Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients

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Abstract. We are concerned with analyzing hyperbolic equations with distributional coefficients. We focus on the case of coefficients with jump discontinuities considered earlier by Hurd-Sattinger in their proof of breakdown of global distributional solutions. In the framework of Colombeau generalized functions, however, Oberguggenberger showed existence and uniqueness of a global solution. Within this framework we develop further a microlocal analysis to understand the propagation of singularities of such Colombeau solutions. To achieve this we introduce a refined notion of wave front set, extending Hörmander's definition for distributions. We show how the coefficient singularities modify the classical relation of the wave front set of the solution and the characteristic set of the operator, with a generalized notion of characteristic set.

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1. Introduction

In this paper, we present a new approach to the mathematical analysis of wave propagation, in particular propagation of singularities, in complex media. Such analyses have been established in smoothly varying media. However, the situation changes substantially if media with nonsmooth variations of the physical parameters are considered. Various results on the classical solvability of the corresponding hyperbolic equations with singular coefficients could be achieved in certain examples ([4, 5, 6, 25]), including nonexistence in a surprisingly simple configuration as shown by Hurd and Sattinger [13]. Furthermore, semigroup methods ([14]) are often applicable in the case of bounded, measurable coefficients, e.g., if the source and initial values are in L^2 ([18]). However, in various (geo)physical applications delta-like sources together with discontinuous, fractal, or multifractal media occur naturally (for example, in sedimentary sequences in the upper crust of the Earth [16] or in fractured rocks; see also [9, 10]).

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We shall discuss and test a recently developed extension of microlocal analysis by means of an example, in which the implications of the theory are more easily understood. In the example, the medium contains a single step (Heaviside) singularity which includes the special case considered by Hurd and Sattinger [13] for which they proved nonexistence of distributional solutions. However, the analysis we present applies to media with a far higher degree of complexity than the step singularity, viz. media described by distributions of any finite order, in particular also Zygmund classes covering multifractal behavior.

The theory is built on Colombeau algebras [1] of generalized functions (subsection 1.2). Such algebras are not only appropriate for our analysis of (wave) solutions, but also provide an explicit modeling procedure of complex media. The 'generalized function' is represented by a class of regularizations. Each regularization is a representative, on which the analysis is carried out. The regularizations are parameterized by a scale (zoom-in) factor. The function's asymptotic properties with respect to the scale parameter establish an accurate measure of regularity. The original distributions are embedded through regularizations of the type 'convolution with mollifier'.

Acoustic and elastic waves are described by a first-order, symmetric, hyperbolic system of partial differential equations in dimension $m+1$. In this paper, we will model distributional coefficients in such a system of equations, preserving their singularity structure (subsection 3.2). In the context of Colombeau algebras we will establish unique solvability of the resulting system (subsection 3.1) and investigate the solution's microlocal properties such as the propagation of singularities. To this end we introduce the concept of generalized characteristics (subsection 4.2). To evaluate the generalized characteristic set we adapt the method of characteristics to representatives in a Colombeau algebra (subsection 4.1). The propagation of singularities is then manifest in the generalized wavefront set of the (Colombeau) solution of the hyperbolic system of equations (section 5). The generalization is carefully introduced to ensure detectability of all singularities. Throughout the paper, we develop a theory around the $1+1$-dimensional configuration and a distributional coefficient with an isolated singularity (subsection 1.1). For such configuration a physical intuition exists and peculiarities like the 'slowness' (cotangent) vector in an 'interface' (subsection 4.2) can be better understood. Finally, we establish the existence of distributional shadows of the solution (section 6). This implicitly induces a solution mapping from the distributional coefficients to a distributional shadow, which brings us back to the applications. The analysis sheds new light on the Schwartz impossibility [26] result in the context of global solutions of partial differential equations (section 2). We reestablish the existence of distributional solutions in the generic case by adapting the levels in the product hierarchy of Oberguggenberger ([21], p. 69; see also the appendix for a brief summary and a table), but strengthen the nonexistence of the limiting case of zero speed leading to Hurd-Sattinger's example.

The wave equation in $1+1$ dimensions can also be thought of as a 'one-way' wave equation. Such equation is obtained after a directional decomposition procedure has been applied to the full-wave equation. Thus, in the example, we focus on the transmission of transient acoustic waves.

In applications, the propagating singularities contain leading-order information about the medium. Hence, their understanding is of key importance to remote sensing, e.g. in data processing and inverse scattering.

1.1. The one-dimensional problem

We study a $1+1$-dimensional model represented by the following linear initial value problem with a discontinuous coefficient,

$$\partial_t u - \partial_x (R(x)u) = 0 \tag{1}$$

$$u|_{t=0} = a \ . \tag{2}$$

We assume the coefficient to be of the form $R(x) = c_1 H(-x) + c_2 H(x)$ where H is the Heaviside function, i.e., $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$, c_1 and c_2 are real nonnegative constants, and a is some distribution in $\mathbb{R}$.

1.2. Algebras of generalized functions

We recall the basic definitions and notions of Colombeau theory. For a detailed presentation of the general theory, its role in generalized function theory, and a review of applications, we refer to [2, 21, 8, 17].

The parameter set for the regularizations defining Colombeau generalized functions is given by the following cascade of normalized test function sets with vanishing moment conditions: for $q \in \mathbb{N}_0$ define

$$\begin{aligned} \mathcal{A}_q(\mathbb{R}) &= \{\chi \in \mathcal{D}(\mathbb{R}) \mid \int \chi(x)dx = 1, \int x^k \chi(x)dx = 0 \ (1 \leq k \leq q)\} \\ \mathcal{A}_q(\mathbb{R}^n) &= \{\phi \in \mathcal{D}(\mathbb{R}^n) \mid \exists \chi \in \mathcal{A}_q(\mathbb{R}) : \phi(x_1, \ldots, x_n) = \chi(x_1) \cdots \chi(x_n)\} \end{aligned}$$

To incorporate real scaling parameters for judiciously chosen asymptotic conditions, we define for $\phi \in \mathcal{A}_0(\mathbb{R}^n)$

$$\phi_\varepsilon(x) = \varepsilon^{-n}\, \phi(x/\varepsilon) \qquad \varepsilon > 0 \ .$$

Note that, as distributions, $\phi_\varepsilon \to \delta_0$ as $\varepsilon \to 0$; in particular, the support of ϕ_ε shrinks to the single point $\{0\}$. A useful general tool to couple the scaling with the shrinking of the support of ϕ is the *support number*

$$l(\phi) := \sup\{x \in \mathbb{R}^n \mid \phi(x) \neq 0\}$$

which is simply the radius of the smallest closed ball containing $\mathrm{supp}(\phi)$. Note that we have $l(\phi_\varepsilon) = \varepsilon l(\phi)$.

We introduce the set of all maps from the index set $\mathcal{A}_0(\mathbb{R}^n)$ into the smooth functions over an open subset Ω,

$$\mathcal{E}[\Omega] = \{R : \mathcal{A}_0(\mathbb{R}^n) \to \mathrm{C}^\infty(\Omega)\} \ .$$

This is a differential algebra with component-wise operations, i.e. all operations are reduced to their classical counterparts at fixed ϕ. We single out certain subalgebras of nets subject to asymptotic conditions uniformly on compact sets, viz. the moderate and the negligible nets:

$$\mathcal{E}_M[\Omega] = \Big\{ \ R \in \mathcal{E}[\Omega] \ \mid \ \forall K \subset \Omega \text{ compact}, \forall \alpha \in \mathbb{N}_0^n : \exists N \in \mathbb{N} : \quad (3)$$
$$\forall \phi \in \mathcal{A}_N(\mathbb{R}^n) : \sup_{x\in K} |\partial^\alpha R(\phi_\varepsilon, x)| = O(\varepsilon^{-N}) \quad (\varepsilon \to 0) \ \Big\}$$

$$\mathcal{N}[\Omega] = \Big\{ \ R \in \mathcal{E}_M[\Omega] \ \mid \ \forall K \subset \Omega \text{ compact}, \forall \alpha \in \mathbb{N}_0^n : \exists N \in \mathbb{N} : \quad (4)$$
$$\forall \phi \in \mathcal{A}_N(\mathbb{R}^n) \forall q \geq N : \sup_{x\in K} |\partial^\alpha R(\phi_\varepsilon, x)| = O(\varepsilon^{q-N}) \quad (\varepsilon \to 0) \ \Big\}$$

In the definition of $\mathcal{N}[\Omega]$ one may neglect to check the growth conditions for derivatives of order ≥ 1 (as long as R is known to be an element in $\mathcal{E}_M[\Omega]$; cf. [7]).

$\mathcal{N}[\Omega]$ is an ideal in $\mathcal{E}_M[\Omega]$ hence we may form the quotient algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M[\Omega]/\mathcal{N}[\Omega]$$

which is called the *Colombeau algebra* over Ω. It is again a differential algebra where operations are reduced to component-wise operations on representatives. To highlight different aspects of the representatives we will switch between the notations $u(\phi, x)$, $u(\phi, .)$, or $u(\phi)$ and write $U = \mathrm{cl}[(u(\phi, .))_\phi]$ to indicate that $U \in \mathcal{G}$ has the representative $u \in \mathcal{E}_M$. Whenever ϕ is arbitrary but fixed we will temporarily use a short hand notation like $u^\varepsilon(x) := u(\phi_\varepsilon, x)$ in computations.

The Colombeau algebra $\mathcal{G}(\Omega)$ has localization properties which guarantees existence of restrictions to open subsets and a meaningful notion of *generalized support*: if U is a Colombeau function then $\mathrm{supp}_g(U)$ is the complement of the largest open subset $X \subseteq \Omega$ such that $U \mid_X = 0$ in $\mathcal{G}(X)$.

There is a canonical embedding of distributions $\iota : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega)$ (as a linear subspace) which in the case of $\Omega = \mathbb{R}^n$ is simply given by

$$\iota(f) = \mathrm{cl}[(f * \phi)_\phi] \qquad f \in \mathcal{D}'(\mathbb{R}^n) \; .$$

For Ω arbitrary one first defines this map on the space $\mathcal{E}'(\Omega)$ of distributions with compact support in Ω and then extends by suitable cut-off over any given compact set $K \subset \Omega$ (cf. [21], Sect. III.9).

Restricted to smooth functions this map is also consistent with the 'constant' embedding of smooth functions, $\sigma : \mathrm{C}^\infty(\Omega) \to \mathcal{G}(\Omega)$,

$$\sigma(f) = \mathrm{cl}[(f)_\phi] \qquad f \in \mathrm{C}^\infty \quad (\iota(f) = \sigma(f) \text{ in this case}) \; .$$

Requiring this consistency was the motivation for introducing the moment conditions in $\mathcal{A}_q$: the proof employs Taylor expansions in the convolution integral and makes use of the vanishing moments of the mollifier.

Some Colombeau functions can be projected to distributions (these are said to have 'distributional shadows'). We say that $U \in \mathcal{G}$ is *associated* with $w \in \mathcal{D}'$, notation $U \approx w$, if $U = \mathrm{cl}[(u(\phi, .))_\phi]$ and $\forall \psi \in \mathcal{D}$ $\exists N \in \mathbb{N}$:

$$\lim_{\varepsilon \to 0} \int u(\phi_\varepsilon, x)\psi(x)dx = \langle w, \psi \rangle \qquad \forall \phi \in \mathcal{A}_N \; .$$

If $U, V \in \mathcal{G}$ then we define $U \approx V$ if $U - V$ is associated with (the distribution) 0. (This is an equivalence relation.) The following properties are immediate: $\iota(w) \approx w$; if $U \approx V$ then $\partial^\alpha U \approx \partial^\alpha V$ and $fU \approx fV$ for $f \in \mathrm{C}^\infty$.

Since we study Cauchy problems in $\mathbb{R}^{m+1}$ $(m = 1)$ we have to define the restriction of a Colombeau function to a coordinate hyperplane: let $V \in \mathcal{G}(\mathbb{R}^{m+1})$ then if $V = \mathrm{cl}[(v(\phi, .))_\phi]$ and $\phi^{(l)}(x_1, \ldots, x_l) := \phi_0(x_1) \cdots \phi_0(x_l)$ for all $\phi_0 \in \mathcal{A}_0(\mathbb{R})$ we define

$$V\mid_{x_{m+1}=0} = \mathrm{cl}[(v(\phi_0^{(m+1)}, .))_{\phi_0^{(m)}}] \; .$$

2. Global distributional solutions

A classical approach to solve (1)-(2) is to split the problem into two subproblems (with then constant coefficients) on either side of the jump at $x = 0$ and to subject the two associated solutions to boundary conditions at the jump to find a solution of the original problem. However, this in general does not lead to a *global* distributional solution, as already observed in the case $c_1 = 0$, $c_2 = 1$, and $a = 1$ by Hurd and Sattinger [13] with the framework of $\mathrm{L}^1_{\mathrm{loc}}$ as solution space (see also the short discussion in [21], example 1.7). Essentially, the reason is that the combination of the subproblems suggests to consider $u(x,t) = 1 + t\delta(x)$, δ the Dirac delta at 0, as the global solution for $t \geq 0$. But this is not in $\mathrm{L}^1_{\mathrm{loc}}$ and leaves us with the problem of consistently defining $H \cdot \delta$ and then differentiating this object upon checking if u actually solves the differential equation. To do so, we want to use one of the notions of $\mathcal{D}'$-products according to the hierarchy of consistent extensions given in [21], p. 69 (also described in the appendix). Within this hierarchy we find that $H \cdot \delta$ exists only on the most general level of the strict products (and consequently in all higher levels), yielding the value $[H \cdot \delta] = \delta/2$ (cf. [21], examples and exercises 7.12). But a simple computation shows that $\partial_t u - \partial_x([H \cdot u]) = -t\delta'(x)/2 \neq 0$ and therefore u does not solve the equation globally.

In the following we investigate all cases $0 \leq c_1 < c_2$. We will show that almost all initial data associated with remote sensing will necessarily lead to inconsistencies if the properties of the differential operator and the requirements on the solution are stated in a purely microlocal form. On the other hand by allowing for distributional products beyond microlocal conditions we can prove general existence of solutions for the case $0 < c_1$ but generic nonexistence for the case $c_1 = 0$.

In order to focus on the interaction of propagating singularities with the medium jump at $x = 0$ we introduce the following assumption concerning the initial value or source:

(o) $a \in \mathcal{D}'(\mathbb{R})$ and a is smooth near 0.

This is valid in particular if the source of the model experiment is concentrated on one side of the medium jump.

To give a precise meaning to the question of existence of distributional solutions of (1)-(2) we start with a detailed list of requirements for a distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ to be considered a global solution.

In a first version these requirements emphasize microlocal methods and consistencies in accordance with the main issues posed in this paper. Later on variants of requirement (v) will be considered that depart from microlocal considerations.

(i) u is continuous in time, i.e. can be considered to be an element of the space $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}))$; in particular, its initial value at $t = 0$ is well defined in the distributional sense. This is natural in the framework of hyperbolic Cauchy problems and could be weakened. E.g., we could assume u to be restrictable to $t = 0$ in the sense of [11], Cor. 8.2.7, which would imply the same continuity property of u near the x-axis (cf. the remark in [27], after Prop. 6.11).

(ii) $u|_{t=0} := u(0) = a$.

(iii) in the left half space $V_- = \{(x,t) \mid x < 0\}$ we have $\partial_t u = c_1 \partial_x u$ (according to (1) when restricted to V_-). This is a consequence of (v) and (vi) below but we prefer to include this redundancy to emphasize coherence with intuition.

(iv) in the right half space $V_+ = \{(x,t) \mid x > 0\}$ we have $\partial_t u = c_2 \partial_x u$ (according to (1) when restricted to V_+). As with (iii) this will also be a consequence of (v)-(vi) below.

(v) the distributional product $(R(x) \otimes 1) \cdot u$ should exist in $\mathcal{D}'(\mathbb{R}^2)$ in the sense of Hörmander ([11], Thm. 8.2.10); in other words, the wave front sets of $R \otimes 1$ and u must not contain opposite cotangent directions over the same base point. Note that $\mathrm{WF}((R(x) \otimes 1)) = \{0\} \times \mathbb{R} \times (\mathbb{R} \times \{0\}) \setminus \{(0,0)\}$ (cf. [11], Thm. 8.2.9, also p. 269) and has therefore exactly horizontal cotangent directions above the t-axis. We conclude that within the microlocal setting the existence of this product is equivalent to the existence of the restriction $u\mid_{x=0}$ (cf. [11], Cor. 8.2.7). This in turn enables us to reformulate the transmission at the medium singularity as a boundary value problem. It furthermore implies (as remarked similarly in (i)) that locally near $x = 0$ we may consider u to be continuous in x and distributional in time.

(vi) finally, we require the equation to be satisfied globally in the sense of $\mathcal{D}'(\mathbb{R}^2)$: $\partial_t u - \partial_x((R(x) \otimes 1) \cdot u) = 0$.

Theorem 1. Assume $0 \leq c_1 < c_2$ and that a satisfies assumption (o) above. Then there is no distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ satisfying all of the requirements (i)-(vi) above unless $a = 0$ on $\mathbb{R}$ or, in case $c_1 = 0$, $a = 0$ in $(0, \infty)$. In this sense, there is no distributional solution to problem (1)-(2) for nontrivial initial values.

Proof. For the proof of nonexistence we distinguish two cases according to the value of c_1.

Case $c_1 > 0$*:* assume that u is a distribution satisfying (i)-(vi). In the regions $V_1 := \{(x,t) \mid x < 0, x + c_1 t < 0\}$, $V_2 := \{(x,t) \mid x > 0, x + c_2 t > 0\}$ (cf. figure 1) the Cauchy data and the equation determine the solution u to be $a(x + c_1 t)$, $a(x + c_2 t)$ respectively (in the sense of pull-back of a by the functions $(x,t) \mapsto x + c_1 t$, $(x,t) \mapsto x + c_2 t$). These distributions can be considered as smooth maps in the x-variable with distributional values in t-space or vice versa. By (v) the boundary value of u at $x = 0$ exists and can be used to determine u in the open sectors

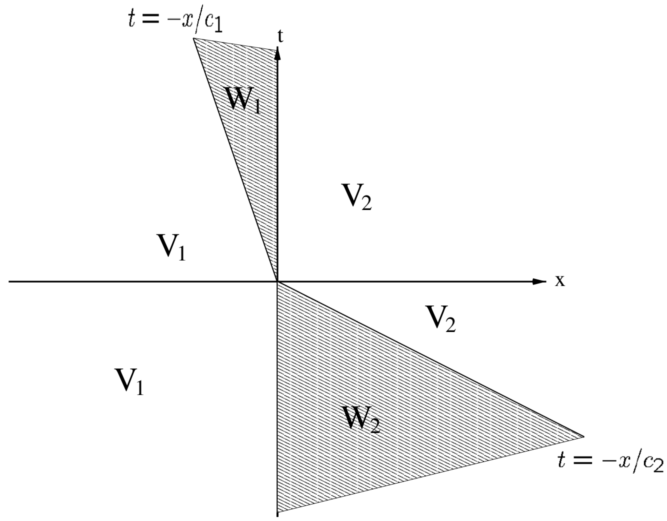

Figure 1. Definition of subdomains

$W_1 := \{(x,t) \mid x < 0, x + c_1 t > 0\}$, $W_2 := \{(x,t) \mid x > 0, x + c_2 t < 0\}$ (cf. figure 1).

We set $b := u \mid_{x=0}$ which by the local continuity with respect to x may be computed as one-sided limits $\lim_{x\to 0^+} u(x,.)$ or $\lim_{x\to 0^-} u(x,.)$ as distributions in the t-variable. For $t > 0$ we take the limit from the right and obtain $b(t) = a(c_2 t)$ as initial value for a Cauchy problem in W_1 where the roles of x and t are interchanged. This yields the formula $u \mid_{W_1} = a((x + c_1 t)c_2/c_1)$ (again, in the sense of the pull-back of a via $(x,t) \mapsto (x + c_1 t)c_2/c_1$). Similarly, we obtain $u \mid_{W_2} = a((x + c_2 t)c_1/c_2)$ by considering for $t < 0$ the limit from the left.

Making use of the coherence properties in the product hierarchy (cf. the appendix) we may compute the product $(R \otimes 1) \cdot u$ as a strict product where only one factor is regularized and then pass to the limit $\varepsilon \to 0$. We choose a net of smooth functions $(R_\varepsilon)_{\varepsilon>0}$ such that $R_\varepsilon \to R$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0$. We will discuss only the upper half space $t > 0$; the case $t < 0$ is completely analogous.

