A NOTE ON SESSION III, THE MATHEMATICAL SIDE OF CAUSALITY, AT THE TIME MACHINE FACTORY 2015

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This is a guide to a specific session at a conference called the Time Machine Factory 2015, which took place at the University of Turin in 2015. We may refer to the online resources of the session programme and material or videos of the talks. Before providing a brief summary of some of the talks delivered in this session of the conference in Section 2 below, we sketch the basic mathematical causality structures in Lorentzian geometry and some related issues of mathematical general relativity in Section 1.

1. Aspects of causality in the spacetimes of mathematical general relativity

The mathematical description of general relativity is fundamentally differential geometric, but also involves sophisticated methods from analysis, in particular when addressing the delicate questions of singularities or rigorous theories of wave propagation. Both of these issues are intimitely coupled to the causality structure on a spacetime, hence depend more specifically on geometric consequences of the possibility or conditions for non-existence of (almost) closed timelike curves. The first subsection summarizes the primarily differential geometric set-up of causality conditions, the second subsection then gives a very brief outlook on the analysis of partial differential equations describing propagation or evolution phenomena in spacetimes with causality structures.

1.1. **Causality in Lorentzian geometry.**

1.1.1. *Spacetimes and causality.* As basic references for the notions discussed here, we refer to [26] for the theory of general relativity and to [17] for smooth differential and Lorentzian geometry, the classic advanced book on *mathematical general relativity* is [12].

Let (M, g) be a pair consisting of a connected smooth manifold M equipped with a symmetric covariant 2-tensor field g, which is non-degenerate at every point and has index 1 (i.e., the tangent space has a one-dimensional maximal subspace where the metric is negative definite). Such a tensor field g is called a *Lorentzian metric* on M.

Let $p \in M$. A tangent vector $v \in T_pM$ is said to be *spacelike*, if $g_p(v, v) > 0$ or $v = 0$. The tangent vector v is called *lightlike* (or *null*), if $g_p(v, v) = 0$ and $v \neq 0$, and it is called *timelike*, if $g_p(v, v) < 0$. A *causal* tangent vector is one that is either lightlike or timelike.

We suppose, in addition, that (M, g) is *time-oriented* by the existence and choice of a nowhere vanishing continuous vector field V on M that is timelike at every point of M. The cone of causal vectors in each tangent space T_pM has two disjoint components, the one containing $V_p \in T_pM$ defines the *future-pointing* causal tangent vectors at p. We define a *spacetime* to be a time-oriented connected Lorentzian manifold (M, g).

Let $I \subseteq \mathbb{R}$ be an interval. A continuous, piecewise C^1 curve $\gamma\colon I \to M$ is said to be a *causal curve*, if the tangent $\dot\gamma(s)$ is causal in $T_{\gamma(s)}M$ for almost every $s \in I$. The causal curve is said to be *future directed*, if each tangent vector is future-pointing.

The most direct and classical implementation of *time-travel* would thus mean to have a timelike curve γ with two distinct parameter values $s_1, s_2 \in I$ such that $\gamma(s_1) = \gamma(s_2)$, i.e., a closed timelike curve. A spacetime (M, g) is said satisfy the *chronology condition*, if there exists no closed timelike curve in M. A *causal spacetime* is a spacetime that possesses no closed causal curves. Clearly, a causal spacetime is also chronological.

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The standard elementary example of a causal spacetime is of course *Minkowski space*, where $M = \mathbb{R}^{n+1}$ and g is the constant metric given in global Cartesian coordinates at the point $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ as bilinear form corresponding to the diagonal matrix $g_{(t,x)} = \operatorname{diag}(-1, 1, \ldots, 1)$. A time orientation is fixed by the constant vector field with value $(1, 0) \in \mathbb{R} \times \mathbb{R}^n$ at every point of M.

1.1.2. *Causal relation and global hyperbolicity.* Two points p, q in a spacetime M are said to be *causally related*, denoted by $p \leq q$, if $p = q$ or there exists a causal future directed curve from p to q. The *causal past* of q is defined as $J^-(q) := \{x \in M : x \leq q\}$ and the *causal future* of p is the subset $J^+(p) := \{y \in M : p \leq y\}$. For example, in Minkowski space, the causal past set $J^-(q)$ is a closed cone with tip at q and extending toward negative time direction, whereas the causal future set $J^+(p)$ is a closed cone with tip at p and extending toward positive time direction. We define the *chronological future* $I^+(p)$ and the *chronological past* $I^-(q)$ in complete analogy with $J^+(p)$ and $J^-(q)$, simply based on *timelike* future directed curves.

