilon - v_\varepsilon) = \tau([v_\varepsilon]_\rho). \tag{5.2}$$

Therefor the first diagram of 5.1 commutes for $({}^\rho\mathcal{GC}^\infty, [-]_\rho, (D_i))$ instead of $(G, \pi, (D_i)_i)$. By assumption, $(D_i)_i$ and $(\delta_i)_i$ are compatible with $(\partial_i)_i$. Then for all $[u_\varepsilon]_\rho \in {}^\rho\mathcal{GC}^\infty$ and for all i we have

$$\tau \circ D_i[u_\varepsilon]_\rho = \tau([\partial_i u_\varepsilon]_\rho) = \lambda(\partial_i u_\varepsilon) = \delta_i \circ \lambda(u_\varepsilon) = \delta_i \circ \tau([u_\varepsilon]_\rho) \tag{5.3}$$

which proves the commutative property of the second diagram of 5.1 for $({}^\rho\mathcal{GC}^\infty, [-]_\rho, (D_i))$ instead of $(G, \pi, (D_i)_i)$. □

We prove now the second co-universal property of ${}^\rho\mathcal{GC}^\infty$ as a quotient algebra. As in the case of Colombeau AG algebra and the case of Robinson-Colombeau ring of generalized numbers, infinite nets should be fixed, otherwise a co-universal solution would not exists.

Definition 37. We say that (G, π) is a *quotient algebra of* $\tilde{\mathcal{E}}_M$, and we write $(G, \pi) \in \mathrm{QALG}(\tilde{\mathcal{E}}_M)$, if

(i) $G : \mathcal{P}({}^\rho\widetilde{\mathbb{R}}^\infty)^{op} \longrightarrow \mathbf{ALG}_\mathbb{R}$ is a functor
(ii) $\pi : \tilde{\mathcal{E}}_M \longrightarrow G$ is natural transformation such that $\pi_A : \tilde{\mathcal{E}}_M(A) \longrightarrow G(A)$ is an epimorphism of ${}^\rho\widetilde{\mathbb{R}}$-algebras for all $A \in \mathcal{P}({}^\rho\widetilde{\mathbb{R}}^\infty)$.

Let (F, λ) be another element of $\mathrm{QALG}(\tilde{\mathcal{E}}_M)$. Then a *morphism of quotient algebra* $j : (G, \pi) \longrightarrow (F, \lambda,)$ is given by

$$j : \mathrm{Ker}(\pi) \hookleftarrow \mathrm{Ker}(\lambda).$$

Theorem 38. $({}^\rho\mathcal{GC}^\infty, [-]_\rho)$ *is co-universal among all the quotient differential algebras of* $\tilde{\mathcal{E}}_M$ *such that every zero-net is a potential infinitesimal.*

In this new framework, "every zero-net is a potential infinitesimal" means that if $\pi_A(u_\varepsilon) = 0$, then $\forall [x_\varepsilon] \in A, u_\varepsilon(x_\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$.

Proof. The fact that $({}^\rho\mathcal{GC}^\infty, [-]_\rho)$ is a quotient algebra such that every zero-net is a potential infinitesimal follows directly from the definition of ${}^\rho\mathcal{GC}^\infty$. Let (F, λ) be another quotient algebra such that every zero-net is a potential infinitesimal and suppose that $ker(\lambda)/ker([-]_\rho) \neq \emptyset$. Thus

$$\exists (u_\varepsilon) \in \tilde{\mathcal{E}}_M \exists [x_\varepsilon] \in A, \exists n \in \mathbb{N}, \forall \varepsilon \in L \subseteq_0 I : \rho_\varepsilon^{-n} |u_\varepsilon(x_\varepsilon)| > 1 \qquad \text{and} \qquad \lambda(u_\varepsilon(x_\varepsilon)) = 0. \tag{5.4}$$

Applying λ we obtain

$$0 = \rho_\varepsilon^{-n} |\lambda(u_\varepsilon(x_\varepsilon))| \geq_L \lambda(1) \geq_L 0$$

because λ is an order-preserving map. This implies that $1e_L \in Ker(\lambda)$ which leads to a contradiction because $1e_L$ is not a potential infinitesimal. □