Since $R \otimes 1$ is constant away from the t-axis we focus on the more interesting part of the upper half space near the axis. Let ψ be a test function on $\mathbb{R}^2$ having support near the positive t-axis and not intersecting V_1. Then using the above formulae for u and considering it as a continuous map $x \mapsto u(x,.)$, $\mathbb{R} \to \mathcal{D}'(\mathbb{R})$, we have

$$\langle (R_\varepsilon \otimes 1) \cdot u, \psi \rangle = \langle u(x,t), R_\varepsilon(x)\psi(x,t)\rangle$$
$$= \int_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x+c_1.)), \psi(x,.)\rangle R_\varepsilon(x)\, dx + \int_{0}^{\infty} \langle a(x+c_2.), \psi(x,.)\rangle R_\varepsilon(x)\, dx \ .$$

Both integrands are continuous functions (with respect to x), have support in a fixed compact set independent of ε, and have a pointwise limit as $\varepsilon \to 0$. By dominated convergence we get

$$\langle (R \otimes 1) \cdot u, \psi \rangle = c_1 \int_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1.)), \psi(x,.)\rangle\, dx$$
$$+ c_2 \int_{0}^{\infty} \langle a(x + c_2.), \psi(x,.)\rangle\, dx \ .$$

The verification whether u is a solution is requirement (vi). For a test function ψ as above we recall that

$$\langle u, \psi \rangle = \int_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1.)), \psi(x,.)\rangle\, dx + \int_{0}^{\infty} \langle a(x + c_2.), \psi(x,.)\rangle\, dx$$

and therefore (with $\partial_j \psi$ $(j = 1, 2)$ denoting the derivative of ψ with respect to its first or second argument)

$$\langle \partial_t u, \psi \rangle = -\int_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1.)), \partial_2\psi(x,.)\rangle\, dx$$
$$- \int_{0}^{\infty} \langle a(x + c_2.), \partial_2\psi(x,.)\rangle\, dx$$

and

$$\langle \partial_x\big((R \otimes 1) \cdot u\big), \psi \rangle = -c_1 \int_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1.)), \partial_1\psi(x,.)\rangle\, dx$$
$$- c_2 \int_{0}^{\infty} \langle a(x + c_2.), \partial_1\psi(x,.)\rangle\, dx \ .$$

We rewrite the first integrand appearing in the expression for $\langle \partial_t u, \psi \rangle$ as follows:

$$
\begin{aligned}
- \langle a(\frac{c_2}{c_1}(x + c_1.)), \partial_2 \psi(x,.)\rangle &= c_1 \frac{c_2}{c_1} \langle a'(\frac{c_2}{c_1}(x + c_1.)), \psi(x,.)\rangle \\
&= c_1 \frac{d}{dx} \langle a(\frac{c_2}{c_1}(x + c_1.)), \psi(x,.)\rangle - c_1 \langle a(\frac{c_2}{c_1}(x + c_1.)), \partial_1 \psi(x,.)\rangle \ .
\end{aligned}
$$

If we transform the second integrand in $\langle \partial_t u, \psi \rangle$ in the same way and compare the result with $\langle \partial_x ((R \otimes 1) \cdot u), \psi \rangle$ we find

$$
\langle \partial_t u - \partial_x ((R \otimes 1) u), \psi \rangle = c_1 \langle a(c_2.), \psi(0,.)\rangle - c_2 \langle a(c_2.), \psi(0,.)\rangle \ .
$$

But this implies that

$$
\partial_t u - \partial_x ((R \otimes 1) u) = (c_1 - c_2)\, \delta \otimes a(c_2.)
$$

which is not zero unless $a = 0$ in $(0, \infty)$.

The reasoning near the negative t-axis is analogous and yields that u cannot satisfy (vi) unless $a = 0$ in $(-\infty, 0)$. Since a is assumed to be smooth near 0 we conclude that this would enforce $a = 0$ globally.

Case $c_1 = 0$*:* assume that u is a distribution satisfying (i)-(vi) and without loss of generality that $c_2 = 1$. Set $c(x,t) = x + t$ then $a \otimes 1$, i.e. $a(x)$, and $c^* a$, i.e. $a(x + t)$, solve the problem in the half spaces V_1 and V_2 respectively (notation as at the beginning in the proof; note that now $W_1 = \emptyset$). By smoothness of a near 0 and since $\mathrm{WF}(c^* a)$ cannot contain horizontal cotangent directions we can define the distributions

$$
\begin{aligned}
u_1 &= (H(-x) \otimes 1) \cdot (a \otimes 1) \\
u_2 &= (H(x) \otimes 1) \cdot c^* a \\
w &= u_1 + u_2 \ .
\end{aligned}
\tag{5}
$$

We have $u \mid_{V_j} = u_j \mid_{V_j}$ $(j = 1, 2)$ and therefore $u - w \mid_{\{t \geq 0, x \neq 0\}} = 0$. Furthermore, the u_j are smooth functions of time with values in $\mathcal{D}'$ over x-space, hence w and $u - w$ are continuous with respect to time.

The arguments are similar to the ones used in the first case. By assumption (v) u can be considered to be continuous in x and distributional in t near the t-axis. In particular, upon approaching the positive t-axis from the left, this implies that $a(x)$ is a continuous function for small x which clearly is consistent with assumption (o). Let ψ be a test function with support concentrated near the positive t-axis so that u can be considered continuous in x and distributional in t there. Then we have

$$
\langle u, \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{0} a(x) \psi(x,t)\, dx dt + \int_{0}^{\infty} \langle a(x + .), \psi(x,.)\rangle\, dx \ .
$$

As in the first case we may evaluate the product $(R \otimes 1) \cdot u$ as a strict product with one factor regularized

$$\langle (R \otimes 1) \cdot u, \psi \rangle = \int_0^\infty \langle a(x + .), \psi(x, .) \rangle \, dx \ .$$

It follows that

$$\langle \partial_t u, \psi \rangle = - \int_0^\infty \langle a(x + .), \partial_2 \psi(x, .) \rangle \, dx$$

$$\langle \partial_x \big((R \otimes 1) \cdot u \big), \psi \rangle = - \int_0^\infty \langle a(x + .), \partial_1 \psi(x, .) \rangle \, dx \ .$$

Rewriting

$$- \langle a(x + .), \partial_1 \psi(x, .) \rangle = - \frac{d}{dx} \langle a(x + .), \psi(x, .) \rangle - \langle a(x + .), \partial_2 \psi(x, .) \rangle$$

we arrive at

$$\langle \partial_t u - \partial_x \big((R \otimes 1) \cdot u \big), \psi \rangle = - \langle a, \psi(0, .) \rangle = - \langle \delta \otimes a, \psi \rangle$$

which is nonzero unless $a = 0$ in $(0, \infty)$. □

Note that within M. Oberguggenberger's hierarchy of distributional products condition (v) refers to 'WF favorable' which is on the third level only. The following remarks illustrate the restrictive character of the purely microlocal condition and motivate extensions to be studied below.

Remark 2.

(i) Consider the discussion of the original Hurd-Sattinger example at the beginning of this section. We note that weakening of requirement (v) within the coherent product hierarchy does not remove the ambiguities, even if the initial values were smooth.

(ii) However, in the case $c_1 > 0$ there are global distributional solutions to (1)-(2) which satisfy requirements (i)-(iv) and (vi) but not (v). The following example is due to M. Oberguggenberger [23]: consider the measurable bounded function

$$u = 1 + \Big(\frac{c_2}{c_1} - 1 \Big) \Big(H(-x) H(x + c_1 t) + H(x) H(-x - c_2 t) \Big) \quad (6)$$

which is a solution to (1)-(2) when $a = 1$. (This can be checked by carrying out the product $R \cdot u$ within $\mathrm{L}^\infty(\mathbb{R}^2)$.) Note that the wave front set of u contains exactly the horizontal cotangent directions over the t-axis. A similar construction of a solution is valid for an initial value a equal to the characteristic function of an interval. In the latter case we obtain a strip changing its slope and jumping by c_2/c_1 at $x = 0$. We will prove below that a generalization of this example leads to a general existence result. In addition, we will show at the end of this paper how to obtain these distributions as classical shadows of the (unique) Colombeau solution to the appropriately transfered Cauchy problem.

(iii) Note that condition (v) ensures that the problem can be reformulated as boundary value problem.

(iv) The reader might think that the last (local) result in the above proof, $\partial_t u - \partial_x(R \cdot u) = -\delta \otimes a$, is inconsistent with the discussion of the original Hurd-Sattinger example at the beginning of this section, where we obtained $\partial_t u - \partial_x(R \cdot u) = -\delta' \otimes t/2$ if $a = 1$. But note that in the above proof we assumed the product $R \cdot u$ to exist on a much lower level in the product hierarchy: we assumed the WF-condition to hold and then used the simplest type of so-called strict product to conveniently compute it for general a. In the Hurd-Sattinger example we noticed that we have to use at least the highest level of strict products. Unlike in the Hurd-Sattinger example ($a = 1$) the initial condition is more general in the above proof.

We will show that if we weaken the product requirement (v) within Oberguggenberger's product hierarchy we have the following situation: in case $c_1 > 0$ a general existence result holds while for $c_1 = 0$ we have generic nonexistence even if we substitute the highest hierarchy level in requirement (v).

We begin with the existence case and consider the variant

(v)′ the distributional product $(R(x) \otimes 1) \cdot u$ exists in $\mathcal{D}'(\mathbb{R}^2)$ as a 'strict product (1)-(3)' in the sense of Oberguggenberger's hierarchy.

Theorem 3. Let $c_1 > 0$ then for any initial value $a \in \mathcal{D}'(\mathbb{R})$ satisfying assumption (o) there is a distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ which meets requirements (i)-(iv), (v)′, and (vi). In this sense it is a global distributional solution to the Cauchy problem (1)-(2). It is given explicitly by

$$u(x,t) = H(-x)H(-x - c_1 t)a(x + c_1 t) + H(x)H(x + c_2 t)a(x + c_2 t) + \frac{c_1}{c_2} H(x)H(-x - c_2 t)a(\frac{c_1}{c_2}(x + c_2 t)) + \frac{c_2}{c_1} H(-x)H(x + c_1 t)a(\frac{c_2}{c_1}(x + c_1 t)) \quad (7)$$

where the appearing distributional products are defined in the following way: we recognize in each term that the second and third factor have a pull-back in common and thus form a product of the kind 'disjoint singular support'; the remaining multiplication with the first factor is justified by the 'WF favorable' in the hierarchy. Furthermore, if $a \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ then formula (7) is valid within measurable functions and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$.

Proof. We give the details for the region $\{t > 0\} \setminus \overline{V_1}$ only which includes the forward transmission across the medium discontinuity. The reasoning for the other sectors in $\mathbb{R}^2$ are similar or even simpler (e.g., a classical Cauchy problem in V_1).

Therefore let $\psi \in \mathcal{D}(\mathbb{R}^2)$ have its support in $\{t > 0\} \setminus \overline{V_1}$. Considering (7) as a weakly measurable map in x into distributions in t the action of u on ψ can be written as

$$\langle u, \psi \rangle = \frac{c_2}{c_1} \int\limits_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1 .)), \psi(x, .) \rangle \, dx + \int\limits_{0}^{\infty} \langle a(x + c_2 .), \psi(x, .) \rangle \, dx \ . \tag{8}$$

For the verification of (v)$'$ we regularize $R \otimes 1$ as in [21], (7.1), and use formula (8) to calculate the limit explicitly. We obtain

$$\langle (R \otimes 1) u, \psi \rangle = c_2 \int\limits_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1 .)), \psi(x, .) \rangle \, dx + c_2 \int\limits_{0}^{\infty} \langle a(x + c_2 .), \psi(x, .) \rangle \, dx \ . \tag{9}$$

Thus we have

$$\langle \partial_x ((R \otimes 1) u), \psi \rangle = -c_2 \int\limits_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1 .)), \partial_1 \psi(x, .) \rangle \, dx - c_2 \int\limits_{0}^{\infty} \langle a(x + c_2 .), \partial_1 \psi(x, .) \rangle \, dx$$

and

$$\langle \partial_t u, \psi \rangle = -\frac{c_2}{c_1} \int_{-\infty}^{0} \langle a(\frac{c_2}{c_1}(x + c_1.)), \partial_2 \psi(x, .) \rangle \, dx$$

$$- \int_{0}^{\infty} \langle a(x + c_2.), \partial_2 \psi(x, .) \rangle \, dx \ .$$

The integrands in the latter equation can be rewritten in the same way as in the proof of Thm. 1 to yield

$$\begin{aligned} \langle \partial_t u, \psi \rangle &= \langle \partial_x((R \otimes 1)u), \psi \rangle - \\ & c_2 \langle a(\frac{c_2}{c_1}(x + c_1.)), \psi(x, .) \rangle \mid_{-\infty}^{0} - c_2 \langle a(x + c_2.), \psi(x, .) \rangle \mid_{0}^{\infty} \\ &= \langle \partial_x((R \otimes 1)u), \psi \rangle - c_2 \langle a(c_2.), \psi(0, .) \rangle + c_2 \langle a(c_2.), \psi(0, .) \rangle \\ &= \langle \partial_x((R \otimes 1)u), \psi \rangle \end{aligned}$$

which proves (vi) (and in turn (iii)-(iv) once this is verified for all of $\mathbb{R}^2$).

(i) and (ii) follow directly by using formula (7) for computing the action of u on test functions. □

The case $c_1 = 0$ is fundamentally different. We will prove a generic nonexistence result for the following even weaker variant of the product condition (v):

(v)″ the distributional product $(R(x) \otimes 1) \cdot u$ exists in $\mathcal{D}'(\mathbb{R}^2)$ as a 'model product (4)' in the sense of Oberguggenberger's hierarchy.

As a preparation we recall some product formulas needed in the proof of the theorem below.

Lemma 4.

(i) If $\chi \in \mathcal{D}'(\mathbb{R})$ and $c(x, t) = x + t$ then the distributional product $(\delta \otimes 1) \cdot c^*\chi$ exists in $\mathcal{D}'(\mathbb{R}^2)$ in the sense of Hörmander, i.e. 'WF favorable', and we have

$$(\delta \otimes 1) \cdot c^*\chi = \delta \otimes \chi \ .$$

(ii) Let $\check{H}(x) = H(-x)$ then $H \cdot \check{H} = 0$ and $H \cdot H = H$ by $\mathrm{L}^2_{\mathrm{loc}}$-duality.

(iii) $H \cdot \delta^{(k)}$ exists as 'model product (4)' if and only if $k = 0$ and we have $H \cdot \delta = \delta/2$.

Proof.
ad (i): we note that $\mathrm{WF}(\delta\otimes 1) = \{0\}\times\mathbb{R}\times\mathbb{R}\times\{0\}\setminus 0$ and $\mathrm{WF}(c^*\chi) = \{(x,t,\eta,\eta) \mid (x+t,\eta) \in \mathrm{WF}(\chi)\}$ (apply [11], Thm.8.2.4, to $c^*\chi$ and $d^*c^*\chi$ with $d(x) = (x - d_0, d_0)$ successively) are in favorable positions and therefore the product exists in the sense of Hörmander. We use the coherence within Oberguggenberger's product hierarchy and the parameter product as in [21], 7.12 (d), with the roles of x and t interchanged: observe that $c^*\chi$ can be interpreted as a weakly smooth map $v\colon x \mapsto \chi(x + .)$, $\mathbb{R} \to \mathcal{D}'(\mathbb{R})$, so we have $(\delta \otimes 1)\cdot v = \delta \otimes v(0) = \delta \otimes \chi$.
ad (ii): this is a special case of [21], Prop. 5.2, and the verification is immediate by direct computation.
ad (iii): let $\psi \in \mathcal{D}(\mathbb{R})$ and $(\varphi_\varepsilon)_{\varepsilon>0}$ is a model delta net ([21], (7.9)) then

$$\langle (H * \varphi_\varepsilon)\cdot(\delta^{(k)} * \varphi_\varepsilon), \psi\rangle = \frac{1}{\varepsilon^k}\int_{-\infty}^{\infty}\int_{-\infty}^{x}\varphi(y)\,dy\,\varphi^{(k)}(x)\,\psi(\varepsilon x)\,dx$$

is convergent for all ψ as $\varepsilon \to 0$ if and only if $k = 0$ in which case the stated formula follows by dominated convergence (cf. also [21], 7.12 (a)). □

Theorem 5. Let $a \in \mathcal{D}'(\mathbb{R})$ with $\mathrm{supp}(a) \cap (0,\infty) \neq \emptyset$ and satisfying assumption (o). Then in case $c_1 = 0$ there is no distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ having all the properties (i)-(iv), (v)$''$, and (vi). In this sense no distributional solution to (1)-(2) can exist.

Proof. We use again the decomposition (5) and its basic properties. By construction for all $t > 0$ the distribution $u(t) - w(t) \in \mathcal{D}'(\mathbb{R})$ has support contained in $\{0\}$ and $u(0) - w(0) = 0$. This implies that

$$u_3(t) := u(t) - w(t) = \sum_{k=0}^{\infty} c_k(t)\,\delta^{(k)} \qquad \text{if } t \geq 0$$

where c_k are continuous functions with $c_k(0) = 0$ and for t varying in compact sets only finitely many $c_k(t)$ are nonzero. In the upper half space $t \geq 0$ we may therefore write $u = u_1 + u_2 + u_3$.

The computation of $R \otimes 1 \cdot u$ is reduced to the investigation of the products $H \otimes 1 \cdot u_j$ for $j = 1, 2, 3$ separately . By smoothness of a near 0 and Lemma 4(ii) we have $H \otimes 1 \cdot u_1 = 0$. Furthermore, since

$$\langle u_2, \psi\rangle = \int_{-\infty}^{0} \langle a, \psi(x, . - x)\rangle\,dx$$

one easily verifies that $H \otimes 1 \cdot u_2 = u_2$ by directly inspecting the limit of $\langle u_2, (H * \varphi_\varepsilon)\psi\rangle$ as $\varepsilon \to 0$ for some model delta net $(\varphi_\varepsilon)_\varepsilon$. Finally, by

Lemma 4(iii) $H \otimes 1 \cdot u_3$ exists as model product if and only if $c_k = 0$ for $k > 0$, in which case it attains the value $\delta \otimes c_0/2$. Hence we have

$$R \cdot u = H(x)a(x+t) + \frac{1}{2}\delta(x)c_0(t)$$

and therefore

$$\begin{aligned}\partial_x((R \otimes 1 \cdot u)) &= \delta(x)a(x+t) + H(x)a'(x+t) + \frac{1}{2}\delta'(x)c_0(t) \\ &= \delta(x)a(t) + H(x)a'(x+t) + \frac{1}{2}\delta'(x)c_0(t)\end{aligned}$$

where we used Lemma 4(i) in the last step. Subtracting this from

$$\partial_t u = H(x)a'(x+t) + \delta(x)c_0'(t)$$

yields

$$\partial_t u - \partial_x(Ru) = \delta \otimes (c_0' - a) + \frac{1}{2}\delta' \otimes c_0 \ .$$

This is equal to zero in the forward half space if and only if $c_0' = a$ and $c_0 = 0$ in $(0, \infty)$ which contradicts $\mathrm{supp}(a) \cap (0,\infty) \neq \emptyset$. □

3. Colombeau modeling for hyperbolic equations and refined microlocal analysis

3.1. Existence and uniqueness of global Colombeau solutions

It was shown by Oberguggenberger ([19], see also [21], example 17.6) that there is a remedy for the dissatisfying situation discussed in the previous section in the framework of Colombeau algebras. We depart from our specific example and discuss the general case in space dimension m. Consider the following hyperbolic Cauchy problem in $\mathbb{R}^{m+1}$.

$$\partial_t U - \sum_{j=1}^{m} A_j(x,t)\partial_{x_j} U - B(x,t)U = F(x,t) \tag{10}$$

$$U(x,0) = G(x) \tag{11}$$

where A_j $(j = 1, \ldots, m)$, B are real valued generalized functions in $\mathcal{G}(\mathbb{R}^{m+1})$ (in the sense that all representatives are real valued smooth functions) and initial value $G \in \mathcal{G}(\mathbb{R}^m)$. We mention that all statements in this subsection are valid for symmetric hyperbolic systems (which brings us back at the two-way wave equation) but here we will only relate to the scalar case.

The coefficients will be subject to some restriction on the allowed divergence in terms of ε-dependence. A Colombeau function $V \in \mathcal{G}(\mathbb{R}^n)$ is said to be of *logarithmic type* if it has a representative $v(\phi)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$, $\eta > 0$ with

$$\sup_{y \in \mathbb{R}^n} |v(\phi_\varepsilon, y)| \leq N \log\left(\frac{C}{\varepsilon}\right) \qquad 0 < \varepsilon < \eta \, . \tag{12}$$

(This property then holds for any representative.)

Theorem 6 (Lafon-Oberguggenberger [15]). Assume that A_j and B are constant for large $|x|$ and that $\partial_{x_k} A_j$ $(k = 1, \ldots, m)$ as well as B are of logarithmic type. Then given initial data $G \in \mathcal{G}(\mathbb{R}^m)$, problem (10)-(11) has a unique solution $U \in \mathcal{G}(\mathbb{R}^{m+1})$.

We also mention the following consistency result which shows that Colombeau theory includes the classically solvable cases.

Proposition 7 (Lafon-Oberguggenberger [15]). In the above Theorem, assume additionally that the coefficients A_j, B are smooth.

(i) If F and G are smooth then the generalized solution $U \in \mathcal{G}(\mathbb{R}^{m+1})$ is equal (in $\mathcal{G}$) to the classical smooth solution.

(ii) If $F \in \mathrm{L}^2(\mathbb{R}, H^s(\mathbb{R}^m))$ and $G \in H^s(\mathbb{R}^m)$ for some $s \in \mathbb{R}$, then the generalized solution $U \in \mathcal{G}(\mathbb{R}^{m+1})$ is associated to the classical solution belonging to $C(\mathbb{R}, H^s(\mathbb{R}^m))$.