Roughly speaking, the famous singularity theorems of general relativity show the existence of black holes and the necessity of a big bang for a large class of "generic universes". In addition to the condition of causality, the proofs of these theorems depend in a crucial way on several constructions involving *causal geodesics*. Recall that a curve γ in the spacetime (M, g) is a *geodesic*, if $\dot\gamma$ is parallel, i.e., $D_{\dot\gamma}\,\dot\gamma = 0$ holds. Here, D denotes the covariant derivative, or Levi-Civita connection, induced by the Lorentzian metric g. In particular, the mentioned constructions required for the singularity theorems rely on a positive answer to the question whether any two causally related points $p \leq q$, $p \neq q$ in M can be connected by a causal geodesic curve from p to q. As it turns out, an additional topological condition on the causal relation guarantees this desired property and leads to the notion of a so-called *globally hyperbolic* spacetime (M, g), that is,
(i) (M, g) is causal,
(ii) for every $p, q \in M$, the set $J^+(p) \cap J^-(q)$ is compact in M.

The classical form of the definition uses the *strong causality condition* in place of (i), that is, it requires that there are no "almost closed curves" in M in the following sense: For every $p \in M$ and for every neighborhood U of p there exists a neighborhood V of p with $V \subseteq U$ and such that no causal curve that starts and ends in V can leave U. The above variant of defining global hyperbolicity is possible due to [4], where it is shown that (i) and (ii) imply that M is strongly causal. As another interesting relationship with topology we may mention that the following can be shown (cf. [20, Theorem 4.24]): A spacetime (M, g) is strongly causal, if and only if the family of open sets $\{I^+(p) \cap I^-(q) \mid p, q \in M\}$ is a basis for the manifold topology of M.

1.1.3. *Cauchy hypersurfaces and metric splitting.* A *Cauchy hypersurface* S in a spacetime M is a subset $S \subset M$ such that every inextendible timelike curve intersects S exactly once (see [17, Chapter 14, Definition 28]). It was shown in [8] that global hyperbolicity of a spacetime is equivalent to the existence of a Cauchy hypersurface S. Moreover, it follows that in this case M is homeomorphic to $\mathbb{R} \times S$. A further refinement of this classical result on Cauchy hypersurfaces is the so-called *globally hyperbolic metric splitting*, established in [3], which provides a characterization of global hyperbolicity by a detailed "normal form description": A spacetime (M, g) is globally hyperbolic if and only if it is isometric with a spacetime $(\mathbb{R} \times S, \lambda)$ such that $\lambda = -\theta\, dt^2 + \rho_t$, where
(a) each $\{t\} \times S$ is a (smooth spacelike) Cauchy hypersurface,
(b) $(\rho_t)_{t \in \mathbb{R}}$ is a smoothly parametrized family of Riemannian metrics on S,
(c) $\theta \in C^\infty(\mathbb{R} \times S)$ and positive.

1.1.4. *Continuous causal curves.* A *continuous* curve $\gamma \colon [0, 1] \to M$ is said to be *causal*, if for any convex open subset U of M and $s_1, s_2 \in [0, 1]$, $s_1 \leq s_2$ with $\gamma([s_1, s_2]) \subset U$, we have $\gamma(s_1) \leq \gamma(s_2)$ (relative U). It can be shown that continuous causal curves are, in fact, *Lipschitz continuous* (see [2, pages 75-76] or [13, pages 365-366]), hence they are rectifiable and differentiable almost everywhere. In particular, continuous causal curves may be parametrized by arclength. Global hyperbolicity of a smooth spacetime (M, g) has been described in [6] in terms of compactness of the set of causal curves connecting two causally related points and parametrized proportionally to arclength on the interval $[0, 1]$ with respect to the compact-open topology. Alternatively, the so-called C^0-topology on the set of equivalence classes or images of (future directed) causal curves is being employed instead (e.g. in [13, 20, 26]; see also the review article [22]).

Let $C(p,q)$ be the set of all *classes of continuous future directed causal curves* from p to q modulo continuously differentiable parameter transforms with positive derivative. As discussed in [26, Section 8.3] (and in more detail in [21]), we may identify classes of curves in $C(p,q)$ with their images as subsets of M and it is easy to show that we obtain a basis for a topology τ on $C(p,q)$ by specifying $O(U) := \{\widetilde{\lambda} \in C(p,q) \mid \text{the image of } \widetilde{\lambda} \text{ is contained in } U\}$ for any open subset $U \subseteq M$. The topology τ satisfies the Hausdorff separation axiom, if (M,g) is causal. In terms of the topological space obtained in this way, we have the following characterization of global hyperbolicity (see [26, Section 8.3], and [21] for the case of a continuous metric g):
A spacetime (M,g) is globally hyperbolic, if and only if
(i) (M,g) is causal and
(ii) for all $p,q \in M$, the topological space $(C(p,q),\tau)$ is compact.