6. Conclusions

References

[1] Borceux, F., *Handbook of Categorical Algebra 3. Category of sheaves.* Encyclopedia of Mathematics and its applications. Cambridge University Press, 1994.
[2] Colombeau, J.F., *New generalized functions and multiplication of distributions.* North-Holland, Amsterdam, 1984.
[3] Colombeau, J.F., *Elementary introduction to new generalized functions*. North-Holland, Amsterdam, 1985.
[4] Colombeau, J.F., *Multiplication of distributions - A tool in mathematics, numerical engineering and theoretical Physics*. Springer-Verlag, Berlin Heidelberg, 1992.
[5] Cutland, N., Kessler, C., Kopp, E., Ross, D. On Cauchy's notion of infinitesimal. *British J. Philos. Sci.*, 39(3):375-378, 1988.
[6] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., *Geometric theory of generalized functions*, Kluwer, Dordrecht, 2001.
[7] Oberguggenberger, M., *Multiplication of Distributions and Applications to Partial Differential Equations.* Vol. 259. Pitman Research Notes in Mathematics. Harlow, U.K.: Longman, 1992.
[8] Delcroix, A., Topology and functoriality in $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras. Application to singular differential problems. J. Math. Anal. Appl. 359 (2009) 394–403.
[9] Delcroix, A., Pilipović, S., Hasler, M.F., Valmorin, V., Sequence spaces with exponent weights. Realizations of Colombeau type algebras. Dissertationes Mathematicae 447, pp. 1 - 73, 2007.
[10] Delcroix, A., Scarpalezos, D., Asymptotic scales-asymptotic algebras. Integral Transforms and Special Functions, 1998, Vol. 6, No. 1-4, pp. 181-190.
[11] Delcroix, A., Scarpalezos, D., Topology on Asymptotic Algebras of Generalized Functions and Applications. Monatshefte für Mathematik 129, pp. 1-14, 2000.
[12] Giordano P., Kunzinger M., Vernaeve H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56-71.
[13] Giordano P., Kunzinger M., Vernaeve H., The Grothendieck topos of generalized functions I: basic theory. See arXiv:2101.04492.
[14] Giordano P., Luperi Baglini L., Asymptotic gauges: generalization of Colombeau type algebras. In revision for Mathematische Nachrichten. See arXiv:1408.1585.
[15] Giordano P., Nigsch E., Unifying order structures for Colombeau algebras. Mathematische Nachrichten, 1–17 (2015). DOI 10.1002/mana.201400277.
[16] Giordano P., Wu E., *Categorical frameworks for generalized functions*. To appear in Arabian Journal of Mathematics, 2014.
[17] Hasler, M.F., On the relation between $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras and asymptotic algebras. Document de travail 2011-07. Décembre 2011. Centre d'Etude et de Recherche en Economie, Gestion, Modélisation et Informatique Appliquée.
[18] Hadamard, J., *The Psychology of Invention in the Mathematical Field*, 1st edition, Princeton University Press, Princeton, NJ. 2nd edition, 1949. Reprinted, Dover Publications, New York, NY, 1954.
[19] Leinster, T., *Basic Category Theory*, Cambridge Studies in Advanced Mathematics, 143, 2014.
[20] Ligeza, J., Generalized periodic solutions of ordinary differential equations in the Colombeau algebra. Annales Mathematicae Silesianae 11. Katowice, pp. 67 - 87, 1997.
[21] Ligeza, J., Remarks on generalized solutions of ordinary differential equations in the Colombeau algebra. Mathematica Bohemica 123, No. 3, pp. 301-316, 1998.
[22] Meloni, G., Munarini, E., Purely algebraic theories for elementary operational calculus and generalized functions. Rendiconti del circolo matematico di Palermo, Serie II, Suppl. 64, pp. 29-42, 2000.
[23] Schmidhuber, J., Simple Algorithmic Principles of Discovery, Subjective Beauty, Selective Attention, Curiosity & Creativity. Proc. 10th Intl. Conf. on Discovery Science (DS 2007) pp. 26–38, LNAI 4755, Springer, 2007. See also arXiv:0709.067.
[24] Schwartz, L., *Théorie des Distributions, I et II*, Hermann, Paris, 1950, 1951.

[25] Sebastiao e Silva, Sur l'Axiomatique des Distributions et ses Possibles Modèles, CIME 2 ciclo, 1961.

[26] Steinbauer, R., Vickers, J.A., *On the Geroch–Traschen class of metrics*, Class. Quantum Grav. **26**, 2009.

[27] Todorov, T., An axiomatic approach to the non-linear theory of generalized functions and consistency of Laplace transforms, Integral Transforms and Special Functions Volume 22, 2011 - Issue 4-5: The International Conference on Generalized Functions, GF2009, Vienna, 2009.

[28] Todorov, T.D., Vernaeve, H., Full algebra of generalized functions and non-standard asymptotic analysis. *Log. Anal.* **1** (2008), 205-234.

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