The sample differential equation (1)-(2) can be modeled as an equation with coefficient $R(x) = R(x) \otimes 1(t) \approx \Lambda$ in $\mathcal{G}(\mathbb{R}^2)$ and be rewritten in the form

$$\partial_t U - \Lambda\, \partial_x U = (\partial_x \Lambda) U \, . \tag{13}$$

A general initial condition, viz.

$$U|_{t=0} = A \in \mathcal{G}(\mathbb{R}) \tag{14}$$

can then be prescribed. The Colombeau function Λ is given by a representative $(\lambda(\phi))_\phi \in \mathcal{E}_{\mathrm{M}}$, i.e. a family of smooth functions parameterized by mollifiers $\phi \in \mathcal{A}_0$, with the property $\lambda(\phi_\varepsilon) \to R \otimes 1$ in $\mathcal{D}'$ as $\varepsilon \to 0$. $\lambda(\phi, x, t)$ is constructed by choosing a (real valued) test function $\chi \in \mathcal{D}(\mathbb{R})$ with $\int \chi = 1$ and using a combined convolution and scaling regularization (recall from the introduction that each $\phi \in \mathcal{A}_0(\mathbb{R}^2)$ is of the form $\phi_0 \otimes \phi_0$):

$$\lambda(\phi, x, t) = R \overset{(x)}{*} (\mu(\phi)\chi(\mu(\phi).))(x)$$
$$= c_1\, \mu(\phi) \int_x^\infty \chi(\mu(\phi)y)\, dy + c_2\, \mu(\phi) \int_{-\infty}^x \chi(\mu(\phi)y)\, dy \quad (15)$$

where $\mu(\phi_0 \otimes \phi_0) = \log(1/l(\phi_0))$ (here, $l(\phi_0)$ is the support number as defined in the introduction). Note that this gives a scaling factor of $\log(1/\varepsilon) - \log(l(\phi_0))$ when evaluated for ϕ_ε (the reason behind the choice of this scaling will become clear by a short digression into theory in Sect. 3; cf. Rem. 8 and Thm. 6). Clearly $R(x)$ is the weak limit of $\lambda(\phi, x, t)$ as $\varepsilon \to 0$. Since $\lambda(\phi)$ is independent of t we will henceforth often suppress the t-variable (we only have to keep in mind that it is considered to be a Colombeau function in $\mathbb{R}^2$). Substituting $\mu(\phi)y \to y$ in the integrals and rewriting $\int_x^\infty \chi = 1 - \int_{-\infty}^x \chi$ we find

$$\lambda(\phi, x) = c_1 + (c_2 - c_1) \int_{-\infty}^{\mu(\phi)x} \chi(y)\, dy\ . \quad (16)$$

Remark 8. We emphasize (as mentioned already in [15], p. 99) that a slight modification of [20], Prop. 1.5., provides us with the following important result for modeling distributional coefficients: for any w in the Sobolev space $W^{-k,\infty}(\mathbb{R}^{m+1})$ ($k \in \mathbb{Z}$) one can construct a Colombeau function $\tilde{W}$ associated with w and being of logarithmic type. For example, we have used exactly this construction in equation (15) to model the discontinuous coefficient $R \otimes 1$ in a way that its derivative is of logarithmic type.

Furthermore, we will make the following *physical assumptions* about the modeling.

Assumption 9.

(i) $c_2 > c_1 \geq 0$ or in terms of the refraction index $n = c_1/c_2$ we have $0 \leq n < 1$

(ii) all regularized medium approximations have non-negative sound velocities $\lambda(\phi)$, i.e. we have

$$\psi(z) := c_1 + (c_2 - c_1) \int_{-\infty}^{z} \chi(y)\, dy \geq 0$$

(this is guaranteed for example if χ is non-negative).

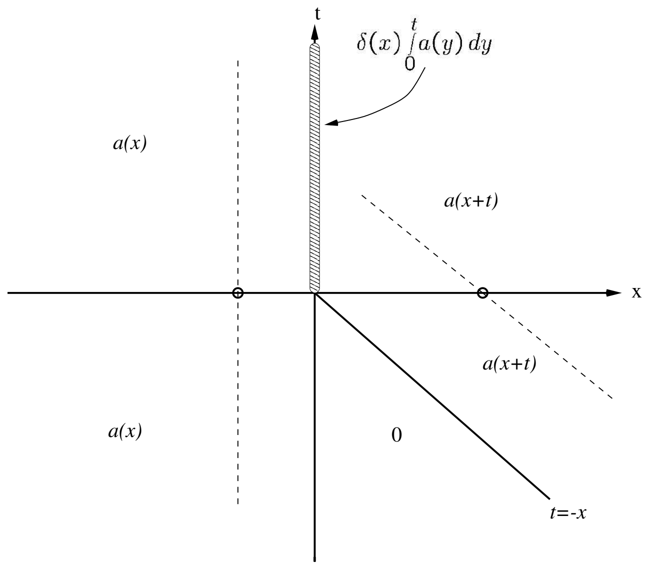

Figure 2. Associated distribution

In the case $c_1 = 0$ the following is shown in [21], ex. 17.6: whenever A is (the canonical image of) a locally integrable function a then U has a distributional shadow which can be computed explicitly and has the properties (i)-(iv) above (formula on p. 163 in [21]; note that the last term there tends to 0 as a distribution in x when $t \to 0$). This is illustrated by figure 2 which is redrawn from [21] (Fig. 4.3, p. 164). Focusing on the possible singularity structure of the Colombeau solution U in the general case $c_1 \geq 0$, this picture is suggestive of expecting different behavior in various regions of space-time (see also figure 1):

(i) in the region V_1 the characteristic flow propagates singularities of the initial data a along the lines parallel to $x + c_1 t = 0$ (vertical if $c_1 = 0$); this should be reflected in spectral (Fourier, cotangent) components of the wave front set of U being parallel to $c_1\xi - \tau = 0$ (or horizontal if $c_1 = 0$) over this region.

(ii) in the region V_2 the characteristic flow propagates initial singularities of a along the lines $t = -x/c_2 + t_0$ ($t_0 > 0$) with cotangent components of the wave front set being perpendicular; note that, eventually, these singularities will hit the positive half of the axis $x = 0$.

(iii) the boundaries of the two sectors W_2 and W_1 (which is empty if $c_1 = 0$) will be part of the singular support as long as a does not vanish of infinite order at 0; e.g., in the case $c_1 = 0$ and if a is continuous at 0 and $a(0) \neq 0$ there will be jumps across the half lines $\{0\} \times \mathbb{R}_-$ and $\{(s, -s/c_2) | s > 0\}$ (with cotangent wave front set vectors typically perpendicular to it).

(iv) the most striking non-classical feature appears along the singular support of the coefficient: for $c_1 = 0$ we saw that there piles up an additional delta-like singularity; this is caused by the coefficient singularity and not by the initial data or a singular right hand side in the equation (note that even if a is smooth this singularity along $x = 0$, $t > 0$ appears); in general, i.e., for arbitrary $0 \leq c_1 < c_2$ and initial value $a \in \mathcal{D}'$, it is not straightforward to guess how the microlocal properties of U are affected by the interaction of the medium singularity at $x = 0$ with the 'arriving' initial singularities propagating in from the right.

The aim of this paper is to analyze the generalized characteristic set and the nonlinear interaction of the singularities at the medium discontinuity from a microlocal point of view.

3.2. Modeling of coefficients with equivalent microlocal properties

Before studying microlocal properties of a Colombeau solution to initial value problems like (13)-(14) we first have to carefully inspect if a transfer or modeling process from given distributional data to appropriate Colombeau objects respects the properties we are interested in. Thereby we want to keep enough flexibility in the modeling methods and also ensure later applicability of more general solvability results to equations of the above type. To this end we not only consider the canonical embedding $\mathcal{D}' \hookrightarrow \mathcal{G}$ but will allow a wider class of related mappings constructed via more general combinations of convolution and scaling, already encountered in (15).

The foundation of an intrinsic regularity theory within Colombeau algebras was laid in [21], Sect. 25, via the definition of the subalgebra $\mathcal{G}^\infty \subseteq \mathcal{G}$ of *regular Colombeau functions*. Its elements are exactly those generalized functions having representatives with the same power of ε-growth in each derivative on compact sets. This is motivated by the classical result that a distribution all derivatives of which are measures, and are therefore of order 0, is a smooth function. The algebra of regular functions has the property

$$\mathcal{G}^\infty \cap \mathcal{D}' = \mathrm{C}^\infty ,$$

hence the notion of (smooth) regularity is consistent with the one in subspace of distributions.

By locality of the $\mathcal{G}^\infty$-property we have a consistent extension of the notion of *singular support* (singsupp_g) of a Colombeau function, defined as the complement of the largest open set on which a generalized function is regular in the above sense. We illustrate this regularity notion in two simple situations.

Example 10.

(i) $\text{singsupp}_g \iota(\delta) = \{0\} = \text{singsupp}\, \delta$: since $\iota(\delta)|_{\mathbb{R}^n \setminus \{0\}} = 0$ we only have to check at $x = 0$; the typical representative of $\iota(\delta)$ is $w(\phi, x) = \phi(x)$ and

$$|\partial^\alpha w(\phi_\varepsilon, 0)| = \varepsilon^{-k-1} |\partial^\alpha \phi(0)| \,;$$

this is a lower bound for the supremum taken over any compact subset containing 0. Whatever $N \in \mathbb{N}_0$, there is a $\phi \in \mathcal{A}_N$ with $\partial^\alpha \phi(0) = c \neq 0$ for infinitely many α. (By the tensor product structure of elements in $\mathcal{A}_N(\mathbb{R}^n)$ for $n > 1$ it suffices to prove this for $\mathcal{A}_N(\mathbb{R})$; one may start with $\psi \in \mathcal{A}_0(\mathbb{R})$, near 0 of the form $\psi(x) = (e^x + e^{-x})/2$, and proceed as in the second part of the proof of Lemma 9.0 in [21].)

(ii) we show that Λ, as defined in (15), is in $\mathcal{G}^\infty(\mathbb{R}^2)$ — this is essentially included already in the remarks preceding [21], Thm. 25.2; we give some details because for our applications this also points out the need for (an obvious) refinement of regularity theory which we will sketch below. Any t-derivative of order ≥ 1 of the representative λ in (15) gives 0 and $\lambda(\phi_\varepsilon, x, t)$ is bounded uniformly for $0 < \varepsilon < 1$. Therefore it is sufficient to check x-derivatives of order $k \geq 1$. Using the shorthand notation $\mu_\varepsilon := \mu(\phi_\varepsilon) = O(\log(1/\varepsilon))$ we obtain the estimate

$$\begin{aligned} |\partial_x^k \lambda(\phi_\varepsilon, x, t)| &= |\partial_x^k \int_{-\infty}^{x} \mu_\varepsilon \chi(\mu_\varepsilon y)\, dy| = \mu_\varepsilon^k |\chi^{(k-1)}(\mu_\varepsilon x)| \\ &\leq \mu_\varepsilon^k \|\chi^{(k-1)}\|_{\mathrm{L}^\infty} = O\big((\log(1/\varepsilon))^k\big) = O(1/\varepsilon)\,, \end{aligned} \tag{17}$$

which tells us that the $\mathcal{G}^\infty$-property is satisfied with uniform ε-power -1.

Localization of a Colombeau function U near a point x_0 can be achieved by using cutoff functions $\varphi \in \mathcal{D}$ with $\varphi(x_0) = 1$. Then φU has compact support and a natural extension of the Fourier transform is available to analyze its singularity spectrum. This was initiated in [3, 12, 17] extending many results from distribution theory in terms of wave front sets ([11], Ch. 8). The *generalized wave front set* WF_g of Colombeau functions is also a consistent extension because we have

$\mathrm{WF}_g(\iota(f)) = \mathrm{WF}(f)$ if $f \in \mathcal{D}'$ (cf. [17], Thm. 3.8, and [12], Cor. 24 and Thm. 25).

In example 10, (ii), above we saw that $\mathrm{singsupp}_g \Lambda = \emptyset$ while $\Lambda \approx H \otimes 1$ and $\mathrm{singsupp} H \otimes 1 = \{0\} \times \mathbb{R}$ (the whole t-axis). The logarithmic rescaling μ_ε in the modeling Colombeau function Λ assures solvability of the equations of interest but also suppresses the original singularity structure. But a closer look at the estimate also shows how this information could be preserved in $\mathrm{singsupp}_g$, viz. by measuring regularity in terms of uniform powers of μ_ε. It turns out indeed that we only need to adapt all basic notions of microlocal analysis to more generally scaled ε-growth to be able to preserve the microlocal information upon modeling the distributional coefficients as above.

The refined (i.e., rescaled) regularity theory for Colombeau functions can be developed by modification of the already existing theory in [3, 12, 17, 21]. For the sake of completeness we give detailed definitions and sketch the proof of 'microlocal invariance' below.

First, we specify admissible scaling functions γ and give an appropriate definition of regularity in terms of a subalgebra $\mathcal{G}^\infty_\gamma(\Omega)$ of $\mathcal{G}(\Omega)$, Ω an open subset of $\mathbb{R}^m$.

Definition 11.

(i) An *admissible scaling* γ is a continuous function $\gamma : (0,1) \to \mathbb{R}_+$ with the following properties:

   (a) $\gamma(r) \to \infty$ and $\gamma(r) = O(1/r)$ as $r \to 0$

   (b) for any $s > 0$: $\gamma(sr) = O(\gamma(r))$ as $r \to 0$

(ii) Let γ be an admissible scaling, then the algebra $\mathcal{G}^\infty_\gamma(\Omega)$ of γ-*regular Colombeau functions* is the set of all $U \in \mathcal{G}(\Omega)$ which have a representative $u \in \mathcal{E}_\mathrm{M}(\Omega)$ with the property: for all compact subsets $K \subset \Omega$ there is (a uniform growth order) $N \in \mathbb{N}$ such that for all (derivative orders) $\alpha \in \mathbb{N}_0^m$ there is $M \in \mathbb{N}$ so that for all mollifiers $\phi \in \mathcal{A}_M$ there are constants $C > 0$, $\eta > 0$ such that it holds

$$\sup_{x \in K} |\partial^\alpha u(\phi_\varepsilon, x)| \le C \gamma(\varepsilon)^N \qquad 0 < \varepsilon < \eta\,. \tag{18}$$

Remark 12. The continuity requirement for scalings in Def. 11, (i), is just a matter of technical convenience (for proofs) and not essential for the regularity property itself; Def. 11, (ii), is exactly Def. 25.1 from [21] if $\gamma(r) = 1/r$.

The *generalized* γ-*singular support* ($\mathrm{singsupp}^\gamma_g$) of a Colombeau function is then defined as the complement of the largest open set where the function is γ-regular in the sense of Def. 11, (ii).

As in the classical situation we can test γ-regularity of a function with compact support by establishing appropriate decay properties of its Fourier transform. Note that a compactly supported Colombeau function U always has a representative $u(\phi, .)$ with support contained in a fixed compact set (use appropriate cut-off by multiplication with a smooth function) and its generalized Fourier transform may be consistently computed by application of the classical formula to the representative (cf. [12], Rem. 19, (i)). The proof of the following theorem coincides with the one of [12], Thm. 18, with ε^{-1} replaced by $\gamma(\varepsilon)$ in the final estimates.

Theorem 13. If U is a Colombeau function of compact support and $\mathcal{F}U$ denotes its generalized Fourier transform then the following statements are equivalent:

(i) U is in $\mathcal{G}_\gamma^\infty$

(ii) $\mathcal{F}U$ is *γ-rapidly decreasing* in $\mathbb{R}^m$, which means that its representative $\widehat{u(\phi, .)}$ (and therefore any representative) has the following property: there is $N \in \mathbb{N}$ such that for all $p \in \mathbb{N}_0$ we can choose $M \in \mathbb{N}_0$ so that for all $\phi \in \mathcal{A}_M$ there are positive constants C, η such that
$$|\widehat{u(\phi_\varepsilon, .)}(\xi)| \leq C\gamma(\varepsilon)^N (1 + |\xi|)^{-p} \tag{19}$$
holds $\forall \xi \in \mathbb{R}^m$, $0 < \varepsilon < \eta$.

To put this result into a microlocal context is straightforward.

Definition 14. Let V be a Colombeau function over Ω. A pair $(x_0, \xi_0) \in \Omega \times \mathbb{R}^m \setminus 0$ ($\Omega \times \mathbb{R}^m$ with the zero section removed) is called *microlocally γ-regular for V* if there is $\varphi \in \mathcal{D}$, $\varphi(x_0) = 1$, and a conic neighborhood Γ of ξ_0 such that for $U = \varphi V$ the estimate (19) holds $\forall \xi \in \Gamma$, i.e., $\mathcal{F}(\varphi V)$ is *γ-rapidly decreasing in the cone* Γ. The generalized wave front set $\mathrm{WF}_g^\gamma(V)$ is the complement (in $T^*\Omega \setminus 0$) of all microlocally γ-regular pairs for V. Clearly, $\mathrm{WF}_g^\gamma(V) = \mathrm{WF}_g(V)$ for $\gamma(r) = 1/r$ and we will use the standard notation $\mathrm{WF}_g(V)$ in this case. We denote by $\Sigma_{x_0}^\gamma(V) = \{\xi \mid (x_0, \xi) \text{ is microlocally } \gamma\text{-irregular}\}$ the cone of γ-irregular (cotangent) directions at x_0 (or the fiber over x_0).

As in [12], Sect. 5, the basic properties of WF_g^γ are easily obtained with the classical procedure. In particular, WF_g^γ is a closed conic subset of $\Omega \times \mathbb{R}^m \setminus 0$ and its projection to Ω gives exactly $\mathrm{singsupp}_g^\gamma$.

Now we are in a position to investigate the microlocal properties of modeling Colombeau functions like Λ given in (15). Observe that Λ is obtained from the distribution $H \otimes 1$ by a (space and time) convolution with the scaled mollifier $(x,t) \mapsto \mu(\phi)^2\chi(\mu(\phi)x)\chi(\mu(\phi)t)$ making use of the identity

$$1 \overset{(t)}{*} \mu(\phi)\chi(\mu(\phi).) = \int \mu(\phi)\chi(\mu(\phi)t)dt = \int \chi = 1\,.$$

If we choose the admissible scaling $\gamma(r) = \log(1/r)$ then the first line in (17) shows that along $x = 0$ we will be able to detect a γ-singularity since $\mu_\varepsilon = \gamma(\varepsilon) - \text{const}$, as noted after (15). In fact, the following stronger statement is true in general for modeling processes of this type.

Theorem 15. Let γ be an admissible scaling and choose $\chi \in \mathcal{D}(\mathbb{R}^m)$ with $\int \chi = 1$. Define the modeling map $\iota_\chi^\gamma : \mathcal{D}' \to \mathcal{G}$ by setting $\iota_\chi^\gamma(w)$ equal to the Colombeau class of the $\mathcal{E}_{\mathrm{M}}(\mathbb{R}^m)$ function

$$(\phi, x) \mapsto (w * \chi^\gamma(\phi,.))(x) := \gamma(l(\phi_0))^m (w * \chi(\gamma(l(\phi_0)).))(x) \qquad (20)$$

where $\phi = \phi_0 \otimes \cdots \otimes \phi_0$ with $\phi_0 \in \mathcal{A}_0(\mathbb{R})$. Then we have for any distribution w invariance of the microlocal properties in the following sense

$$\mathrm{WF}_g^\gamma(\iota_\chi^\gamma(w)) = \mathrm{WF}(w)\,. \qquad (21)$$

In particular, this includes equality of the singular supports

$$\operatorname{singsupp}_g^\gamma \iota_\chi^\gamma(w) = \operatorname{singsupp} w\,.$$

Proof. This is essentially an adaption of the proof of [17], Thm. 3.8.

Step 1: $\quad (x_0, \xi_0) \notin \mathrm{WF}(w) \implies (x_0, \xi_0) \notin \mathrm{WF}_g^\gamma(\iota_\chi^\gamma(w))$

Let $\varphi \in \mathcal{D}$ with $\varphi(x_0) = 1$ and $\psi \in \mathcal{D}$ with $\psi = 1$ in a neighborhood of $\operatorname{supp}\varphi$. Using the notation $\chi^\gamma(\phi, x) = \gamma(l(\phi_0))^m \chi(\gamma(l(\phi_0))x)$ the Fourier transform of $\varphi \cdot \iota_g^\gamma(w)$ has the representative (with $\mathcal{F}$ denoting the classical Fourier transform)

$$(\phi, \xi) \mapsto \mathcal{F}(\varphi \cdot (w * \chi^\gamma(\phi,.)))(\xi)\,.$$

If ε is small enough the support of $\chi^\gamma(\phi_\varepsilon,.)$ will be so small that for x in the support of φ we may rewrite $(w * \chi^\gamma(\phi_\varepsilon,.))(x)$ as $((\psi w) * \chi^\gamma(\phi_\varepsilon,.))(x)$. Hence the above representative evaluated at (ϕ_ε, ξ) can be written in the form

$$\Big(\widehat{\varphi} * (\widehat{(\psi w)} \cdot \widehat{\chi^\gamma(\phi_\varepsilon,.)})\Big)(\xi) = \int \widehat{\varphi}(\xi - \eta)\, \widehat{(\psi w)}(\eta)\, \widehat{\chi}(\frac{\eta}{\gamma_\varepsilon})\, d\eta \qquad (22)$$

where we have used the short notation $\gamma_\varepsilon := \gamma(\varepsilon l(\phi_0)) = O(\gamma(\varepsilon))$. By assumption there exists a conic neighborhood Γ of ξ_0 such that for supports of φ, ψ small enough the function $\widehat{(\psi w)}$ is rapidly decreasing in Γ.