1.1.5. *Singularity theorems.* The celebrated singularity theorems by Hawking and Penrose essentially prove *timelike or null geodesic incompleteness* of any spacetime satisfying a standard energy condition (involving the effect that gravity attracts masses), global hyperbolicity (or a comparably strong causality condition), and some form of initial condition (implementing the fact that the universe is expanding) or boundary condition (existence of so-called *future trapped surfaces*, e.g., related to collapsing stars). *Timelike geodesic incompleteness* means the existence of an *inextendible* timelike geodesic curve $\gamma\colon [0,b[\to M$, i.e., there is a sequence of parameter values $s_n \to b$ such that the sequence of image points $(\gamma(s_n))_{n\in\mathbb{N}}$ does not converge in M; similarly for *null geodesic incompleteness.* Inextendible timelike geodesics are interpreted as evidence for a singularity in the sense of a big bang (past directed) or big crunch (future directed). The existence of inextendible null geodesics related to a trapped surface are an indication for a black hole.

1.2. **Analysis of partial differential equations on spacetimes and gravitational waves.**

1.2.1. *Wave equations and a more general notion of global hyperbolicity.* In a broad sense, well-posedness of Cauchy problems for wave equations can also be considered a “good causality property” of a theory, since this includes that physical fields are propagating into all of space and are determined from appropriate initial conditions.

The wave operator $\Box$ on the spacetime (M,g) is defined as the semi-Riemannian Laplacian Δ_g correspoding to the Lorentzian metric g, i.e., $\Box f = \Delta_g f = \operatorname{div}(\operatorname{grad} f)$ for a scalar function f on M. Recall that the semi-Riemannian gradient is defined by the requirement that $g(\operatorname{grad} f, v) = df(v) = v(f)$ holds for vector fields v on M, while $\operatorname{div} v$ is the contraction in the covariant differential Dv, a tensor field of type $(1,1)$, of its “new” covariant slot with its “original” contravariant slot. We will use the notion of *wave equation* on a spacetime (M,g) for partial differential equations on M that are of the form $Pu = f$, where the differential operator can be written as $P = \Box + Q$ with Q of order 1 or 0.

One of the fundamental consequences for analysis on a globally hyperbolic spacetime is that we always have global well-posedness of (distributional) Cauchy problems for wave equations with initial data given on spacelike Cauchy hypersurfaces (cf. [1]). Furthermore, the following observation is immediate: If (M,g) is a globally hyperbolic spacetime and $M \cong \mathbb{R} \times S$ as in the metric splitting characterization described above, then $\Box$ is a strictly hyperbolic 2nd-order differential operator with respect to level sets isomorphic to $\{t\} \times S$. Here, we use the notion of a strictly hyperbolic differential operator on a manifold in the sense of [10, Definition 23.2.3].

It is interesting to mention here a different point of view, namely, if P is an arbitrary strictly hyperbolic 2nd-order differential operator P on a manifold M, then we have a *semi-global*[1] well-posedness result for (distributional) Cauchy problems with initial data on level sets in [10, Theorem 23.2.4] and, furthermore, there exists a Lorentzian metric g on M such that the principal part of P coincides with that of $\Box$ (cf. [10, Section 24.1]). However, one can easily give examples showing

[1] i.e., valid on relatively compact open subsets,

that strict hyperbolicity of $\Box$ on M does not imply global hyperbolicity of the spacetime (M, g) (e.g., delete a single point from Minkowski space).

In [7] it has been suggested to replace the standard definition of global hyperbolicity by the requirement of well-posedness for wave equations, in particular, in the realistic situations where the Lorentzian metric fails to be smooth on all of M. The view has been advocated that a "singularity" of a spacetime should be manifest as an obstruction to the Cauchy development of the physical fields in that spacetime, which is different from the standard notion of singularity as an "obstruction to the extension of geodesics". In this spirit, a spacetime (M, g) is called $\Box$-*globally hyperbolic*, if there exists a C^1 spacelike hypersurface $S \subset M$ with unit future pointing (continuous) normal vector field ξ such that $M \setminus S$ is the disjoint union of two open (connected) subsets M^+ and M^-, ξ points from M^- to M^+, and the Cauchy problem

$$\Box\, u = f, \quad u|_S = v, \quad \xi u|_S = w,$$

is well-posed in appropriate Sobolev spaces.

2. Some of the talks delivered in Session III

2.1. **James Vickers: Time, causality, and Einstein's equations.** This key talk started with a discussion of the differences regarding the notions of time, space, and causality in Newtonian Theory, Special and General Relativity. In the latter, the dependence of the concept of past and future on a spacetime point was emphasized and how curvature is reflected in a variation of the lightcones in the tangent spaces from point to point. Moreover, the simplest mathematical model situations of black holes, white holes, and worm holes were described in terms of intuitive geometric visualizations and explicit formulae of model spacetime metrics.