As in the proof of [11], Lemma 8.1.1, we can find a closed conic neighborhood $\Gamma_1 \subset \Gamma \cup \{0\}$ and a constant $c > 0$ such that

$$\xi \in \Gamma_1, \eta \notin \Gamma \implies |\xi - \eta| \geq c|\xi|\,.$$

Then we split the estimation of the integral in (22) at $\xi \in \Gamma_1$ into two parts

$$|\int \widehat{\varphi}(\xi-\eta)\,\widehat{(\psi w)}(\eta)\,\widehat{\chi}(\frac{\eta}{\gamma_\varepsilon})|\,d\eta \leq \int\limits_{\Gamma} |\widehat{\varphi}(\xi-\eta)||\widehat{(\psi w)}(\eta)||\widehat{\chi}(\frac{\eta}{\gamma_\varepsilon})|\,d\eta + \\ + \int\limits_{\mathbb{R}^m\setminus\Gamma} |\widehat{\varphi}(\xi-\eta)||\widehat{(\psi w)}(\eta)||\widehat{\chi}(\frac{\eta}{\gamma_\varepsilon})|\,d\eta =: I_1(\xi) + I_2(\xi)\,.$$

$I_1(\xi)$ contains only rapidly decreasing integrand factors, the first and second having bounds of the form $(1+|\xi-\eta|^2)^{-l}$ and $(1+|\eta|^2)^{-l}$ times some constant. Application of Peetre's inequality yields a bound $(1+|\xi|^2)^{-l}$ times some other constant, l an arbitrary, positive integer. The remaining integral $\int |\widehat{\psi}(\eta/\gamma_\varepsilon)|\,d\eta$ is bounded by a constant times $\gamma_\varepsilon^k \int d\eta/(1+|\eta|)^k$, k large enough but fixed.

In $I_2(\xi)$ we first estimate $|\widehat{\varphi}(\xi-\eta)|$ by (a constant times) $(1+|\xi-\eta|)^{-l}$, l an arbitrary, positive integer. Using the above stated property of Γ_1 this is bounded by $(1+c|\xi|)^{-l}$. The remaining integral involves the polynomially bounded factor $|\widehat{(\psi w)}(\eta)|$ (since ψw is smooth and has compact support) which together with the last rapidly decreasing factor gives again a bound of the form γ_ε^k times some constant, k large enough but fixed.

In summary, the generalized Fourier transform of $\varphi \cdot \iota_\chi^\gamma$ has a representative which can be dominated by $C_{l,\phi,k}\gamma_\varepsilon^k(1+|\xi|)^{-l}$ for l arbitrarily large, ε small, k large enough but fixed, and $\xi \in \Gamma_1$. This completes the first step.

Step 2: $\quad (x_0,\xi_0) \notin \mathrm{WF}_g^\gamma(\iota_\chi^\gamma(w)) \implies (x_0,\xi_0) \notin \mathrm{WF}(w)$

¿From the assumption it follows that there is a conic neighborhood Γ of ξ_0 and $\varphi \in \mathcal{D}$, $\varphi(x_0) = 1$, and $N \in \mathbb{N}_0$ such that for arbitrary $p \in \mathbb{N}_0$, we can find $M \in \mathbb{N}_0$ so that for all mollifiers $\phi \in \mathcal{A}_M$ with appropriate positive constants C and ε_0,

$$|\mathcal{F}(\varphi \cdot (w * \chi^\gamma(\phi_\varepsilon,.)))(\xi)| \leq C\gamma_\varepsilon^N(1+|\xi|)^{-p} \qquad \forall \xi \in \Gamma, 0 < \varepsilon < \varepsilon_0\,.$$

Let $\psi \in \mathcal{D}$ with $\psi = 1$ in a neighborhood of $\operatorname{supp}\varphi$. We have

$$|\widehat{(\varphi w)}(\xi)| \leq |\mathcal{F}(\varphi\cdot(w - w * \chi^\gamma(\phi_\varepsilon,.)))(\xi)| + |\mathcal{F}(\varphi\cdot(w * \chi^\gamma(\phi_\varepsilon,.)))(\xi)|$$

where we can estimate the second term on the right hand side for $\xi \in \Gamma$ as above. As noted in the first step of the proof for ε small we may insert

ψ as additional factor for w in the above convolutions and can therefore rewrite the first term in the form

$$|\mathcal{F}(\varphi \cdot ((\psi w) * (\delta_0 - \chi^\gamma(\phi_\varepsilon, .))))(\xi)| = |\widehat{\varphi} * (\widehat{(\psi w)} \cdot (1 - \widehat{\chi^\gamma(\phi_\varepsilon, .)}))(\xi)|$$
$$\leq \int |\widehat{\varphi}(\eta)||\widehat{(\psi w)}(\xi - \eta)||1 - \widehat{\chi}(\frac{\xi - \eta}{\gamma_\varepsilon})|\, d\eta\, .$$

We can use a polynomial bound (in $\xi - \eta$) for the second factor and by Taylor expansion a bound $|\xi - \eta|/\gamma_\varepsilon$ times a constant for the third factor (note that $\widehat{\chi}$ is real analytic and $\widehat{\chi}(0) = 1$). Altogether the first two factors can be bounded by $C\gamma_\varepsilon^{-1}(1 + |\xi - \eta|^2)^L$, $L \in \mathbb{N}$ fixed, which is in turn bounded by $C'\gamma_\varepsilon^{-1}(1 + |\xi|^2)^L(1 + |\eta|^2)^L$. Since $\widehat{\varphi}$ is rapidly decreasing $\int |\widehat{\varphi}(\eta)|(1 + |\eta|^2)^L\, d\eta$ is bounded by some constant which yields finally a bound of the form $\gamma_\varepsilon^{-1}(1 + |\xi|^2)^L$ times some constant.

In summary, we may state that there are $N \in \mathbb{N}$ and $M \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ there is some positive constant C so that

$$|\widehat{(\varphi w)}(\xi)| \leq C\Big(\frac{(1 + |\xi|)^{M+1}}{\gamma_\varepsilon} + \frac{\gamma_\varepsilon^N}{(1 + |\xi|)^p}\Big)$$

is valid for all $\xi \in \Gamma$. For $|\xi| \geq 1$ we may further rewrite this with some positive constants c', c'' in the form

$$|\xi|^p\gamma_\varepsilon^{-N}|\widehat{(\varphi w)}(\xi)| \leq c'\gamma_\varepsilon^{-N-1}|\xi|^{p+M+1} + c''\, . \tag{23}$$

We now proceed using exactly the idea at the end of the proof of [21], Thm. 25.2. We assert that

$$|\xi|^{\frac{p-(M+1)N}{N}}|\widehat{(\varphi w)}(\xi)| \quad \text{is bounded uniformly for} \quad \xi \in \Gamma, |\xi| \geq 1\, .$$

Since p may be chosen arbitrarily large this will complete step 2.

We prove the above assertion by contradiction assuming that one can find a sequence ξ_j ($j \in \mathbb{N}$) in Γ such that $|\xi_j| \to \infty$ and

$$|\xi_j|^{\frac{p-(M+1)N}{N}}|\widehat{(\varphi w)}(\xi_j)| \to \infty$$

as $j \to \infty$. For each j we can choose ε_j such that $\gamma_{\varepsilon_j} = |\xi_j|^{\frac{p-(M+1)N}{N}}$; we may assume that $\varepsilon_j \to 0$ as $j \to 0$. If we insert ξ_j and ε_j into inequality (23) and send $j \to \infty$ we arrive at the contradiction that the right hand side of (23) stays bounded while the left hand side tends to ∞. □

Remark 16. As an immediate application of Thm.15 we observe that for $\Lambda \in \mathcal{G}(\mathbb{R}^2)$, as constructed in (15), and $\gamma(r) = \log(1/r)$ we have

$$\mathrm{WF}_g^\gamma(\Lambda) = \mathrm{WF}(R \otimes 1) = \{0\} \times \mathbb{R} \times \big(\mathbb{R} \times \{0\} \setminus \{(0,0)\}\big)$$

while we had $\mathrm{WF}_g(\Lambda) = \emptyset$.

4. The generalized characteristic set

This section is devoted to the analysis of the differential operator in equation (13). Subsection 4.1 collects useful formulae for the solution in our example in particular. In subsection 4.2 we investigate in detail the generalized characteristic set of the operator which in the classical case is the key to the study of propagation of singularities.

4.1. Representations of the generalized solution

In our simple example (13)-(14) we can make use of the special structure to obtain explicit formulae for a representative of the unique Colombeau solution U (along the lines of [21], Ex. 17.6).

Consider the representative $\lambda(\phi)$ of Λ given by equation (15). Let $a(\phi)$ be a representative of $A \in \mathcal{G}(\mathbb{R})$. We want to specify a convenient representative $u(\phi)$ of $U \in \mathcal{G}(\mathbb{R}^2)$ given for fixed ϕ as the unique solution of

$$\partial_t u(\phi) - \lambda(\phi)\partial_x u(\phi) = \lambda'(\phi)u(\phi) \tag{24}$$

$$u(\phi, x, 0) = a(\phi, x)\ . \tag{25}$$

Since $\lambda(\phi)$ is real valued and smooth we can employ the method of characteristics to determine $u(\phi)$: for ϕ fixed denote by $\sigma(\phi, x, t; s)$ the unique smooth and (by boundedness of $\lambda(\phi)$) global solution of the initial value problem ($\dot\sigma$ denoting $\frac{d}{ds}\sigma$)

$$\dot\sigma(\phi, x, t; s) = -\lambda(\phi, \sigma(\phi, x, t; s)) \tag{26}$$

$$\sigma(\phi, x, t; t) = x\ . \tag{27}$$

Then $u(\phi)$ is given by (note again that $\phi \in \mathcal{A}_0(\mathbb{R}^2)$ is of the form $\phi_0 \otimes \phi_0$ for some $\phi_0 \in \mathcal{A}_0(\mathbb{R})$)

$$u(\phi, x, t) = \underbrace{a(\phi_0, \sigma(\phi, x, t; 0))}_{a_\sigma(\phi,x,t)} \underbrace{\exp\Big(\int_0^t \lambda'(\phi, \sigma(\phi, x, t; s))\, ds\Big)}_{e(\phi,x,t)} \tag{28}$$

which we can interpret as the product of the two Colombeau functions $A_\Sigma = \mathrm{cl}[(a_\sigma(\phi, x, t))_\phi]$ and $E = \mathrm{cl}[(e(\phi, x, t))_\phi]$, i.e., $U = A_\Sigma E$.

If we strengthen the physical non-negativity assumption 9(ii) on $\lambda(\phi)$ to $\chi \geq 0$ and use (26) we deduce that $\dot\sigma(\phi, x, t; s) < 0$ unless $\lambda(\phi, \sigma(\phi, x, t; s))$ vanishes and is stationary in which case the integrand in the definition of e vanishes. Hence $s \mapsto \sigma(\phi, x, t; s)$ is strictly monotone for the relevant values. If $c_1 > 0$ then $\lambda'(\phi, r)/\lambda(\phi, r)$ is always well-defined, smooth, and integrable with respect to r. This allows for the substitution $r = \sigma(\phi, x, t; s)$, $dr = -\lambda(\phi, r)ds$, in the integral and yields

$$e(\phi, x, t) = \exp\,(\int\limits_{\sigma(\phi,x,t;0)}^{x} \frac{\lambda'(\phi, r)}{\lambda(\phi, r)}\, dr) = \frac{\lambda(\phi, \sigma(\phi, x, t; 0))}{\lambda(\phi, x)} \qquad \text{if } c_1 > 0\,. \tag{29}$$

In doing estimates upon inserting ϕ_ε later on it is convenient to introduce the shorthand notation $\lambda^\varepsilon(r) = \lambda(\phi_\varepsilon, r)$ and a^ε, σ^ε, u^ε, a^ε_σ, e^ε, in the same way. Then the solution formula for the representative reads

$$u^\varepsilon(x, t) = a^\varepsilon(\sigma^\varepsilon(x, t; 0)) \exp\,(\int\limits_0^t \lambda^{\varepsilon\prime}(\sigma^\varepsilon(x, t; s))\, ds) = a^\varepsilon_\sigma(x, t)\, e^\varepsilon(x, t)\,. \tag{30}$$

The characteristic coordinates $\sigma(\phi, x, t; s)$ depend smoothly on (x, t). For example, differentiating the equations (26)-(27) with respect to x gives

$$\begin{aligned} &\partial_x \dot{\sigma^\varepsilon}(x, t; s) = -\lambda^{\varepsilon\prime}(\sigma^\varepsilon(x, t; s))\, \partial_x \sigma^\varepsilon(x, t; s) \\ &\partial_x \sigma^\varepsilon(x, t; t) = 1\,. \end{aligned}$$

This in turn yields

$$\partial_x \sigma^\varepsilon(x, t; s) = \exp(\int\limits_s^t \lambda^{\varepsilon\prime}(\sigma^\varepsilon(x, t; z))\, dz) \tag{31}$$

which in the case $c_1 > 0$, $\chi \geq 0$, can be rewritten as above into the simple expression $\partial_x \sigma^\varepsilon(x, t; s) = \lambda^\varepsilon(x)/\lambda^\varepsilon(\sigma^\varepsilon(x, t; s))$. If we apply ∂_t to (26)-(27) then the initial condition becomes

$$\partial_t \sigma^\varepsilon(x, t; t) = -\dot{\sigma^\varepsilon}(x, t; t) = \lambda^\varepsilon(\sigma^\varepsilon(x, t; t)) = \lambda^\varepsilon(x)$$

yielding

$$\partial_t \sigma^\varepsilon(x, t; s) = \lambda^\varepsilon(x) \exp(\int\limits_s^t \lambda^{\varepsilon\prime}(\sigma^\varepsilon(x, t; z))\, dz) \tag{32}$$

or if $c_1 > 0$, $\chi \geq 0$, then $\partial_t \sigma^\varepsilon(x, t; s) = \lambda^{\varepsilon 2}(x)/\lambda^\varepsilon(\sigma^\varepsilon(x, t; s))$.

We make use of the above observations to derive a simple formula for the distributional action of u^ε (for ε fixed) on an arbitrary test function $\psi \in \mathcal{D}(\mathbb{R}^2)$. As a smooth function, u^ε acts on ψ via the usual integral formula where we insert equation (30)

$$\langle u^\varepsilon, \psi \rangle = \iint a^\varepsilon(\sigma^\varepsilon(x,t;0)) \, \exp(\int_0^t \lambda^{\varepsilon\prime}(\sigma^\varepsilon(x,t;s)) \, ds) \, \psi(x,t) \, dx \, dt \ . \tag{33}$$

We change coordinates $(x,t) \mapsto (y,t)$ with $y = \sigma^\varepsilon(x,t;0)$, equivalently $x = \sigma^\varepsilon(y,0;t)$, and upon substituting (31) into $dx = \partial_y \sigma^\varepsilon(y,0;t) dy$ arrive at the simple expression

$$\langle u^\varepsilon, \psi \rangle = \iint a^\varepsilon(y) \, \psi(\sigma^\varepsilon(y,0;t),t) \, dy \, dt \tag{34}$$

where we also used the flow property $\sigma^\varepsilon(y,0;s) = \sigma^\varepsilon(x,t;s)$ of the characteristic lines.

Alternatively, we find a formula analogous to (28) by tracing the characteristic flow back to the boundary values of U at $x = 0$. From (28) we may directly compute a representative of $B := U\mid_{x=0} \in \mathcal{G}(\mathbb{R})$ (cf. the restriction formula in the introduction)

$$b(\phi_0, t) = a(\phi_0, \sigma(\phi, 0, t; 0)) \exp\,(\int_0^t \lambda'(\phi, \sigma(\phi, 0, t; s)) \, ds) \ . \tag{35}$$

In the integral (33) above we can also change the coordinates by $x = \sigma^\varepsilon(0,r;t)$ with $\sigma^\varepsilon(x,t,r) = 0$, which means to trace back x to the boundary point $(0,r)$, and use (32). By the definition of r and the flow property of σ^ε we have $\sigma^\varepsilon(x,t;s) = \sigma^\varepsilon(0,r;s)$ and therefore

$$\begin{aligned} &\langle u^\varepsilon, \psi \rangle \\ &= \lambda^\varepsilon(0) \iint a^\varepsilon(\sigma^\varepsilon(0,r;0)) \, \exp(\int_0^r \lambda^{\varepsilon\prime}(\sigma^\varepsilon(0,r;s)) \, ds)) \psi(\sigma^\varepsilon(0,r;t),t) \, dr \, dt \\ &= \lambda^\varepsilon(0) \iint b^\varepsilon(r) \, \psi(\sigma^\varepsilon(0,r;t),t) \, dr \, dt \ . \end{aligned} \tag{36}$$

4.2. The generalized characteristic set and the characteristic flow

Referring to [3], Def. 3, and adapting it to the full Colombeau algebra we will restate the definition of *generalized characteristic set* for a differential operator with Colombeau functions as coefficients. Assume that $\Omega \subseteq \mathbb{R}^n$ open and that the operator $P : \mathcal{G}(\Omega) \to \mathcal{G}(\Omega)$ is given by (notation: for $\alpha \in \mathbb{N}_0^n$ we write $|\alpha| = \alpha_1 + \ldots \alpha_n$ and $D^\alpha = (-i\partial_{x_1})^{\alpha_1} \cdots (-i\partial_{x_n})^{\alpha_n}$)

$$P(x,D) = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha$$

where $A_\alpha \in \mathcal{G}(\Omega)$ and $A_{\alpha_0} \neq 0$ for some α_0 with $|\alpha_0| = m$. If we denote by $\Xi = (\Xi_1, \ldots, \Xi_n)$ the Colombeau class of the smooth coordinate functions $(\xi_1, \ldots, \xi_n)$ and set $\Xi^\alpha = \Xi_1^{\alpha_1} \cdots \Xi_n^{\alpha_n}$ then the *principal symbol* $P_m \in \mathcal{G}(\Omega \times \mathbb{R}^n)$ can be defined by

$$P_m = \sum_{|\alpha|=m} A_\alpha \Xi^\alpha$$

which in terms of representatives $a_\alpha(\phi)$ for A_α reads more familiarly as

$$p_m(\phi, x, \xi) = \sum_{|\alpha|=m} a_\alpha(\phi, x)\xi^\alpha \ .$$

Here, $p_m(\phi)$ is a representative of P_m.

The following definition describes a microlocal ellipticity condition for P and the characteristic set as the regions of non-ellipticity.

Definition 17. A point $(x_0, \xi_0) \in T^*\Omega \setminus 0$ is called *non-characteristic* for P if there exist a neighborhood V of x_0 and a conic neighborhood W of ξ_0 such that for some $r \geq 0$ there is $N \in \mathbb{N}_0$ such that for all $\phi \in \mathcal{A}_N$ there are $C > 0$, and $1 > \eta > 0$ yielding the estimate

$$|p_m(\phi_\varepsilon, x, \xi)| \geq C\varepsilon^r |\xi|^m \quad \text{for all } x \in V, \xi \in W, 0 < \varepsilon < \eta \ . \tag{37}$$

The complement (in $T^*\Omega \setminus 0$) of all non-characteristic points defines the *generalized characteristic set* $\mathrm{Char}_g P$.

Remark 18. In case the coefficients A_α are smooth functions, e.g., are represented by $a_\alpha(\phi) = a_\alpha \in C^\infty$, then $\mathrm{Char}_g P$ reproduces exactly the classical definition of $\mathrm{Char} P$ as the zero set of the principal symbol. One can try to restate this is in the spirit of generalized point values, as described in [24]. There it is proved that Colombeau functions over an open set are characterized by their evaluations on so-called compactly supported generalized points in the open set Ω — a Colombeau function can be identified with its graph in this sense. By the above definition we observe the following: assume that a generalized pointvalue $(\tilde{x_0}, \tilde{\xi_0})$ (in the notation of [24], Def. 2.2) is a zero of the principal symbol P_m, i.e., $P_m(\tilde{x_0}, \tilde{\xi_0}) = 0$ as a generalized Colombeau number; if it also happens to be the class of a classical point $(x_0, \xi_0) \in T^*\Omega \setminus 0$ then we must have $(x_0, \xi_0) \in \mathrm{Char}_g P$. In other words, the classical shadow of the generalized zero set of the principal symbol is contained in $\mathrm{Char}_g P$. A further investigation of this relation could lead to a more geometric description or even provide alternative definitions for the notion of a generalized characteristic set.

We will compute the generalized characteristic set for our example operator in (x,t)-space with Colombeau coefficients

$$P(x,t,D_x,D_t) = i(D_t - \Lambda D_x) - \partial_x \Lambda \ . \tag{38}$$

First, we note that the principal symbol $P_1(x,t,\xi,\tau)$ has the representative

$$p_1(\phi,x,t,\xi,\tau) = i(\tau - \lambda(\phi,x,t)\xi) \ , \tag{39}$$

where λ is given by (15). Then a point $(x_0,t_0;\xi_0,\tau_0) \in T^*\mathbb{R}^2 \setminus 0$ is not in $\mathrm{Char}_g P$ if there exist a neighborhood V of (x_0,t_0) and a conic neighborhood W of (ξ_0,τ_0) such that for some $r \geq 0$ there is $N \in \mathbb{N}_0$ such that for all $\phi \in \mathcal{A}_N$ there are $C > 0$, and $1 > \eta > 0$ yielding the estimate

$$|\tau - \lambda(\phi_\varepsilon,x,t)\xi| \geq C\varepsilon^r(|\xi| + |\tau|) \quad \forall (x,t) \in V, (\xi,\tau) \in W, 0 < \varepsilon < \eta \ . \tag{40}$$

Substituting (16) we find

$$|p_1(\phi_\varepsilon,x,t,\xi,\tau)| = |\tau - \xi\Big(c_1 + (c_2 - c_1)\int\limits_{-\infty}^{\mu_\varepsilon x} \chi(y)\,dy\Big)|$$

with the notation $\mu_\varepsilon = \mu(\phi_\varepsilon)$ as in example 10, (ii). This is independent of t so we will only distinguish the cases $x_0 < 0$, $x_0 = 0$, and $x_0 > 0$ in the following.

As expected we reproduce the classical behavior on either side of the medium discontinuity: if $x_0 < 0$ then $\int_{-\infty}^{\mu_\varepsilon x} \chi = 0$ if ε is small for x near x_0 giving $|P_1(\phi_\varepsilon,x,t,\xi,\tau)| = |\tau - c_1\xi|$; similarly because $\int_{-\infty}^{\mu_\varepsilon x} \chi = 1$ near $x_0 > 0$ and small ε we obtain $|P_1(\phi_\varepsilon,x,t,\xi,\tau)| = |\tau - c_2\xi|$.

For the case $x_0 = 0$ we choose an interval $[-\alpha,\alpha]$ $(\alpha > 0)$ as neighborhood and have to estimate

$$\min_{|x|\leq\alpha} |\tau - \xi(c_1 + (c_2 - c_1)\int\limits_{-\infty}^{\mu_\varepsilon x} \chi(y)\,dy)| = \min_{|x|\leq\alpha} |\tau - \xi\psi(\mu_\varepsilon x)| \tag{41}$$

from below. By homogeneity in (ξ,τ) it is sufficient to restrict to the situation $|\xi| + |\tau| = 1$ and estimate the above expression by some constant times some power of ε to detect non-characteristic directions.

We divide the investigation of the case $x_0 = 0$ in further subcases concerning the (ξ,τ)-directions in the cotangent part of $\mathrm{Char}_g P$:

- if $\xi = 0$ we have $|\tau| = 1$ and $|\tau - \xi\psi| = 1 > 0$; hence the directions $(0,\pm 1)$ are non-characteristic

- if $\tau\xi < 0$ we have $|\tau - \xi\psi| = (-\tau\xi + \xi^2\psi)/|\xi| \geq |\tau| > 0$ (independent of x and ε) so the second and fourth open quadrants in the (ξ, τ)-plane consist of non-characteristic directions only

- if $\xi\tau \geq 0$ we have $\xi = \pm 1 - \tau$ and can rewrite

$$|\tau - \xi\psi| = |\tau(1+\psi) \mp \psi| = \underbrace{(1+\psi)}_{1 \leq . \leq C_\chi} |\tau \mp \frac{\psi}{1+\psi}|$$

  where $C_\chi = c_1 + (c_2 - c_1)\|\chi\|_{\mathrm{L}^1}$. This is bounded from below by $C\varepsilon^r$ if and only if $\tau \neq 0$ and $|\tau| \geq C\varepsilon^r + \frac{\psi}{1+\psi}$ or $\tau = 0$ and $\psi \geq C\varepsilon^r$ for $|x| \leq \alpha$ and ε small enough. If we define the χ-dependent quantities $-\|\chi\|_{\mathrm{L}^1} \leq \chi_0 := \min_z \int_{-\infty}^z \chi(y)\, dy \leq 0$ and $1 \leq \chi_1 := \max_z \int_{-\infty}^z \chi(y)\, dy \leq \|\chi\|_{\mathrm{L}^1}$ (note that χ has compact support and $\int \chi = 1$) then the corresponding quantities $\psi_0 := \inf \psi$ and $\psi_1 := \sup \psi$ can be bounded as follows

$$0 \leq \psi_0 = c_1 + (c_2 - c_1)\chi_0 \leq c_1 < c_2 \leq \psi_1 = c_1 + (c_2 - c_1)\chi_1 \leq C_\chi .$$

  Hence we can be sure that $\tau \neq 0$ defines a non-characteristic direction if $|\tau| > \frac{\psi_1}{1+\psi_1}$ (note that $\psi_1/(1+\psi_1) < 1$) and that $(\pm 1, 0)$ is non-characteristic if and only if $\psi_0 > 0$.

  On the other hand if we assume that for $|\tau| < 1$ the equation

$$\psi(z) = c_1 + (c_2 - c_1) \int_{-\infty}^{z} \chi(y)\, dy = \frac{|\tau|}{1 - |\tau|}$$

  is solvable for some $z \in \mathbb{R}$ then $p_1(\phi_\varepsilon, x, t, \pm 1 - \tau, \pm|\tau|)$ vanishes identically on the set $\{(x, t, \varepsilon) \mid t \in \mathbb{R}, |x| \leq \alpha, \varepsilon > 0 : \mu_\varepsilon x = z\}$ which includes x arbitrary close to $x_0 = 0$ and ε arbitrary small. Hence in this case the directions $(1 - |\tau|, |\tau|)$ in the first quadrant and $(-1 + |\tau|, -|\tau|)$ in the third quadrant are characteristic. This situation appears if $\frac{\psi_0}{1+\psi_0} \leq |\tau| \leq \frac{\psi_1}{1+\psi_1}$.

To summarize, in $\mathrm{Char}_g P$ we have the following possible cotangent directions (ξ, τ) over a base point (x, t).

Proposition 19. Assume that $(x, t, \xi, \tau) \in \mathrm{Char}_g P$ for P as given in (38). Then if $x < 0$ then $\tau = c_1\xi$ and if $x > 0$ then $\tau = c_2\xi$, if $x = 0$ the characteristic directions cover the cones $\psi_0\xi \leq \tau \leq \psi_1\xi$ with $\xi > 0$ and $\psi_0\xi \geq \tau \geq \psi_1\xi$ with $\xi < 0$. In particular, this is also true for the case $c_1 = 0$.

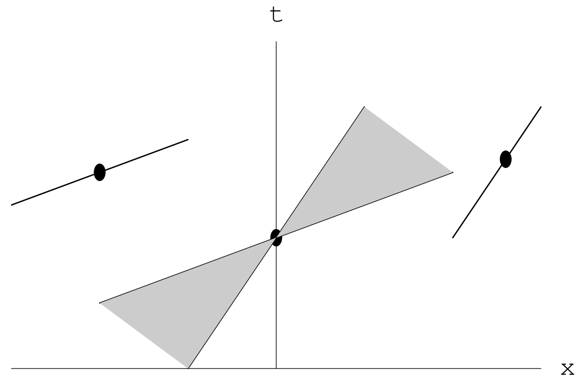

Figure 3. Cotangent directions in the characteristic set

Example 20. If the modeling mollifier χ is chosen to be nonnegative we have $\psi_0 = c_1$ and $\psi_1 = c_2$ and we see that the characteristic cotangent directions at points $(0, t)$ 'interpolate' between the characteristic directions on either side of the axis $x = 0$. The three different cases $x < 0$, $x = 0$, and $x > 0$ are illustrated in figure 3.

Note that by inspection of the possible values of ψ_0 and ψ_1 one easily verifies that these are actually the *minimal cones* appearing as χ varies in the set of real valued test functions with integral 1.

In the situation of the last example — that is $\chi \geq 0$ — we will also give a more detailed picture about the behavior of the characteristic flow $(x, t, s) \mapsto (\sigma^\varepsilon(x, t; s), s)$ in space-time as $\varepsilon \to 0$. For $\varepsilon > 0$ fixed this represents the global flow according to the smooth and bounded vector field $(-\lambda^\varepsilon(x), 1)$ in $\mathbb{R}^2$. By $-c_2 \leq -\lambda^\varepsilon \leq -c_1 \leq 0$ the space component $\sigma^\varepsilon(x, t; s)$ is non-increasing with respect to the flow parameter s, i.e., the flow never turns to the right. Since the characteristics are given globally as $s \mapsto (\sigma^\varepsilon(x, t; s), s)$ and never intersect we also have monotonicity properties in x and t separately (at fixed ε)

$$\sigma^\varepsilon(x', t'; s) \leq \sigma^\varepsilon(x, t; s) \qquad \text{if } x' \leq x \text{ and } t' \leq t \, . \tag{42}$$

Consider any compact set K contained in the open left half space V_-. K is contained in some closed box $[x_0', x_0] \times [t_0', t_0]$ with $x_0 < 0$. We clearly have $\sigma^\varepsilon(x, t; s) \leq x_0$ for all $(x, t) \in K$ and s such that

$(\sigma^\varepsilon(x,t;s),s) \in K$. This implies that uniformly for all such $(x,t;s)$

$$\dot{\sigma^\varepsilon}(x,t;s) = -\lambda^\varepsilon(\sigma^\varepsilon(x,t;s)) = -c_1 - (c_2 - c_1) \int\limits_0^{\mu_\varepsilon \sigma^\varepsilon(x,t;s)} \chi(y)\, dy$$

$$\geq -c_1 - (c_2 - c_1) \int\limits_0^{\mu_\varepsilon x_0} \chi(y)\, dy = -c_1$$

as long as ε is small enough, say $\varepsilon < \varepsilon_0$. On the other hand, since $\chi \geq 0$, the opposite estimate $\dot{\sigma^\varepsilon}(x,t;s) \leq -c_1$ is always true, which implies that for these (x,t), s, and $0 < \varepsilon < \varepsilon_0$ we have $\dot{\sigma^\varepsilon}(x,t;s) = -c_1$ and $\sigma^\varepsilon(x,t;t) = x$. This implies that on any compact set $K \subset V_-$ we have eventually (if ε is small enough, e.g., if $|\mu_\varepsilon x_0| > l(\chi)$) $\sigma^\varepsilon(x,t;s) = x + c_1(t-s)$ as long as $(\sigma^\varepsilon(x,t;s),s)$ stays in K.

The case that K is contained in the open right half space V_+ is completely analogous. Therefore we have proved

Lemma 21. Assume $\chi \geq 0$ and let K_-, K_+ be compact subsets of V_-, V_+ respectively. Then there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$

$$\sigma^\varepsilon(x,t;s) = x + c_1(t-s) \text{ on } \{(x,t,s) \in K_- \times \mathbb{R} \mid (\sigma^\varepsilon(x,t;s),s) \in K_-\}$$
$$\sigma^\varepsilon(x,t;s) = x + c_2(t-s) \text{ on } \{(x,t,s) \in K_+ \times \mathbb{R} \mid (\sigma^\varepsilon(x,t;s),s) \in K_+\}.$$

From this we can easily identify the *domains of dependence* on initial or boundary values for compact subsets within the various open regions defined in figure 1. For example, if K is a compact subset of V_1 then the tubular set $K_0 = \{(x + c_1(t-s),s) \mid st \geq 0, |s| \leq |t|\}$ is also a compact subset of $V_1 \subseteq V_-$. The lemma implies that eventually all characteristic lines joining K with the x-axis will be lines of slope $-1/c_1$ and stay within K_0. Hence in the solution formula (30) only strictly negative arguments in a^ε and $\lambda^{\varepsilon\prime}$ will occur if (x,t) varies in a compact set in V_1. Similarly, for compact subsets of W_1 the characteristic flow eventually will only trace back to boundary values on the positive t-axis, bounded away from $(0,0)$. We summarize this behavior in the following figure 4. Note that all this is also valid for $c_1 = 0$ with the only change that W_1 does not appear — the characteristic lines in the left half space are vertical then.

We now come to the most interesting part of the characteristic flow: what happens when the propagating signals cross the singularities of the medium? Since for all ε the characteristic curves cross the x-axis at a certain point we may simply restrict to initial points of the form $(x,0)$ if all values of s are considered. The case $c_1 = 0$ is already discussed in [22] and we summarize it in the following

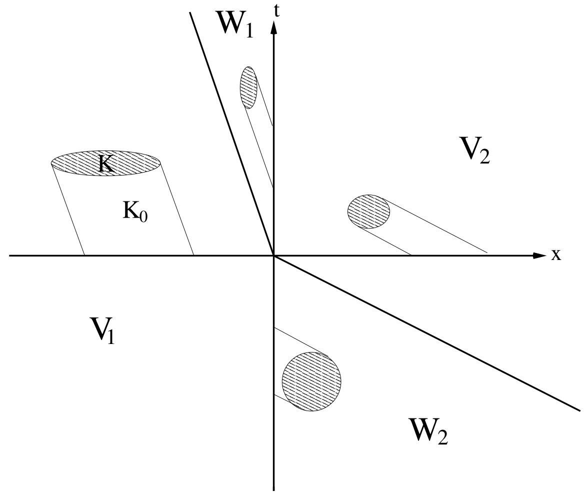

Figure 4. Domains of dependence

Proposition 22. Assume $\chi \geq 0$ then the family of smooth functions $(x,s) \mapsto \sigma^\varepsilon(x,0;s)$ $(\varepsilon > 0)$ on $\mathbb{R}^2$ converges almost everywhere to a continuous function $(x,s) \mapsto \sigma(x,0;s)$. If $c_1 > 0$ then

$$\sigma(x,0;s) = \begin{cases} x - c_1 s & \text{if } c_1 s \geq x \text{ and } x \leq 0 \\ x - c_2 s & \text{if } c_2 s \leq x \text{ and } x \geq 0 \\ \frac{c_2}{c_1} x - c_2 s & \text{if } c_1 s \leq x \text{ and } x \leq 0 \\ \frac{c_1}{c_2} x - c_1 s & \text{if } c_2 s \geq x \text{ and } x \geq 0 \end{cases} \tag{43}$$

and if $c_1 = 0$ then

$$\sigma(x,0;s) = \begin{cases} x & \text{if } x \leq 0 \\ x - c_2 s & \text{if } c_2 s \leq x \text{ and } x \geq 0 \\ 0 & \text{if } 0 \leq x \leq c_2 s \end{cases} . \tag{44}$$

Proof. The detailed proof for the case $c_1 = 0$ is given in [22], Prop. 3. Note that it does not even assume that χ is nonnegative.

We assume $c_1 > 0$. Following the idea in [22], p. 263, we define $\eta_\varepsilon = \max\{|x| \mid \chi(\mu_\varepsilon x) \neq 0\}$ which is bounded by $0 \leq \eta_\varepsilon \leq l(\chi)/\mu_\varepsilon$ (where $l(\chi)$ is the support number, cf. subsect. 1.2). Therefore $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. Note that $\lambda^\varepsilon(x) = c_1$ if $x \leq -\eta_\varepsilon$ and $\lambda^\varepsilon(x) = c_2$ if $x \geq \eta_\varepsilon$ and therefore

$$\sigma^\varepsilon(x,0;s) = \begin{cases} x - c_1 s & x \leq -\eta_\varepsilon \text{ and } s \geq \frac{x+\eta_\varepsilon}{c_1} \\ x - c_2 s & x \geq \eta_\varepsilon \text{ and } s \leq \frac{x-\eta_\varepsilon}{c_2} \end{cases} .$$

As $\varepsilon \to 0$ this shows pointwise convergence in the regions $x < 0$, $s > x/c_1$ and $x > 0$, $s < x/c_2$ as stated in (43).

Next we consider $x \leq -\eta_\varepsilon$ and follow the characteristic flow through $(x, 0)$ backwards, i.e., for decreasing values of s. Clearly $\sigma^\varepsilon(x, 0; s) = x - c_1 s$ as long as $s \geq (x + \eta_\varepsilon)/c_1$. On the other hand if s_ε marks the entrance point into the region at distance η_ε to the right of the t-axis, i.e., $\sigma^\varepsilon(x, 0; s_\varepsilon) = \eta_\varepsilon$ then we have by the above lemma

$$\sigma^\varepsilon(x, 0; s) = \sigma^\varepsilon(\eta_\varepsilon, s_\varepsilon; s) = \eta_\varepsilon - c_2(s - s_\varepsilon) \qquad \forall s \leq s_\varepsilon \ .$$

Clearly $s_\varepsilon \leq (x + \eta_\varepsilon)/c_1$ because $(\sigma^\varepsilon(x, 0; s), s)$ always moves to the lower right when s is decreasing since $-c_2 \leq \dot{\sigma^\varepsilon} \leq -c_1 < 0$. By the same estimate for $\dot{\sigma^\varepsilon}$ we can also estimate s_ε from below by the intersection time s^* of the line $(x - c_1 s, s)$ with the vertical line $(-\eta_\varepsilon, s)$. We have $s^* = (x - \eta\varepsilon)/c_1$ which shows that $s_\varepsilon \to x/c_1 =: s_1$ as $\varepsilon \to 0$. Hence if $x < 0$ and $s < x/c_1$ we have that $\sigma^\varepsilon(x, 0; s) \to 0 - c_2(s - s_1)$ as $\varepsilon \to 0$ which proves the assertion in the third line of (43).

For the subcase $x \geq \eta_\varepsilon$ we follow the flow through $(x, 0)$ into the future, i.e. for increasing values of s. Let $s_\varepsilon \geq (x + \eta_\varepsilon)/c_2 =: \bar{s_\varepsilon}$ mark the event $\sigma^\varepsilon(x, 0; s_\varepsilon) = -\eta_\varepsilon$ then

$$\sigma^\varepsilon(x, 0; s) = \sigma^\varepsilon(-\eta_\varepsilon, s_\varepsilon; s) = -\eta_\varepsilon - c_1(s - s_\varepsilon) \qquad \forall s \geq s_\varepsilon \ .$$

We obtain an upper bound for s_ε by considering the time s^* when the line $(\eta_\varepsilon - c_1(s - \bar{s_\varepsilon}), s)$ (this is the line with slope $-1/c_1$ issuing from the entrance point of the flow into the vertical strip of width $2\eta_\varepsilon$ around the t-axis from the right) intersects the vertical $(-\eta_\varepsilon, s)$. Here, we used again the fact that $\dot{\sigma^\varepsilon} \leq -c_1$. We have $s^* = (x + \eta_\varepsilon)/c_2 + 2\eta_\varepsilon/c_1$ which proves that $s_\varepsilon \to x/c_2 =: s_2$ as $\varepsilon \to 0$. Therefore if $x > 0$, $s > x/c_2$ then $\sigma^\varepsilon(x, 0; s)$ tends to $-0 - c_1(s - s_2)$ which proves the assertion in the fourth line of (43). □

We thus see that in the limit the characteristic flow produces a kink according to the change of velocity upon transmission through the medium jump. In case $c_1 = 0$ the flow becomes trapped in the singularity as already shown in the figure in subsection 3.1.

5. Microlocal properties of the generalized solution – propagation of singularities

In this section we will study the interplay of the wave fronts sets of the initial data and the medium coefficients and the characteristic set of the partial differential operator. Such interplay induces the propagation of singularities. First, we will recover classical relations between the wave front and characteristic sets in the case of $\mathcal{G}^\infty$-regular medium coefficients. We recall from example 10, (ii), that the Colombeau model coefficient Λ given by equation (15) is in $\mathcal{G}^\infty(\mathbb{R}^2)$ — we remind that this does not imply that Λ is γ-regular when $\gamma(r) \neq 1/r$ — and hence the operator P in (38) is a differential operator with $\mathcal{G}^\infty$ coefficients.

Theorem 23. Let $A \in \mathcal{G}(\mathbb{R})$ and U be the unique Colombeau solution to problem (13)-(14). Then we have

(i) $\mathrm{WF}_g(U) \subseteq \mathrm{Char}_g P$.

(ii) If A is in $\mathcal{G}^\infty(\mathbb{R})$ then U is in $\mathcal{G}^\infty(\mathbb{R}^2)$.

Proof. *ad (i):* This follows directly from the general theorem about propagation of singularities for linear operators with $\mathcal{G}^\infty$ coefficients proved in [3], Thm. 4. Although the proof is given in detail there for the so-called 'simplified' version of Colombeau algebras (where only ε-parameterization is used) an inspection of the arguments shows that the proof transfers to the 'full' version we are dealing with.

ad (ii): This follows actually from the fact that we have $\mathcal{G}^\infty$-regularity of the solution U to the general hyperbolic problem (10)-(11) if A is $\mathcal{G}^\infty$. This can be seen by inspection of the estimates in the existence proof from [15]. □

Extending this insight to the detailed influence of the step medium coefficients requires the earlier introduced refined notion of wave front set (the γ-scaled variant). As shown in subsect. 3.2 this notion preserves the singularity structure of the medium. On the other hand, with respect to its pure geometrical nature we leave the notion of generalized characteristic set of the differential operator intact, i.e., we stick to Def. 37.