The second part gave some examples of solutions to Einstein's equations which contain closed timelike curves (CTCs), which means that by locally always travelling within ones forward light-cone in spacetime, one can actually travel into ones own past on the global level. The examples illustrated were the van Stockum rotating cylinder, Gödel's universe, the rotating black hole, wormhole solutions, the Gott 2-string solution, and Ori's solution. The possibility of travelling into ones own past leads to potential causal paradoxes, although not necessarily is the existence of CTCs incompatibile with the laws of physics. One crucial question is whether the Cauchy problem for a scalar field in such a spacetime can still be well-posed. In particular, the evolution of a scalar field in a spacetime which contains an initially globally hyperbolic region which evolves into one containing CTCs was considered. It was discussed how these results relate to several aspects of "Hawking's Chronology Protection Conjecture", stating essentially that in a finite region of spacetime CTCs cannot form without violating the so-called averaged weak energy condition. Support for the chronology protection is provided by some evidence that quantum mechanical effects preclude the creation of any putative "time machine" in form of CTCs. For example, the spacetime with closed timelike curves due to Gott will, when quantum mechanical effects are taken into account, be distorted in such a way that the region with closed timelike curves will no longer form.

Alternatively, there is Novikov's "Principle of Self-Consistency", observing that events on a CTC influence one another in a cyclic and self-consistent way and the Cauchy problem can be solved locally. One can then try to extend the local solution to obtain a global solution. In the well-known Polchinski Paradox, where a billard ball enters a wormhole at one end, but returns at an earlier time through the other end, choosing only self-consistent initial data, the earlier returned billard ball will hit the original ball only in a way so that it still enters the worm hole.

Considering as a test case the wave equation on the spacetime of a rotating cosmic string, which allows for CTCs, one finds that global solutions are possible, but the wave operator changes type, since it is hyperbolic in the chronological regions, but elliptic in the other parts of spacetime. In particular, the initial data have to obey further restrictions in the elliptic region.

A few references for James Vicker's talk are [5, 9, 11, 16, 18, 24, 25].

2.2. **Michael Kunzinger: Singularity theorems in low regularity.** The talk began by explaining the basic structure and concepts of the singularity theorems of general relativity, which were milestones in the understanding of solutions to the Einstein equations, a line of research initiated by Penrose and continued by Hawking, Penrose, Ellis, Geroch, and others. Still the investigation of singularity theorems constitutes a central topic of active research in mathematical relativity. In essence, the singularity theorems show that under realistic assumptions on the spacetime (and independently of any symmetries) there necessarily exist incomplete timelike or null geodesics, which may be interpreted as singularities of the spacetime. One weakness of the classical singularity theorems is that they do not make any statement on the actual nature of the singularities themselves. In particular, they do not imply that the curvature blows up where a causal geodesic ceases to exist. Thus, in principle, they allow the possibility that the spacetime might be singular in the above sense merely due to the fact that the differentiability of the spacetime metric drops below C^2 (twice continuously differentiable). The talk reported on recent progress in low-regularity causality theory and gave precise statements and sketches of proofs of the fact that both the Penrose and the Hawking theorem remain valid for Lorentzian metrics of differentiability class $C^{1,1}$ (i.e., the first-order derivatives are Lipschitz continuous), which is the maximal class where the geodesic differential equation with initial conditions still has unique solutions.

A few references for Michael Kunzinger's talk are [12, 14, 15, 19, 23].

2.3. **Clemens Sämann: Global hyperbolicity for spacetimes with continuous metric.** The talk opened with a brief review of the definition and the reasons why global hyperbolicity is a commonly used causality condition in general relativity: It ensures well-posedness of the Cauchy problem for the wave equation; globally hyperbolic spacetimes are the class of spacetimes used in the initial value formulation of Einstein's equations; it plays an important role in the formulation and in the proofs of singularity theorems. Classically, i.e., with smooth Lorentzian metric, there are four equivalent notions of global hyperbolicity: Compactness of the "causal diamonds" $J^+(p) \cap J^-(q)$ and (strong) causality, compactness of the space of causal curves connecting two points and causality, existence of a Cauchy hypersurface, and the metric splitting of the spacetime. The talk described techniques enabling one to show that the definition of global hyperbolicity in terms of compactness of the "causal diamonds" plus the non-total imprisonment condition (causal curves eventually leave any compact subset) can be extended to spacetimes with *continuous* Lorentzian metrics, while retaining the first three equivalences above. Furthermore, global hyperbolicity in this sense implies properties such as causal simplicity (closedness of the causality relation $\leq$), stable causality ("small" perturbations of the metric with respect to the C^0 open topology do still not produce closed timelike curves), and the existence of maximal causal curves connecting any two causally related points. We may refer to [21] for more details mentioned in the talk, in particular, also regarding a clear analysis of comparability of several topologies on the space of continuous causal curves.

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