Led by Thm. 15, we expect to get a more refined detection of singularities of U through the analysis of the generalized wave front set $\mathrm{WF}_g^\gamma(U)$ with $\gamma(r) = \log(1/r)$. Note that we have

$$\mathrm{WF}_g^\gamma(U) \supseteq \mathrm{WF}_g(U)\ ;$$

a critical question is whether the generalized wave front set can still be bounded by the characteristic set of the operator P. In this respect the determination of $\mathrm{WF}_g^\gamma(U)$ also requires that, according to (28), we have to deal with the product of two γ-singular generalized functions $U = A_\Sigma E$, and have to give estimates for $\mathrm{WF}_g^\gamma(A_\Sigma E)$ in terms of their respective wave front sets.

To obtain sensible results we also want to exclude any pathologies arising from the mixing of regularity scales. For example, taking the generalized constant A_0 given by $(1/l(\phi))_\phi$ as initial value A would always produce an (x,t)-independent overall factor $1/\varepsilon$ in the expressions for U. But $1/\varepsilon$ can never be dominated by powers of $\log(1/\varepsilon)$ and would thus 'simulate' singular behavior everywhere in the solution of a kind that does not relate to the propagation and interaction process. Of course this is mathematically correct because A_0 is an element in $\mathcal{G}^\infty \setminus \mathcal{G}^\infty_\gamma$ but it is completely misleading if we aim to observe propagating singularities caused by distributional initial values. Therefore we will assume the initial value A to have the following property.

Definition 24. $A \in \mathcal{G}(\mathbb{R})$ is said to be of *γ-type* if it has a representative $(a(\phi))_\phi$ with the property that for every compact set K there is $N \in \mathbb{N}_0$ such that $\sup_{x\in K} |a(\phi_\varepsilon, x)| = O(\gamma(\varepsilon)^N)$ as $\varepsilon \to 0$.

The methodology underlying the specification of the wave front set is as follows. $\mathrm{WF}^\gamma_g(U)$ is local with respect to the base space: the cones of irregular directions over a certain base point (x_0, t_0) are detected according to Def. 14 by choosing test functions φ with support near (x_0, t_0) and $\varphi(x_0, t_0) = 1$ and then investigating the decay properties of the Fourier transform $\widehat{(\varphi u^\varepsilon)}(\xi, \tau)$ where as usual $u^\varepsilon = u(\phi_\varepsilon)$ denotes an arbitrary representative of U.

If we use the representative obtained by the method of characteristics in subsect. 4.1 then by (30) an explicit expression for $\widehat{(\varphi u^\varepsilon)}$ is given by

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \iint e^{-i(\xi x + \tau t) + \int_0^t \lambda^{\varepsilon\prime}(\sigma^\varepsilon(x,t;s))\, ds}\, a^\varepsilon(\sigma^\varepsilon(x, t; 0))\varphi(x, t)\, dx\, dt \; . \tag{45}$$

Using equation (34) with $e^{-i(\xi x+\tau t)}\varphi(x, t)$ in place of $\psi(x, t)$ this can be written with characteristic coordinates in the alternative form

$$\widehat{(\varphi u^\varepsilon)}(\xi, \tau) = \iint e^{-i(\xi \sigma^\varepsilon(y,0;t)+\tau t)} a^\varepsilon(y)\varphi(\sigma^\varepsilon(y, 0; t), t)\, dy\, dt \; . \tag{46}$$

As with the computation of $\mathrm{Char}_g P$ in the further investigation we divide $\mathbb{R}^2$ into several sectors according to the geometry of the generalized characteristic flow (cf. figure 1). For the sake of brevity we will focus now on that part of the forward time domain which includes the transmission from one medium into the other. Thus we will investigate the singular behavior at points $(x_0, t_0) \in \overline{V_2 \cup W_2}$. The exact determination of the microlocal properties at the discontinuity $x = 0$ for general initial values A seem to require a more systematic development of the following tools: a generalized stationary phase method and an analysis of the behavior of the wave front set under generalized pull-backs.

$\boxed{(x_0, t_0) \in V_2}$ If (x, t) varies in a small neighborhood of (x_0, t_0) then by the results of subsect. 4.2 we have $\sigma^\varepsilon(x, t; s) = x + c_2(t - s)$ for small ε as long as $(\sigma^\varepsilon(x,t;s),s)$ stays within a compact subset of V_2. Hence if $\mathrm{supp}(\varphi)$ is concentrated in such a neighborhood of (x_0,t_0) we have for small $\varepsilon > 0$

$$\widehat{(\varphi u^\varepsilon)}(\xi,\tau) = \iint e^{-i(\xi x+\tau t)} a^\varepsilon(x+c_2 t)\varphi(x,t)\,dx\,dt\ .$$

Without loss of generality assume that φ is of the form $\varphi_1(x+c_2t)\varphi_2(t)$ where φ_1, resp. φ_2, is concentrated near $x_0+c_2t_0$, resp. t_0. This follows upon appealing to property (13) (or rather its γ-analogue) in [12] and noting that $(x+c_2t,t)$ are coordinates in $\mathbb{R}^2$. Then we change coordinates according to $y = x+c_2t$ in the integral and obtain

$$\begin{aligned} &\iint e^{-i(\xi y+t(\tau-c_2\xi))} a^\varepsilon(y)\,\varphi_1(y)\,\varphi_2(t)\,dy\,dt \\ &= \int e^{-i\xi y} a^\varepsilon(y)\,\varphi_1(y)\,dy \int e^{-it(\tau-c_2\xi)}\varphi_2(t)\,dt = \widehat{(\varphi_1 a^\varepsilon)}(\xi)\,\widehat{\varphi_2}(\tau-c_2\xi)\,. \end{aligned}$$

Here, by assumption (A) the first factor is bounded by

$$\|\varphi_1 a^\varepsilon\|_{\mathrm{L}^1} \le \|\varphi_1\|_{\mathrm{L}^1} \sup_{y\in\mathrm{supp}(\varphi_1)} |a^\varepsilon(y)| = O(\gamma(\varepsilon)^N) \text{ for some } N\in\mathbb{N}_0$$

and the second factor is rapidly decreasing in (ξ,τ) if $\tau\neq c_2\xi$. On the other hand if $\tau = c_2\xi$ then, since φ_1 is concentrated near $x_0+c_2t_0$, the whole expression can only be non-γ-rapidly decreasing if this point belongs to $\mathrm{singsupp}_g^\gamma(A)$. Hence we have proved the following

Proposition 25. If A is of γ-type then

$$\mathrm{WF}_g^\gamma(U\mid_{V_2}) \subseteq \{(x,t)\in V_2 \mid x+c_2t\in \mathrm{singsupp}_g^\gamma(A)\} \\ \times\{(r,c_2r)\mid r\neq 0\}.$$

We point out that this is valid for any Colombeau initial value A of γ-type. In particular, any distribution of finite order can be modeled in this way by convolution with a γ-scaled delta net of mollifiers.

It is worth noting that we can even recover the exact shape of $\mathrm{WF}_g^\gamma(U)$ in V_2 if we model a distributional initial value $a\in\mathcal{D}'(\mathbb{R})$ in an appropriate way. As we saw above in this region we have for ε small

$$u^\varepsilon(x,t) = a^\varepsilon(x+c_2t) = c_2^* a^\varepsilon(x,t)$$

with the slight abuse of notation c_2^* for the pull-back by the map $c_2(x,t) = x+c_2t$. Assume that A is modeling a distribution a via $a(\phi) = a*\chi_2^\gamma(\phi)$ as in Thm. 15 which we will denote after ε-insertion by $a^\varepsilon = a*\chi_2^\varepsilon$. Then we may rewrite

$$c_2^* a(\phi,x,t) = a*\chi_2^\gamma(\phi)(x+c_2t) = \langle a(y),\chi_2^\gamma(\phi,x+c_2t-y)\rangle\ .$$

We would like to consider this as a γ-modeling of the distribution $c_2^* a \in \mathcal{D}'(\mathbb{R}^2)$ to be able to apply Thm. 15. This can be achieved by the following construction. Choose $\beta_0 \in \mathcal{A}_0(\mathbb{R})$ arbitrary and define

$$\chi_2 = \beta_0 * \frac{1}{c_2}\beta_0(\frac{.}{c_2}) \; . \tag{47}$$

χ_2 is smooth with compact support and by the simple property [11], (1.3.4), it follows that $\int \chi_2 = 1$. Finally, we set

$$\beta_2(x,t) = \beta_0(x)\frac{1}{c_2}\beta_0(\frac{t}{c_2})$$

which defines a test function in $\mathbb{R}^2$ with integral 1. In the following we use the notation $\beta_0^\gamma(\phi)$, $\beta_2^\gamma(\phi)$, and $\chi_2^\gamma(\phi)$ as in the modeling map of Thm. 15.

Consider the two-dimensional convolution

$$c_2^* a * \beta_2^\gamma(\phi)(x,t) = \langle c_2^* a(y,s), \beta_2^\gamma(\phi, x-y, t-s)\rangle \; .$$

By formula [11], (6.1.1), for the pull-back (e.g., with the function $h(z,r) = (z-r, r/c_2)$ in the notation of the cited equation), this can be rewritten as

$$\begin{aligned} &\langle (a \otimes 1)(y,s), \beta_2^\gamma(\phi, x-y+s, t-s/c_2)/c_2\rangle \\ &\qquad = \langle a(y), \int \beta_2^\gamma(\phi, x-y+s, t-s/c_2)\, ds/c_2\rangle \; . \end{aligned}$$

Substituting $t - s/c_2 = r/c_2$ in the integral and using the definition of β_2 via β_0 we finally arrive at

$$c_2^* a * \beta_2^\gamma(\phi)(x,t) = \langle a(y), \Big(\beta_0^\gamma(\phi, .) * \frac{1}{c_2}\beta_0^\gamma(\phi, \frac{.}{c_2})\Big)(x + c_2 t - y)\rangle$$

which matches exactly the above expression for $c_2^* a(\phi, x, t)$ if χ_2 is given by (47) (observe that a simple computation shows that indeed $\chi_2^\gamma(\phi) = \beta_0^\gamma(\phi) * \beta_0^\gamma(\phi, ./c_2)/c_2$).

To summarize this construction we may state the following.

Proposition 26. If the initial value A models a distribution a over $\mathbb{R}_+$ by $A = \iota_{\chi_2}^\gamma(a)$ (in the notation of Thm. 15) with χ_2^γ given by (47) then in the region V_2 we have

$$U = \iota_{\beta_2}^\gamma(c_2^* a) \quad \text{and} \quad \mathrm{WF}_g^\gamma(U) = \mathrm{WF}(c_2^* a) \; .$$

Note that adapting the arguments of [11], p. 270, it is easy to compute $\mathrm{WF}(c_2^* a)$ explicitly in terms of $\mathrm{WF}(a)$ which recovers the classically expected result in V_2

$$\mathrm{WF}_g^\gamma(U) = \mathrm{WF}(c_2^* a) = \{(x,t;\eta, c_2\eta) \mid (x + c_2 t, \eta) \in \mathrm{WF}(a)\} \; .$$

Note that the analysis for $(x_0, t_0) \in V_1$ is equivalent. In stating the corresponding results one only has to replace c_2, resp. V_2, by c_1, resp. V_1, in all the propositions.

$\boxed{(x_0, t_0) \in W_1}$ Note that this case only occurs if $c_1 > 0$ (otherwise $W_1 = \emptyset$). We now use formula (46) where we assume that $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$. Let K be some fixed compact set containing $\{y \mid \exists t \in \operatorname{supp}(\varphi_2) : \varphi_1(\sigma^\varepsilon(y,0;t)) \neq 0\}$ (this is possible by the properties of the characteristic flow established in 3.3). Then we may insert an additional factor $\psi(y)$ in the integrand of (46) where ψ is a test function with $\psi = 1$ on K without changing the value of the integral. We interpret $\varphi_1(\sigma^\varepsilon(y,0;t))$ as the inverse Fourier transform of $\widehat{\varphi_1}$ evaluated at $\sigma^\varepsilon(y,0;t)$ giving

$$\widehat{(\varphi u^\varepsilon)}(\xi,\tau) = \frac{1}{2\pi} \iiint e^{-i\left((\xi-\eta)\sigma^\varepsilon(y,0;t)+\tau t\right)} \psi(y)\, a^\varepsilon(y)\, \widehat{\varphi_1}(\eta)\, \varphi_2(t)\, d\eta\, dy\, dt\ .$$

Since we may assume that ε is small and φ has support concentrated in a small neighborhood of (x_0, t_0) we know that y will vary near $y_0 = \sigma^\varepsilon(x_0, t_0; 0) > 0$. Hence y will stay strictly positive in the support of the integrand. Furthermore, $y_0 = \sigma^\varepsilon(x_0, t_0; 0) \leq x_0 + c_2 t_0 < c_2 t_0$ since $x_0 < 0$. Therefore we can choose the support of φ so small that $0 < y < c_2 t$ whenever $\varphi(\sigma^\varepsilon(y,0;t), t) \neq 0$. Using the notation from the proof of Prop. 22, in case $x > \eta_\varepsilon$, we thus get

$$\sigma^\varepsilon(y,0;t) = c_1 s_\varepsilon(y) - \eta_\varepsilon - c_1 t$$

on the support of the (original) integrand. We insert this equality into the above integral formula and interchange the order of integration to obtain

$$\frac{e^{i\xi\eta_\varepsilon}}{2\pi} \int e^{-i\eta_\varepsilon\eta} \widehat{\varphi_1}(\eta)\widehat{\varphi_2}(c_1\eta + \tau - c_1\xi) \underbrace{\int e^{-ic_1(\xi-\eta)s_\varepsilon(y)} \psi(y) a^\varepsilon(y)\, dy}_{f_\varepsilon(\xi,\eta)}\, d\eta$$

where the smooth function $f_\varepsilon(\xi,\eta)$ has the property

$$|\partial_\eta^k f_\varepsilon(\xi,\eta)| \leq c_1^k \int |s_\varepsilon(y)|^k |\psi(y) a^\varepsilon(y)|\, dy \leq C_k \gamma(\varepsilon)^N$$

for C_k independent of ε and some N independent of k, because A is assumed to be of γ-type and ψ has compact support. In the integral above we now use $\widehat{\varphi_2}(c_1\eta + \tau - c_1\xi) = e^{i\eta(\tau - c_1\xi)}\widehat{\varphi_2}(c_1\eta)$ and obtain

$$\widehat{(\varphi u^\varepsilon)}(\xi,\tau) = \frac{e^{i\xi\eta_\varepsilon}}{2\pi} \int e^{-i\eta(\eta_\varepsilon - (\tau - c_1\xi))} \underbrace{\widehat{\varphi_1}(\eta)\widehat{\varphi_2}(c_1\eta) f_\varepsilon(\xi,\eta)}_{\gamma(\varepsilon)^N g_\varepsilon}\, d\eta\ .$$

Here, $(g_\varepsilon)_\varepsilon$ constitute a bounded family of functions in $\mathcal{S}(\mathbb{R})$ (w.r.t. the variable η) — this means that this set is bounded w.r.t. every seminorm on $\mathcal{S}$ (cf. [11], Def. 7.1.2) independent of ε. This is guaranteed by the property of f_ε together with the fact that $\varphi_j \in \mathcal{S}$. Therefore, if we interpret the above integral as

$$\gamma(\varepsilon)^{-N}\widehat{(\varphi u^\varepsilon)}(\xi,\tau) = \frac{e^{i\xi\eta_\varepsilon}}{2\pi}\widehat{g^\varepsilon}(\eta_\varepsilon - (\tau - c_1\xi)) ,$$

by continuity of the Fourier transform on $\mathcal{S}$ the family $(\widehat{g^\varepsilon})_\varepsilon$ is bounded in the same sense and we may estimate for arbitrary $k \in \mathbb{N}_0$

$$\gamma(\varepsilon)^{-N}|\widehat{(\varphi u^\varepsilon)}(\xi,\tau)| \leq C_k(1 + |\eta_\varepsilon - (\tau - c_1\xi)|)^{-k}$$

with a constant C_k independent of ε. If (ξ,τ) varies in closed cones disjoint to $\tau = c_1\xi$ then since $\eta_\varepsilon \to 0$ ($\varepsilon > 0$) this estimate proves γ-rapid decrease for $\widehat{(\varphi u^\varepsilon)}(\xi,\tau)$. Thus no cotangent directions different from $\tau = c_1\xi$ can occur in the γ-wave front set in the region W_1.

As for the singular support we use the viewpoint of considering the Colombeau Cauchy problem in W_1 with 'initial' value $U\mid_{x=0} = B$ where a representative $(b(\phi))_\phi$ of B is given by (35). Then we know that by the convergence properties of σ^ε in W_1 we have, as ε is getting small enough (let $(x+c_1(t-s), s)$ flow back to the right until its first argument becomes 0)

$$u^\varepsilon(x,t) = b^\varepsilon(t + \frac{x}{c_1}) .$$

This shows that any γ-singular behavior of B around a point $r > 0$ can only be transported parallel to the line $x + c_1t = c_1r$ into the region W_1. To summarize we have proved the following

Proposition 27. If A is of γ-type, then

$$\mathrm{WF}_g^\gamma(U\mid_{W_1}) \subseteq \{(x,t) \in W_1 \mid \frac{x + c_1t}{c_1} \in \mathrm{singsupp}_g^\gamma(B)\} \times \{(r, c_1r) \mid r \neq 0\} .$$

We illustrate Prop. 25-27 in the simple qualitative figure 5 (dashed lines denote propagating singularities and solid arrows indicate cotangent directions of the wave front sets).

$\boxed{x_0 = 0,\ t_0 > 0}$ Here we prove that for certain initial values A and $0 < c_1 < c_2$ the γ-scaled wave front set of the solution U will contain noncharacteristic cotangent directions. Those directions are indeed caused by the medium singularity and not by the characteristic flow or the initial value (cf. the remarks in item (iv) towards the end of subsection 3.1).

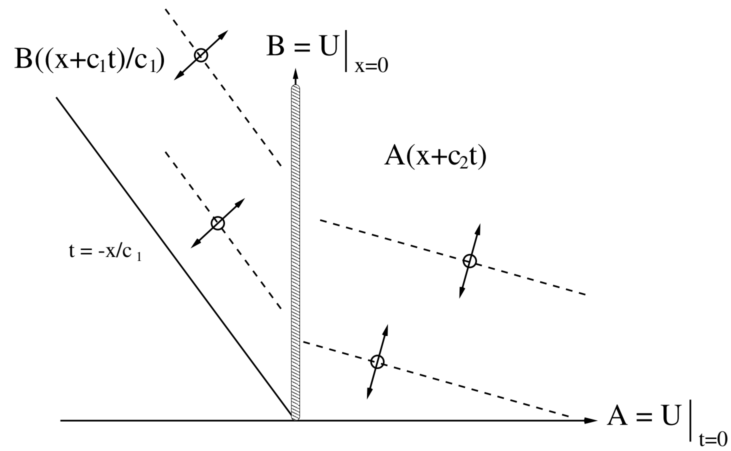

Figure 5.

Remark 28.

(i) In case $c_1 = 0$ according to Prop. 19 the generalized characteristic set does contain $\{0\} \times \mathbb{R} \times \mathbb{R} \setminus 0 \times \{0\}$ and therefore also includes the singular spectrum, $\mathrm{WF}_g^\gamma(\Lambda)$, of the medium.

(ii) Note that unlike in the regions V_2 and W_1 now the formula $U = A_\Sigma \cdot E$ shows several difficulties at the same time: first, the singular support of E lies exactly in the focus of our zoom into $(0, t_0)$; second, the inner structure of A_Σ as a pullback of the Colombeau function A by the γ-irregular Colombeau representative $(\phi, x, t) \mapsto \sigma(\phi, x, t; 0)$ directly contains the medium singularity since (x, t) varies in a neighborhood of its singular support.

We prepare for the estimates of the integrand in formula (46) by stating useful properties related to the generalized characteristic flow.

Lemma 29. Let $S \in \mathcal{G}(\mathbb{R}^2)$ be represented by $(\phi, y, t) \mapsto \sigma(\phi, y, 0, t)$.

(i) For any $\phi \in \mathcal{A}_0$ we have

$$(\lambda(\phi, y)\partial_y + \partial_t)\sigma(\phi, y, 0; t) = 0$$

which implies $\Lambda(y)\partial_y S + \partial_t S = 0$ in $\mathcal{G}(\mathbb{R}^2)$.

(ii) Let $\chi \geq 0$, $\phi \in \mathcal{A}_0$ and define for α, $\varepsilon > 0$

$$K_{\varepsilon,\alpha}(t_0) := \{(y, t) \mid |t_0 - t| \leq \alpha, |\sigma^\varepsilon(y, 0; t)| \leq \alpha\}\,.$$

Then for every $\beta > 0$ there is $\alpha > 0$ such that for all $\varepsilon > 0$

$$\mathrm{pr}_1\big(K_{\varepsilon,\alpha}(t_0)\big) \subseteq [c_1 t_0 - \beta, c_2 t_0 + \beta]$$

where pr_1 denotes the projection onto the first coordinate.

(iii) Let $0 < \beta < c_1 t_0$ and $\phi \in \mathcal{A}_0$ and choose $\alpha > 0$ as in (ii). If $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\mathrm{supp}(\varphi) \subseteq [-\alpha, \alpha] \times [t_0 - \alpha, t_0 + \alpha]$ then for every $k \in \mathbb{N}_0$ and $(y, t) \in \mathrm{supp}(\varphi)$

$$(\lambda^\varepsilon(y)\partial_y + \partial_t)^k (a^\varepsilon(y)\varphi(\sigma^\varepsilon(y, 0; t), t)) = \sum_{j=0}^{k} \binom{k}{j} \lambda^\varepsilon(y)^j \, (\frac{d}{dy})^j a^\varepsilon(y) \, \partial_2^{k-j} \varphi(\sigma^\varepsilon(y, 0; y), t) \ ,$$

where ∂_2 means differentiation w.r.t. the second argument in φ.

Proof. ad (i): Let $r(\phi, y, t) := \big(\lambda(\phi, y)\partial_y + \partial_t\big)\sigma(\phi, y, 0; t)$ then according to (26) and (31) we have

$$r(\phi, y, t) = \lambda(\phi, y) e^{-\int_0^t \lambda'(\phi, \sigma(\phi, y, 0, s))ds} - \lambda(\phi, \sigma(\phi, y, 0; t).$$

By (27) we have $r(\phi, y, 0) = 0$ which together with $\partial_t r(\phi, y, t) = -\lambda'(\phi, \sigma(\phi, y, 0; t)) r(\phi, y, t)$ implies $r(\phi, ., .) = 0$.

ad (ii): Making α small enough we may assume that $t_0 - \alpha > 0$. Due to the monotonicity and flow properties of σ^ε at ε fixed we have for $y \in \mathrm{pr}_1(K_{\varepsilon,\alpha}(t_0)$ the bounds $\sigma^\varepsilon(-\alpha, t_0 - \alpha; 0) \le y \le \sigma^\varepsilon(\alpha, t_0 + \alpha; 0)$. The assertion follows from simple estimates using the defining differential equation for σ^ε if α is chosen small enough:

$$\sigma^\varepsilon(-\alpha, t_0 - \alpha; 0) = -\alpha + \int_0^{t_0 - \alpha} \lambda^\varepsilon(\sigma^\varepsilon(-\alpha, t_0 - \alpha, s))ds \ge -\alpha + c_1(t_0 - \alpha)$$

and

$$\sigma^\varepsilon(\alpha, t_0 + \alpha; 0) = \alpha + \int_0^{t_0 + \alpha} \lambda^\varepsilon(\sigma^\varepsilon(\alpha, t_0 + \alpha, s))ds \le \alpha + c_2(t_0 + \alpha) \, .$$

ad (iii): $L = \lambda^\varepsilon(y)\partial_y + \partial_t$ is a combination of the three operators ∂_y, ∂_t, and multiplication by $\lambda^\varepsilon(y)$. Since $y > c_1 t_0 - \beta > 0$ in $\mathrm{supp}\,\big((y, t) \mapsto \varphi(\sigma^\varepsilon(y, 0; t), t)\big)$ we have for ε small enough $\lambda^{\varepsilon\prime} = 0$ in the considered domain. Therefore all three operators commute, in particular $\partial_y(\lambda^\varepsilon f) = \lambda^\varepsilon \partial_y f$. Furthermore, by (i) and the chain rule $L \cdot \varphi(\sigma^\varepsilon(y, 0; t), t) = \partial_2\varphi(\sigma^\varepsilon(y, 0; t), t)$. Therefore the stated formula follows using Leibniz rule and binomial expansion or induction. □

Proposition 30. Assume $\chi \ge 0$ and define

$$\Gamma = \{(\xi, \tau) \mid \tau \ne 0 \text{ and either } \xi/\tau < c_1 \text{ or } \xi/\tau > c_2\} \, .$$

(i) If $A \in \mathcal{G}_\gamma^\infty$ near the interval $(c_1 t_0, c_2 t_0)$ then

$$\Sigma^\gamma_{g,(0,t_0)}(U) \subseteq \mathbb{R}^2 \setminus \Gamma\,.$$

(ii) If $c_1 > 0$ and A is constant and nonzero near $(c_1 t_0, c_2 t_0)$ then

$$\Sigma^\gamma_{g,(0,t_0)}(U) = \mathbb{R} \times \{0\} \setminus \{(0,0)\}\,.$$

Proof. ad (i): Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\phi(0,t_0) = 1$ and $\operatorname{supp}(\varphi) \subseteq [-\alpha,\alpha] \times [t_0-\alpha, t_0+\alpha]$ where $\alpha > 0$ is chosen as in Lemma (iii) above. Therefore a^ε represents a γ-regular Colombeau function inside the integral (46). Let $(\xi,\tau) = \omega(\xi_0,\tau_0) \in \Gamma$ with $\omega \geq 1$ then

$$\widehat{(\varphi u^\varepsilon)}(\xi,\tau) = \iint e^{-i\omega(\xi_0 \sigma^\varepsilon(y,0;t) + \tau_0 t)} a^\varepsilon(y) \varphi(\sigma^\varepsilon(y,0;t),t)\, dy\, dt\,.$$

We set $p_\varepsilon(y,t) = \xi_0 \sigma^\varepsilon(y,0;t) + \tau_0 t$ and note that $\operatorname{grad} p_\varepsilon = (\xi_0 \partial_y \sigma^\varepsilon, \tau_0 - \xi_0 \lambda^\varepsilon)$ stays away from 0 by the defining inequalities of Γ. If

$$L^t = \frac{i}{\omega\tau_0}(\lambda^\varepsilon(y)\partial_y + \partial_t)$$

then $L^t\big(\exp(-i\omega p_\varepsilon)\big) = \exp(-i\omega p_\varepsilon)$ by Lemma (i) above and integration by parts for k times gives with $L = -i(\lambda^\varepsilon(y)\partial_y + \partial_t)/\omega\tau_0$

$$\widehat{(\varphi u^\varepsilon)}(\omega(\xi_0,\tau_0)) = \iint e^{-i\omega p_\varepsilon(y,t)} L^k\big(a^\varepsilon(y)\varphi(\sigma^\varepsilon(y,0;t),t)\big)\, dy\, dt\,.$$

Since $L^k = (-i/\omega\tau_0)^k (\lambda^\varepsilon(y)\partial_y + \partial_t)^k$ we can apply part (iii) of the Lemma and rewrite the integral into

$$(\frac{-i}{\omega\tau_0})^k \sum_{j=0}^k \binom{k}{j} \iint e^{-i\omega p_\varepsilon(y,t)} \lambda^\varepsilon(y)^j\, (\frac{d}{dy})^j a^\varepsilon(y)\, \partial_2^{k-j} \varphi(\sigma^\varepsilon(y,0;t),t)\, dy\, dt\,.$$

By the regularity assumption about A there is an N, independent of j and k, such that for some $C' > 0$ we have $\sup_y |(\frac{d}{dy})^j a^\varepsilon(y)| \leq C' \gamma_\varepsilon^N$. Clearly, $\sup_y |\lambda^\varepsilon(y)^j| \leq c_2^j$ and since the support of $\varphi(\sigma^\varepsilon(y,0;t),t)$ stays in some compact set independent of ε each integral can be estimated by some constant times $c_2^j \gamma_\varepsilon^{-N} \|\partial_2^{k-j}\varphi\|_{\mathrm{L}^\infty}$.

Altogether we see that $|\widehat{(\varphi u^\varepsilon)}(\omega(\xi_0,\tau_0))|$ is bounded by $C\gamma_\varepsilon^N \omega^{-k}$ for k arbitrary, N independent of k, and the constant C depending only on k, derivatives of φ of order $\leq k$, τ_0, c_2, $\alpha > 0$, and is valid for $\varepsilon > 0$ small enough.

This proves that $\widehat{(\varphi u^\varepsilon)}(\xi,\tau)$ is rapidly decreasing in Γ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with support near $(0,t_0)$.

ad (ii): By smooth cut-off we may assume that (x,t) varies only in a small neighborhood Ω of $(0,t_0)$. As noted in the proof above $\sigma^\varepsilon(x,t;0)$ will stay to the right of 0, that is $\exists\alpha > 0$ such that $\sigma^\varepsilon(x,t;0) \geq \alpha$ for all $(x,t) \in \Omega$ and ε small. Therefore $\lambda^\varepsilon(\sigma^\varepsilon(x,t;0)) = c_2$ and using (29) in (30) gives

$$u^\varepsilon(x,t) = c_2 \frac{a^\varepsilon(\sigma^\varepsilon(x,t;0))}{\lambda^\varepsilon(x)} \, .$$

yielding in turn that

$$\Sigma^\gamma_{(0,t_0)}(U) = \Sigma^\gamma_{(0,t_0)}(A_\Sigma/\Lambda) \, .$$

It is easy to determine $\Sigma^\gamma_{(0,t_0)}(1/\Lambda)$: clearly, $1/\Lambda$ is γ-regular outside the t-axis and by evaluating the derivatives at points $(0,t)$ one proves that $\mathrm{singsupp}^\gamma_g(1/\Lambda) = \{0\} \times \mathbb{R}$; by the t-independence of Λ the cotangent wave front set directions can only be horizontal; using the fact that the typical representative is real-valued a symmetry argument for the Fourier transform (as in [12], at the end of Ex. 26) yields that both the positive and negative horizontal directions must be included, hence

$$\Sigma^\gamma_{(0,t_0)}(1/\Lambda) = \mathbb{R} \setminus 0 \times \{0\} \, .$$

Since A is constant on an interval containing the set $\{\sigma^\varepsilon(x,t;0) \mid (x,t) \in \Omega\}$ the assertion follows. $\square$

Remark 31. We can give an alternative argument to show that horizontal cotangent directions may appear at the t-axis in the case $c_1 > 0$. Use the general relation $\mathrm{WF}^\gamma_g(U) \supseteq \mathrm{WF}^\gamma_g(\partial_t U)$ and the fact that U is a solution to $\partial_t U = \partial_x(\Lambda U)$. By differentiating the representative using $\partial_x \sigma^\varepsilon(x,t;0) = \lambda^\varepsilon(x)/\lambda^\varepsilon(\sigma^\varepsilon(x,t;0))$ (stated after (31)) and $\lambda^{\varepsilon\prime}(\sigma^\varepsilon(x,t;0)) = 0$ for $(x,t) \in \Omega$ we get

$$\Sigma^\gamma_{(0,t_0)}(U) \supseteq \Sigma^\gamma_{(0,t_0)}(A'_\Sigma \cdot \Lambda) \, .$$

Now A' plays the role A played in the proof above and we obtain immediately that

$$\Sigma^\gamma_{(0,t_0)}(U) \supseteq \Sigma^\gamma_{(0,t_0)}(\Lambda) = \mathbb{R} \setminus 0 \times \{0\}$$

if A' is constant near $c_2 t_0$.

Corollary 32. Let $U \in \mathcal{G}(\mathbb{R}^2)$ be the generalized solution to the hyperbolic equation $\partial_t U - \partial_x(\Lambda U) = 0$ with $U\mid_{t=0} = A \in \mathcal{G}(\mathbb{R})$ and let $\gamma(r) = \log(1/r)$.

(i) If A is γ-regular then

$$\mathrm{WF}^\gamma_g(U) \subseteq \mathrm{Char}_g P \cup \mathrm{WF}^\gamma_g(\Lambda) \, .$$

(ii) If $c_1 > 0$ there are solutions U such that

$$\mathrm{WF}_g^\gamma(U) \not\subseteq \mathrm{Char}_g P \quad \text{but} \quad \mathrm{WF}_g^\gamma(U) \subseteq \mathrm{Char}_g P \cup \mathrm{WF}_g^\gamma(\Lambda)\ .$$

These results illustrate that the medium singularities can be 'visible' in the singularity structure of the solution even if the initial values are regular. It suggests that in search for upper bounds for the propagation of singularities by linear hyperbolic PDOs with $\mathcal{G}$-coefficients in general one has to combine the following ingredients: for the classical part, the characteristic set of the operator and the wave front set of the right hand side (which is empty in our example); in addition, a certain set generated by the wave front sets of the coefficients seems to be necessary.

However, as can be seen from the following example by the interaction of singularities at the medium discontinuity an unexpected variety of irregular directions may occur. We investigate the case of a point source at some $s_0 > 0$.

Proposition 33. Let $\rho \in \mathcal{D}(\mathbb{R})$ with $\int \rho = 1$. Define A to be the class of $a(\phi, y) = \gamma(l(\phi))\rho(\gamma(l(\phi)(y - s_0))$, so that $A \approx \delta_{s_0}$. Then we have

$$\Sigma^\gamma_{g,(0,s_0/c_2)}(U) = \mathbb{R}^2 \setminus \{(0,0)\}$$

which implies that

$$\mathrm{WF}_g^\gamma(U) \not\subseteq \mathrm{Char}_g P \cup \mathrm{WF}_g^\gamma(\Lambda)\ .$$

Proof. Let $d_1 > 0$ such that $\mathrm{supp}(\rho) \subseteq [-d_1, d_1]$ and set $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$ where $\varphi_j \in \mathcal{D}(\mathbb{R})$ $(j = 1,2)$ with $\varphi_1(0) = \varphi_2(s_0/c_2) = 1$ and $\varphi_j \geq 0$. Let $d_2 > 0$ such that $\mathrm{supp}(\varphi_2) \subseteq [s_0/c_2 - d_2, s_0/c_2 + d_2]$.

Let $(\xi_0, \tau_0) \neq (0,0)$ arbitrary. We will show that $\widehat{(\varphi u^\varepsilon)}$ does not satisfy the γ-rapid decay property (19) in the direction of (ξ_0, τ_0), that is for $(\xi, \tau) = \omega(\xi_0, \tau_0)$ where $\omega \geq 1$.

Given any $N \in \mathbb{N}_0$ choose $\beta > 1$ and set $p = \beta(N+1)$. Fix $\phi \in \mathcal{A}_0(\mathbb{R}^2)$ arbitrary and let ε be small enough such that $\gamma_\varepsilon = \log(1/l(\phi_\varepsilon)) \geq 1$. Setting $\omega = \gamma_\varepsilon^{1/\beta}$ we obtain

$$\begin{aligned}
\frac{\omega^p}{\gamma_\varepsilon^N} |\widehat{(\varphi u^\varepsilon)}(\omega(\xi_0, \tau_0))| &= \gamma_\varepsilon |\widehat{(\varphi u^\varepsilon)}(\gamma_\varepsilon^{1/\beta}(\xi_0, \tau_0))| \\
&= \gamma_\varepsilon \iint e^{-i\gamma_\varepsilon^{1/\beta}(\tau_0 t + \xi_0 \sigma^\varepsilon(y,0;t))}\, \gamma_\varepsilon\, \rho(\gamma_\varepsilon(y - s_0))\, \varphi(\sigma^\varepsilon(y,0;t), t)\, dydt
\end{aligned}$$

Here we change the integration variables to $(z, r) = \gamma_\varepsilon(y - s_0, t - s_0/c_2)$ and pull the factor $\exp(-i\tau_0 s_0 \gamma_\varepsilon^{1/\beta})$, which is of modulus 1, out of the integral to arrive at the expression

$$|\int \rho(z) \int e^{-i(\gamma_\varepsilon^{1/\beta-1}\tau_0 z+\xi_0\gamma_\varepsilon^{1/\beta}\sigma^\varepsilon(\frac{z}{\gamma_\varepsilon}+s_0,0;\frac{r}{\gamma_\varepsilon}+\frac{s_0}{c_2}))} \cdot \varphi_1(\sigma^\varepsilon(\frac{z}{\gamma_\varepsilon}+s_0,0;\frac{r}{\gamma_\varepsilon}+\frac{s_0}{c_2}))\,\varphi_2(\frac{r}{\gamma_\varepsilon}+\frac{s_0}{c_2})\,drdz|\,. \qquad (48)$$

Assertion: For all $r \in \mathbb{R}$ we have as ε tends to 0

$$\gamma_\varepsilon^{1/\beta} \cdot \sigma^\varepsilon(\frac{z}{\gamma_\varepsilon}+s_0,0;\frac{r}{\gamma_\varepsilon}+\frac{s_0}{c_2}) \to 0 \qquad \text{uniformly for } z \in [-d_1,d_1]\,. \qquad (49)$$

If ε is small then $z/\gamma_\varepsilon+s_0 \geq s_0-d_1/\gamma_\varepsilon \geq s_0/2 > 0$ hence we may refer to the subcases corresponding to $x \geq \eta_\varepsilon$ in the proof of Prop. 22. Sticking to the notation of that proof recall that for $x > 0$ we defined $s_\varepsilon(x)$ by $\sigma^\varepsilon(x,0;s_\varepsilon(x)) = -\eta_\varepsilon$. Using the shorthand notations $z_\varepsilon = z/\gamma_\varepsilon + s_0$ and $r_\varepsilon = r/\gamma_\varepsilon + s_0/c_2$ we consider the following three cases

$c_2 r_\varepsilon \leq z_\varepsilon - \eta_\varepsilon$: we obtain $0 \leq \sigma^\varepsilon(z_\varepsilon,0;r_\varepsilon) = z_\varepsilon - c_2 r_\varepsilon = (z-r/c_2)/\gamma_\varepsilon$

$z_\varepsilon - \eta_\varepsilon \leq c_2 r_\varepsilon \leq s_\varepsilon(z_\varepsilon)$: by the monotonicity properties of σ^ε and the definition of s_ε we have $|\sigma^\varepsilon(z_\varepsilon,0;r_\varepsilon)| \leq \eta_\varepsilon$

$s_\varepsilon(z_\varepsilon) \leq c_2 r_\varepsilon$: here $0 > \sigma^\varepsilon(z_\varepsilon,0;r_\varepsilon) = -\eta_\varepsilon - c_1(r_\varepsilon - s_\varepsilon(z_\varepsilon))$; as noted in the proof of Prop. 22 a lower bound for $s_\varepsilon(z_\varepsilon)$ is given by $(z_\varepsilon + \eta_\varepsilon)/c_2 \geq (-d_1/\gamma_\varepsilon + s_0 + \eta_\varepsilon)/c_2$ and hence $0 > \sigma^\varepsilon(z_\varepsilon,0;r_\varepsilon) \geq (c_1/c_2-1)\eta_\varepsilon - c_1(d_1/c_2+r)/\gamma_\varepsilon$.

Since $\eta_\varepsilon = O(1/\gamma_\varepsilon)$ we deduce that in all cases for r fixed and $z \in [-d_1,d_1]$ we obtain $|\sigma^\varepsilon(z_\varepsilon,0;r_\varepsilon)| = O(\gamma_\varepsilon^{-1})$. Therefore $\gamma_\varepsilon^{1/\beta}|\sigma^\varepsilon(z_\varepsilon,0;r_\varepsilon)|$ is $O(\gamma_\varepsilon^{1/\beta-1})$ as $\varepsilon \to 0$ which proves (49).

Since $\varphi_j \geq 0$ the inner integrand in (48) is of the form $f_\varepsilon(r,z) \cdot \exp(ip_\varepsilon(r,z))$ where $f_\varepsilon \geq 0$ and $f_\varepsilon \to 1$, $p_\varepsilon \to 0$ as $\varepsilon \to 0$ pointwise in r and uniformly in z. If ε is small then its real part $\cos(p_\varepsilon)f_\varepsilon$ is nonnegative and hence by Fatou's lemma the real part of the whole integral tends to ∞ (since $\int \rho = 1$ is real and nonzero).

Therefore if N and p are as above then for any $n \in \mathbb{N}$ and $\varepsilon_0 > 0$ we can find $\varepsilon < \varepsilon_0$ and $\omega \geq 1$ such that $|\widehat{(\varphi u^\varepsilon)}(\omega(\xi_0,\tau_0))| \geq n\gamma_\varepsilon^N\omega^{-p}$. $\square$

6. Distributional shadows

We give a concise discussion of the situation in which the initial value A models a given distribution a. Away from the coefficient singularity the initial values and their singularities propagate as expected from the classical ray theory.

Proposition 34. If $\mathcal{G}(\mathbb{R}) \ni A \approx a \in \mathcal{D}'(\mathbb{R})$ then we have

$$U\mid_{V_1} \approx c_1^* a \qquad U\mid_{V_2} \approx c_2^* a$$

where c_j^* denotes the distributional pullback via $(x,t) \mapsto x + c_j t$.

Proof. This follows from the general consistency result presented in 3.1 when considering the Cauchy problems in the regions V_1 and V_2 separately. However, it is straightforward to prove it directly. If ψ is a test function with $\mathrm{supp}(\psi) \subset V_j$ then we have for the representative $(u(\phi))_\phi$ given in (28) and ε small enough

$$\langle u^\varepsilon, \psi\rangle = \iint a^\varepsilon(x+c_j t)\,\psi(x,t)\,dx\,dt = \langle c_j^* a^\varepsilon, \psi\rangle \to \langle c_j^* a, \psi\rangle \qquad (\varepsilon \to 0)\,.$$

□

If we assume that $a \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ we can take full advantage of Prop. 22 combined with formula (34). The case $c_1 = 0$ is completely covered in [19] and [21], Ex. 17.6, and we presented the result already in subsection 2.1. Therefore we assume $c_1 > 0$. Since $a^\varepsilon \to a$ in the sense of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and by the uniform boundedness of $\psi(\sigma^\varepsilon(y,0;t),t)$ together with Prop. 22 we conclude that as $\varepsilon \to 0$

$$\langle u^\varepsilon, \psi\rangle \to \iint a(y)\,\psi(\sigma(y,0;t),t)\,dy\,dt\,.$$

Here we can split the integration according to the different regions defined in (43) yielding

$$\int_{-\infty}^{0}\int_{-\infty}^{c_1 t} a(y)\,\psi(y-c_1t,t)\,dy\,dt + \int_{0}^{\infty}\int_{-\infty}^{0} a(y)\,\psi(y-c_1t,t)\,dy\,dt$$
$$+ \int_{-\infty}^{0}\int_{0}^{\infty} a(y)\,\psi(y-c_2t,t)\,dy\,dt + \int_{0}^{\infty}\int_{c_2 t}^{\infty} a(y)\,\psi(y-c_2t,t)\,dy\,dt$$
$$+ \int_{-\infty}^{0}\int_{c_1 t}^{0} a(y)\,\psi(\frac{c_2}{c_1}(y-c_1t),t)\,dy\,dt + \int_{0}^{\infty}\int_{0}^{c_2 t} a(y)\,\psi(\frac{c_1}{c_2}(y-c_2t),t)\,dy\,dt$$

(where the the first two pairs of integrals correspond to the first two lines in (43) respectively). Upon adjusting the integration variables appropriately in each integral, the integrals will have the factor $\psi(x,t)$ in common. Carefully inspecting the integral limits the sum of integrals can be rewritten as (combining the first two pairs of integrals)

$$\iint H(-x)H(-x-c_1t)a(x+c_1t)\psi(x,t)\,dx\,dt$$
$$+\iint H(x)H(x+c_2t)a(x+c_2t)\psi(x,t)\,dx\,dt$$
$$+\frac{c_1}{c_2}\iint H(x)H(-x-c_2t)a(\frac{c_1}{c_2}(x+c_2t))\psi(x,t)\,dx\,dt$$
$$+\frac{c_2}{c_1}\iint H(-x)H(x+c_1t)a(\frac{c_2}{c_1}(x+c_1t))\psi(x,t)\,dx\,dt\ .$$

Thus we have proved the following result

Proposition 35. If $\mathcal{G}(\mathbb{R}) \ni A \approx a \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ then the unique Colombeau solution U to problem (13)-(14) admits a distributional shadow $w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$ which is given by

$$w(x,t) = H(-x)H(-x-c_1t)a(x+c_1t) + H(x)H(x+c_2t)a(x+c_2t)$$
$$+\frac{c_1}{c_2}H(x)H(-x-c_2t)a(\frac{c_1}{c_2}(x+c_2t))+\frac{c_2}{c_1}H(-x)H(x+c_1t)a(\frac{c_2}{c_1}(x+c_1t))\ .$$

(Here all products are to be understood as products of measurable functions.)

Note that Prop. 35 recovers the distribution given in Thm. 3 concerning a global distributional solution to (1)-(2). On the other hand it shows explicitly the reason for the nonexistence of global distributional solutions in Thm. 1: it is not continuous in x as a distribution in t.

If a is an approximation to a delta-like source to the right of the medium singularity, e.g. a function with small support concentrated around $x_0 > 0$. Then we observe that the distributional shadow of the corresponding Colombeau solution looks like a refraction of an incoming signature at the medium discontinuity (cf. figure 6). Due to the scaling factor c_2/c_1 the support of the signal will be compressed while it is amplified by the same amount.

Remark 36. Assume that $\mathrm{supp}(a) \subseteq (0,\infty)$. Then we observe that for the wave front set of the limit distribution w, the distributional shadow of U, we have

$$\begin{aligned}
\mathrm{WF}(w\mid_{V_2}) &= \mathrm{WF}(c_2^* a\mid_{V_2}) \\
&= \{(x,t;\eta,c_2\eta) \in V_2 \times \mathbb{R}^2 \mid (x+c_2t,\eta) \in \mathrm{WF}(a)\} \\
\mathrm{WF}(w\mid_{W_1}) &= \mathrm{WF}(c_1^*(a(\frac{c_2}{c_1}.))\mid_{W_1}) \\
&= \{(x,t;\eta,c_1\eta) \in W_1 \times \mathbb{R}^2 \mid (\frac{c_2(x+c_1t)}{c_1},\eta) \in \mathrm{WF}(a)\}
\end{aligned}$$

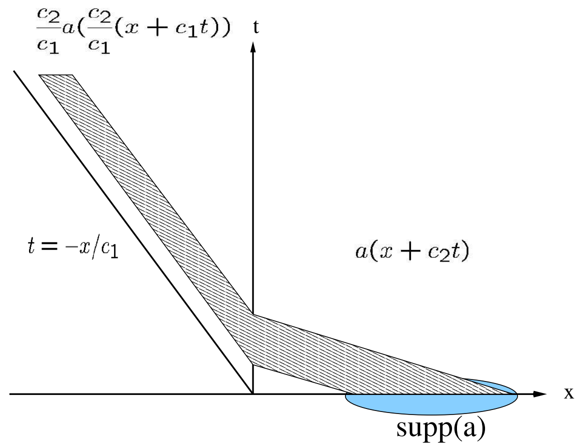

Figure 6. Transmission of highly spatially localized signal

and for the restriction to the submanifold $x = 0$ in the base space

$$\mathrm{WF}(w)\mid_{x=0} \supseteq \{(0, c_2 t; \xi, \tau) \mid \Big(\tau = 0 \text{ and } t \in \mathrm{supp}(a)\Big) \text{ or } \Big((\tau = c_1\xi \text{ or } \tau = c_2\xi) \text{ and } (t,\xi) \in \mathrm{WF}(a)\Big)\}\,.$$

The first two assertions are immediate. For the third one we note that by direct computation similarly to the first part of the proof of Thm. 1 one obtains for a tensor product of test functions $\varphi_1(x)\varphi_2(t)$ the expression

$$\widehat{(\varphi w)}(\xi,\tau) = \frac{1}{c_1}\int_{-\infty}^{0} e^{-ix(c_1\xi-\tau)/c_1}\varphi_1(x)\langle a, \varphi_2(./c_2 - x/c_1)e^{-i\tau./c_2}\rangle\, dx + \frac{1}{c_2}\int_{0}^{\infty} e^{-ix(c_2\xi-\tau)/c_2}\varphi_1(x)\langle a, \varphi_2((.-x)/c_2)e^{-i\tau./c_2}\rangle\, dx\,.$$

This cannot be rapidly decreasing for $\varphi_1(x) = 0$, $\mathrm{supp}(\varphi_2(./c_2)) \cap \mathrm{supp}(a) \neq \emptyset$ if $\tau = c_1\xi$, $\tau = c_2\xi$, or $\tau = 0$ (in the latter case the integrals can be interpreted as one-dimensional Fourier transform of a compactly supported piecewise continuous function with discontinuity at 0). Note that the cotangent direction $\tau = 0$ is a remnant of the wave front set of the proper Colombeau solution which corresponds with an instantaneous delay at the singularity.

Finally we reconsider the case of the initial value $a = \delta_{s_0}$ ($s_0 > 0$) — a point source to the right of the medium singularity (c.f. Prop. 33).

Proposition 37. Let γ be any admissible scaling and $\rho \in \mathcal{D}(\mathbb{R})$ with $\int \rho = 1$. Define A to be the class of $a(\phi, y) = \gamma(l(\phi))\rho(\gamma(l(\phi)(y - s_0)))$, so that $A \approx \delta_{s_0}$. Then the corresponding unique Colombeau solution U to the Cauchy problem (13)-(14) admits an associated distribution $w \in \mathcal{D}'(\mathbb{R}^2)$ which is given by

$$w = \begin{cases} H(-x)\frac{c_2}{c_1}\delta_{s_0}(\frac{c_2}{c_1}(x + c_1 t)) + H(x)\delta_{s_0}(x + c_2 t) & \text{if } c_1 > 0 \\ \delta(x)H(c_2 t - s_0) + H(x)\delta_{s_0}(x + c_2 t) & \text{if } c_1 = 0 \end{cases} . \quad (50)$$

(Note that all appearing products exist within $\mathcal{D}'(\mathbb{R}^2)$ by the wave front set condition.) The strong interaction of the propagating point singularity at s_0 with the medium singularity is seen from the following

$$\begin{aligned} \mathrm{WF}(w) = \{(s_0 - c_2 r, r; \xi, c_2\xi) \mid r \leq s_0/c_2, \xi \neq 0\} \\ \cup \{(-c_1 r, r + s_0/c_2; \xi, c_1\xi) \mid r \geq 0, \xi \neq 0\} \\ \cup \{(0, s_0/c_2)\} \times \mathbb{R}^2 \setminus \{(0,0)\} . \quad (51) \end{aligned}$$

Proof. Let $\psi \in \mathcal{D}(\mathbb{R}^2)$ then according to (34) we have

$$\langle u^\varepsilon, \psi \rangle = \iint \gamma_\varepsilon \rho(\gamma_\varepsilon(y - s_0))\, \psi(\sigma^\varepsilon(y, 0; t), t)\, dy\, dt$$

where we substitute $z = \gamma_\varepsilon(y - s_0)$ to obtain

$$\langle u^\varepsilon, \psi \rangle = \iint \rho(z)\, \psi(\sigma^\varepsilon(s_0 + z/\gamma_\varepsilon, 0; t), t)\, dz\, dt .$$

We assert that for almost all t: $\sigma^\varepsilon(s_0 + z/\gamma_\varepsilon, 0; t) \to \sigma(s_0, 0; t)$ as $\varepsilon \to 0$ uniformly for $z \in \mathrm{supp}(\rho)$. Since $z/\gamma_\varepsilon \to 0$ we have that $s_0 + z/\gamma_\varepsilon$ will stay in a compact interval neighborhood of $s_0 > 0$ for ε small. An inspection of the proof of Prop. 22 shows that the arguments there for the cases $x > 0$ can be extended for almost all t fixed and x varying in a compact interval included in $(0, \infty)$.

Hence by Lebesgue's dominated convergence

$$\langle u^\varepsilon, \psi \rangle \to \int \rho(z)\, dz \int \psi(\sigma(s_0, 0; t), t)\, dt \qquad (\varepsilon \to 0) .$$

By using (43) resp. (44) and splitting the integrals accordingly the relations (50) can be read off.

To prove (51) let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ be real valued with $\varphi(0, s_0/c_2) = 1$ and calculate with $\alpha = (c_2\xi - \tau)/(c_1\xi - \tau)$ (assume $c_1\xi - \tau \neq 0$ otherwise we use α^{-1} in the following)

$$\widehat{(w\varphi)}(\xi,\tau) = \frac{1}{c_2} e^{-i\tau s_0/c_2} \cdot \int e^{-ir(c_2\xi-\tau)/c_2} \big(H(-r)\varphi(r, \frac{s_0-r}{c_2}) + \alpha H(r)\varphi(\alpha\frac{c_1}{c_2}r, \frac{s_0-\alpha r}{c_2})\big)\, dr = \frac{1}{c_2} e^{-i\tau s_0/c_2} \widehat{\psi}(\frac{\tau - c_2\xi}{c_2})$$

for a compactly supported, piecewise smooth, and bounded function ψ on $\mathbb{R}$. $\alpha = 1$ if and only if $\xi = 0$ which implies that $\psi \in C^0(\mathbb{R}) \setminus C^1(\mathbb{R})$ since the left and right sided derivatives at 0 do not coincide and ψ does not vanish near 0. If $\alpha \neq 1$ then ψ has a jump discontinuity at 0. In both cases ψ is real valued which implies $\widehat{\psi}(-\tau) = \overline{\widehat{\psi}(\tau)}$ and hence the cones of irregular directions are symmetric. It follows that $\widehat{(w\varphi)}(\xi,\tau)$ cannot be rapidly decreasing regardless of the direction of (ξ,τ). □

7. Discussion

We have developed a microlocal analysis of Colombeau generalized functions to understand the singularity structure of the global solution of the ‘extended’ Hurd-Sattinger hyperbolic equation. Thereby we have related the propagation of singularities to the generalized characteristic set of the wave operator.

Particular attention had to be paid to the modeling of distributional coefficients, carefully preserving the wave front sets. This led to the notion of scaled, intrinsic, regularity. With a view to this regularity and the existence of classical shadows of the Colombeau solution we conclude with the following remark.

In general, the association relation does not respect wave front sets. For example, the Colombeau class U of $(x\phi(x))_\phi$ is associated to 0 but has wave front set $\mathrm{WF}_g(U) = \{0\} \times \mathbb{R} \setminus 0 \supset \emptyset = \mathrm{WF}(0)$. On the other hand, as shown in Ex. 10 (ii), for the class $\Lambda \approx R \otimes 1$, we have $\mathrm{WF}_g(\Lambda) = \emptyset$ whereas $\mathrm{WF}(R \otimes 1) = \{0\} \times \mathbb{R} \times \mathbb{R} \setminus 0 \times \{0\}$. But, in our refined notion of wave front set we recover $\mathrm{WF}_g^\gamma(\Lambda) = \mathrm{WF}(R \otimes 1)$.

A general inclusion relation exists ([17], Prop. 3.18, p. 130) in case of strong association. If we consider $\mathcal{G}^\infty \ni A \approx a \in \mathrm{L}^1_{\mathrm{loc}} \setminus C^\infty$ then by Thm. 23 we have $\mathrm{WF}_g(U) = \emptyset$ and by Prop. 35 we have $U \approx w \in \mathcal{D}'$, but $\mathrm{WF}(w)$ will be nonempty. Therefore the association of U with w cannot be strong. This indicates that the notion of strong association might be too restrictive for application to hyperbolic equations.

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We would like to thank Michael Oberguggenberger for his valuable comments and several inspiring discussions.

Appendix

A. Hierarchy of distributional products

For convenience of the reader we recall the basic definitions of the coherent distributional products described in Oberguggenberger's book [21], Chapt. II, leading to the hierarchy table referred to several times throughout the paper. All products considered here yield exactly the classical multiplication when restricted to smooth functions and their value, when defined, is always a distribution.

We denote by Ω an open subset of $\mathbb{R}^n$ and by $\widehat{u}$ the Fourier transform of u.

To begin with we mention the most elementary product in this context, i.e., $\mathrm{C}^\infty \cdot \mathcal{D}'$: the product of a smooth function and a distribution is defined as the adjoint of multiplication with the smooth function in the test function space.

A.1. Singular support and wave front set conditions

A.1.0.1. *Disjoint singular support:* Assume that u, v are in $\mathcal{D}'(\Omega)$ with disjoint singular supports. Then for any $x \in \Omega$ there is a neighborhood Ω_x and a function $f_x \in \mathcal{D}(\Omega_x)$ such that either $f_x u$ or $f_x v$ is smooth. Then in Ω_x the product of u and v can be defined in the sense of $\mathrm{C}^\infty \cdot \mathcal{D}'$ and by the localization properties of $\mathcal{D}'$ (cf. [11], subsect. 2.2) this consistently defines a distribution in Ω.

We briefly recall the definition of the wave front set. Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in T^*\Omega \setminus 0 := \{(x,\xi) \mid x \in \Omega, \xi \neq 0\}$ (the cotangent bundle over Ω with the zero section removed). u is said to be *microlocally regular* at (x_0, ξ_0) if there is $\varphi \in \mathcal{D}(\Omega)$, $\varphi(x_0) \neq 0$, and an open cone Γ with axial vector ξ_0 such that $\widehat{\varphi u}$ is rapidly decreasing in Γ. $\mathrm{WF}(u)$ is the closed subset of $T^*\Omega \setminus 0$ where u is not microlocally regular.

A.1.0.2. *WF favorable:* If u, $v \in \mathcal{D}'(\Omega)$ their wave front sets are said to be in favorable position if $(x, \xi) \in \mathrm{WF}(u)$ implies that $(x, -\xi) \notin \mathrm{WF}(v)$. In this case the product of u and v can be defined as the pullback of the tensor product $u \otimes v \in \mathcal{D}'(\Omega \times \Omega)$ by the diagonal map $\Omega \to \Omega \times \Omega$, $x \mapsto (x, x)$ (cf. [11], Thm. 8.2.10).

A.2. The Fourier product

Given two distributions $u, v \in \mathcal{D}'(\Omega)$ we say that their Fourier product exists if for every $x \in \Omega$ there is an open neighborhood Ω_x and $f_x \in \mathcal{D}(\Omega)$, $f_x = 1$ on Ω_x, such that the $\mathcal{S}'$-convolution of $\widehat{f_x u}$ and $\widehat{f_x v}$ exists. Locally near x, the product of u and v is then defined to be the inverse Fourier transform of $\widehat{f_x u} * \widehat{f_x v}$ (for a definition of $\mathcal{S}'$-convolvability see [21], sect. 6).

A.3. Duality products

Let X be a *normal space of distributions*, that is $\mathcal{D} \subseteq X \subseteq \mathcal{D}'$ and $\mathcal{D}$ is dense in X. Assume that the dual space X' is (equipped with a locally convex topology so that it becomes) normal as well and that multiplication with a fixed element in $\mathcal{D}$ induces a continuous linear map both from X into X and from X' into X'.

For any normal space of distributions Y denote by Y_{loc} the set of distributions $v \in \mathcal{D}'$ such that $\psi v \in Y$ for all $\psi \in \mathcal{D}$. If $u \in (X')_{\text{loc}}$ and $v \in X_{\text{loc}}$ then the product of u and v can be defined by

$$\langle u \cdot v, \psi \rangle := \langle \chi u, \psi v \rangle$$

for $\psi \in \mathcal{D}$ and $\chi \in \mathcal{D}$ chosen arbitrarily with $\chi = 1$ on $\operatorname{supp}(\psi)$. Note that in the above definition the left hand side denotes a $(\mathcal{D}', \mathcal{D})$ pairing while the right hand side uses the pairing (X', X).

A.4. Regularization and passage to the limit

The basic idea is to regularize one or both factors by convolution, perform the multiplication in the sense $\mathrm{C}^\infty \cdot \mathcal{D}'$ or $\mathrm{C}^\infty \cdot \mathrm{C}^\infty$, and try to take the limit. The regularizing convolutions are carried out with two principal types of mollifiers.

A.4.0.3. *Strict delta net:* This is a net $(\rho^\varepsilon)_{\varepsilon>0}$ in $\mathcal{D}(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\rho^\varepsilon) \to \{0\} \quad \text{as } \varepsilon \to 0 \tag{52}$$

$$\int \rho^\varepsilon(x)\, dx = 1 \quad \text{for all } \varepsilon > 0 \tag{53}$$

$$\int |\rho^\varepsilon(x)|\, dx \quad \text{is bounded independently of } \varepsilon. \tag{54}$$

A.4.0.4. *Model delta net:* Given $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\int \varphi(x)\, dx = 1$ define the net $(\varphi_\varepsilon)_{\varepsilon>0}$ by $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon^n$.

Consider the following four possibilities to define a product of u and v:

$$u \cdot [v] = \lim_{\varepsilon \to 0} u(v * \rho^{\varepsilon}) \quad (1)$$

$$[u] \cdot v = \lim_{\varepsilon \to 0} (u * \rho^{\varepsilon}) v \quad (2)$$

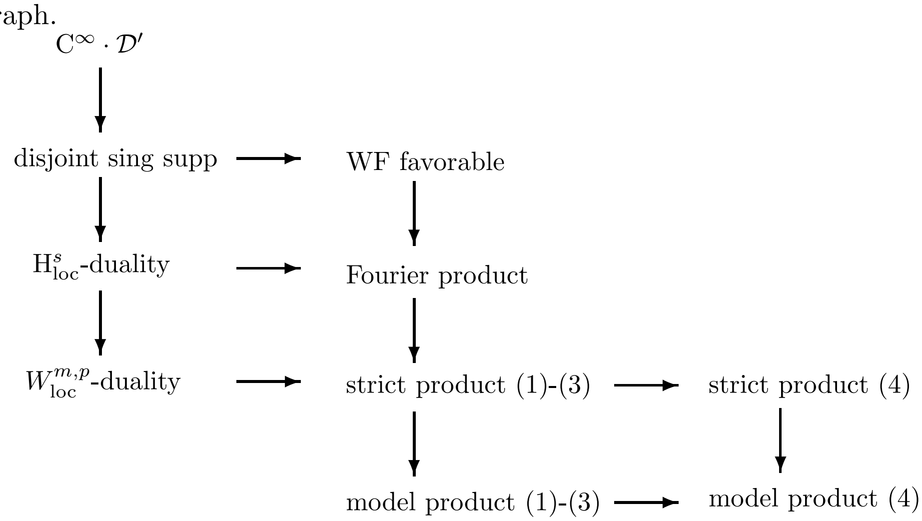
$$[u] \cdot [v] = \lim_{\varepsilon \to 0} (u * \rho^{\varepsilon})(v * \sigma^{\varepsilon}) \quad (3)$$

$$[u \cdot v] = \lim_{\varepsilon \to 0} (u * \rho^{\varepsilon})(v * \rho^{\varepsilon}) \quad (4)$$

where the limit is required to exist in $\mathcal{D}'(R^n)$ and independent of the choice of $(\rho^{\varepsilon})_{\varepsilon>0}$ and $(\sigma^{\varepsilon})_{\varepsilon>0}$ in the class of strict, resp. model, delta nets. This defines 4 types of so called *strict*, resp. *model*, products. Since the definitions (1)-(3) turn out to be equivalent when using strict, resp. model, delta nets (cf. [21], Thms. 7.2 and 7.11) we distinguish only the following four products: strict product (1)-(3), strict product (4), model product (1)-(3), and model product (4).

A.5. Coherence properties

The various products satisfy coherence properties and can be brought into the following hierarchy table. Here, an arrow indicates that a product definition is contained and consistent with its successor in the graph.



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